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Special functions. (English)

Encyclopedia of Mathematics and Its Applications. 71. Cambridge: Cambridge University Press. xvi, 664 p. 55.00; $ 85.00 (1999). [ISBN 0-521-62321-9/hbk]

This book covers a wealth of material on special functions, notably knowledge which was developed by Richard Askey and his co-authors during the several decades of his contributions to this subject, but also material which connects special functions with combinatorial questions collected by George Andrews. These two researchers are well-known for their efforts to support and demand the use of hypergeometric functions in their respective fields, hence hypergeometric functions and q-hypergeometric functions (basic hypergeometric functions) play a prominent role in the book under review. The book covers 12 chapters and 6 appendices. Furthermore, it contains a rich collection of 444 (!) exercises that are distributed among the different chapters. Here are the details:

Chapter 1: The gamma and beta functions. In this chapter the usual material about the gamma and beta functions is covered. Moreover, results for the logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ of the gamma function and for the Hurwitz and Riemann zeta function are developed; in particular, several integral representations are given. The gamma function is characterized by the Bohr-Mollerup theorem, and finally the p-adic gamma function is introduced.

Chapter 2: The hypergeometric functions. The generalized hypergeometric function is introduced, and elementary examples are given. Euler's integral representation, and the usual summation theorems (Gauss, Chu-Vandermonde, Pfaff-Saalschuetz, Dixon) come next. Then the hypergeometric differential equation is treated from the Riemannian point of view that analytic functions are determined to a large extent by their singularities. Next, Barnes type integrals, contiguous relations, and continued fractions of ratios of hypergeometric functions are covered. The Jacobi polynomials as specific hypergeometric polynomials are introduced. Finally, dilogarithms, binomial sums, and fractional integration by parts are treated.

Chapter 3: Hypergeometric Transformations and Identities. This chapter starts with quadratic transformations. Then elliptic integrals are considered as hypergeometric functions, and arithmetic-geometric mean sequences are introduced. Next, transformations for balanced series, Whipple's transformation and Dougall's formula are given. Integral analogues of hypergeometric sums lead to the Wilson polynomials. The Riemannian point of view is reconsidered in connection with quadratic transformations. Gosper's algorithm on indefinite hypergeometric summation is given, and the Wilf-Zeilberger method for proving hypergeometric identities is compared with Pfaff's method, and the question of how these methods are related to contiguous relations is analyzed.

Chapter 4: Bessel functions and Confluent hypergeometric functions. Here, the confluent hypergeometric function is introduced. Then a Barnes type integral is given. As special cases, the Whittaker and the Bessel functions are covered. Recurrence equations, integral representations, and asymptotic expansions are treated. A two-dimensional Fourier transform leads to a generating function of the Bessel functions. Addition theorems and integrals of Bessel functions come next. Finally zeros and monotonicity properties of Bessel functions are discussed.

Chapter 5: Orthogonal polynomials. The elementary properties of general orthogonal polynomials are derived. Next, Gauss quadrature is examined. Then zeros of orthogonal polynomials are discussed, and the connection of orthogonal polynomials with continued fractions is treated. After Parseval's formula, the moment-generating function is introduced.

Chapter 6: Special orthogonal polynomials. Under this heading comes a discussion of the classical hypergeometric type orthogonal polynomials. The Hermite, Laguerre and Jacobi polynomials and their properties are discussed in detail.

