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On solutions of holonomic divided-difference equations on nonuniform lattices. (English)

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The authors study polynomials orthogonal on nonuniform lattices

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3 \text{ if } q \neq 1 \text{ and } x(s) = c_4 s^2 + c_5 s + c_6 \text{ if } q = 1.$$

In order to complete the existing characterization of the classical orthogonal polynomials on this type of lattices, they introduce two new monomial bases for the expansion

$$P_n(x(s) = \sum_{k=0}^{n} a_k x(s)^k,$$

and their formal Stieltjes function

$$\int_a^b \frac{\mathrm{d}\mu(x(s))}{x(z) - x(s)} = \sum_{n=0}^{\infty} \frac{\mu_n}{x(s)^{n+1}};\ \mu_n = \int_a^b x(s)^n \mathrm{d}\mu(x(s)),\ x(z) \notin (a, b).$$

As indicated in the introduction, the first basis $\{F_n\}_n$ is chosen to provide nice operational properties: The basis $\{F_n(x(s))\}_n$ of polynomials of degree n in $x(s)$ satisfy

$$\mathbf{D}_x F_n(x(s)) = a_n n F_{n-1}(x(s)),\ \mathbf{D}_x \frac{1}{F_n(x(s))} = \frac{b_n}{F_{n+1}(x(s))},$$

and

$$\mathbf{S}_x F_n(x(s)) = c_n F_n(x)s)) + d_n F_{n-1}(x(s)),\ \mathbf{S}_x \frac{1}{F_n(x(s))} = \frac{e_n}{F_n(x(s))} + \frac{f_n}{F_{n+1}(x(s))},$$

with given constants a_n, b_n, c_n, d_n, e_n and f_n; the **companion operators** are given by

$$\mathbf{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})},\ \mathbf{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{2}.$$

(these operators transform polynomials of degree n in $x(s)$ into degree $n-1$ resp. n polynomials)

To achieve solutions of arbitrary linear divided-difference equations with polynomial coefficients involving products of $\mathbf{D}_x$ and $\mathbf{S}_x$ only (more suitable for general Askey-Wilson polynomials), the second basis $\{B_n\}_n$ is introduced on the general q-quadratic lattice

$$x(s) = uq^s + vq^{-s}$$

by

$$B_n(a, s) = (2auq^s, q)_n (2avq^{-s}, q)_n,\ n \geq 1;\ B_0(x, s) = 1.$$

(the notation $(\cdots; q)_n$ indicates the customary q-Pochhammer symbol)

The elements of this basis satisfy a host of properties and lead to quite a number of explicit series solutions to the divided difference equations studied.

The connection between the two bases is given in the paper in Proposition 16:

$$F_n(x(s)) = \sum_{j=0}^{n} r_{n,j} B_j(a, s),\ B_n(a, s) = \sum_{j=0}^{n} s_{n,j} F_j(x(s)),$$

with explicit expressions for the connection coefficients $r_{n,j}$ and $s_{n,j}$.

The layout of the paper is as follows:

1. Introduction
2. A new basis compatible with the companion operators
3. Algorithmic series solutions of divided-difference equations
4. Applications and illustrations
5. Conclusions and perspectives

References (35 items)

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MSC:

33D45 Basic orthogonal polynomials and functions (Askey-Wilson polynomials, etc.)

39A13 Difference equations, scaling (q-differences)

Cited in **1** Document

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