Then linearization coefficients are considered, and combinatorial interpretations of the classical systems are given. The Wilson polynomials and their properties come next. Finally a q-generalization of the ultraspherical polynomials is deduced.
Chapter 7: Topics in orthogonal polynomials. Connection coefficients are introduced, and for the classical systems these coefficients are explicitly determined. Nonnegativity results for hypergeometric functions and positive polynomial sums come next. In particular, the Askey-Gasper inequality which was used by de Branges in his proof of the Bieberbach conjecture [*L. de Branges*, Acta Math. 154, 137-152 (1985; Zbl 573.30014)] is deduced using results about connection coefficients. Theorems by Vietoris and Turan are covered. Finally, Apery's irrationality proof of $\zeta(3)$ is given.
Chapter 8: The Selberg integral and its applications. Here, Selberg's and Aomoto's integrals and extensions of these formulas are given. A two-dimensional electrostatic problem studied by Stieltjes connects the zeros of the Jacobi polynomials with Selberg's integral in an interesting way. Siegel's inequality, which is a refinement of the arithmetic-geometric mean inequality, is studied next, and a connection to the Laguerre polynomials is considered. Applications of Selberg's integral to constant-term identities and nearly-poised ${}_3F_2$ identities are given. The Hasse-Davenport relation and a finite-field analog of Selberg's integral finish this chapter.
Chapter 9: Spherical harmonics. Harmonic polynomials and the Laplace equation in three dimensions provide an introduction to the topic of this chapter. Then the harmonic polynomials of degree k and their orthogonality are studied. Their addition theorem yields an addition theorem for ultraspherical polynomials which was used by *L. Weinstein* [Int. Math. Res. Not. 5, 61-64 (1991; Zbl 743.30021)] in his proof of the Bieberbach conjecture. It is shown that Fourier transforms of higher order are still expressible in terms of Bessel functions. Next, finite-dimensional representations of compact groups are studied. Finally, Koornwinder's product formula for Jacobi polynomials is given.
Chapter 10: Introduction to q-series. In this chapter, the theory of q-hypergeometric series (basic hypergeometric series) is motivated by considering non-commutative q-algebra, related with the rule $yx = qxy$. Using this approach, the definition of the q-binomial coefficients and their connection with the standard binomial coefficients are straightforward. The q-integral is defined, and the q-binomial theorem is proved by two different approaches both based on recurrence equations. The q-Gamma function, and Jacobi's triple product identity are next. Ramanujan's summation formula is used to give results about the representations of numbers as sums of squares. Elliptic and theta functions are covered, and q-beta integrals are used to find a q-analog of the Wilson polynomials. Finally, the basic hypergeometric series is studied. Basic hypergeometric identities, the q-ultraspherical polynomials and the Mellin transform finish this chapter.
Chapter 11: Partitions. Partitions are defined, and the connection of partition analysis with q-series is studied. Generating functions, and other results on partitions are obtained by this method. Next, graphical methods are discussed, and congruence properties of partitions are covered.
Chapter 12: Bailey chains. Rogers's second proof of the Rogers-Ramanujan identities is given. Then, Bailey's lemma and Watson's transformation formula are treated. Finally, some applications are given.
Appendices on infinite products, summability and fractional integration, Asymptotic expansions, Euler-Maclaurin summation formula, Lagrange inversion formula, and series solutions of differential equation follow, and a bibliography, an index, a subject index and a symbol index complete the book.
To begin with these last items: For a book of this size, the subject index is

rather small (3 pages). Hence, obviously not every subject can be found here. Just to mention a few, one finds neither addition theorem, nor Bieberbach conjecture, nor irrationality of $\zeta(3)$, nor indicial equation (notation defined on p. 640 in Appendix F, and used on p. 74). Many other topics cannot be found in the subject index either. In my opinion, a book covering such a wealth of information needs a better index. Similarly, the bibliography (on purpose) contains only the articles that are explicitly mentioned in the text, and by no means covers the topic of the book encyclopedically. Another minor irritation is the fact that the notations $[x]$ (e.g. on pp. 203, 314) and $\lfloor x \rfloor$ (e.g. on pp. 279, 340) for the greatest integer in x are used synonymously, but only the latter is defined on p. 15.

On the other hand, the material is written in an excellent manner, and it gives the reader very interesting insights to special functions. On many occasions, theorems are proved by several alternative methods. This gives the reader a much better feeling for what is going on, indicating that 'Special functions' is not a topic which can be taught deductively. Furthermore, the book contains very few typos.

But a book of this size covers thousands of formulas, and by Murphy's law, a few of them should be incorrect. I tried to find such misprints, in particular in the sections 3.11 and 3.12 about summation methods, since there I could use my Maple software for purposes of detection [*W. Koepf*, Hypergeometric summation. An algorithm approach to summation and special function identities (1998; Zbl 909.33001)]. Not surprisingly, this search was successful: Formula (3.11.10) is incorrect by a factor $-n$; both identities in the middle of p. 175 are incorrect restatements of the corresponding contiguous relations (3.11.12) and (3.11.15) on p. 173; furthermore in formula (3.12.1) the upper parameter $z+n-1$ should read $z+n+1$ (I would like to thank George Andrews for sending me the corrected formula).

In spite of these minor shortcomings, I recommend this book warmly as a rich source of information to everybody who is interested in 'Special Functions'.

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- 33-02 Research monographs (special functions)