3.8 Computer Algebra in Education

Twentyfive years ago when I studied Physics, only one of the students who participated in a laboratory course that I took was possessing one of the first *calculators* to process the data we were obtaining. Everybody else, including me, used a *slide-rule* for this purpose. Nowadays, the calculator is used by everybody, by far not only for academic purposes. Hence it is the responsibility of school education, and here in particular of Mathematics education, to take this situation into account, and to teach our children the (intelligent) use of a calculator.

In my opinion, there is no doubt that sooner or later computer algebra systems or hand-held computer algebra tools will be used by everybody in the same way as (numeric) calculators are used today. Obviously this gives us a new responsibility to integrate computer algebra in the Math curriculum and to teach the students the use of them. When I realized this, I began to use DERIVE in my calculus courses at the Free University Berlin, in particular for the Math teacher education [41, 38, 39] as a didactical tool.

Whereas calculators brought more numeric computation into the classroom, computer algebra systems enable the use of more symbolic computation. In particular, the interaction between numeric and symbolic computation can enhance the Math education substantially [40].

The usage of computer algebra in the classroom is increasing worlwide, and slowly these steps are institutionalized and Math curricula are adapted accordingly.

In the articles below, several main contributors come to word, and describe their activities in this direction. These activities are spread from the use of hand-held devices like the TI-89 in undergraduate education to the use of special purpose computer algebra systems in graduate studies.

Wolfram Koepf (Kassel)

3.8.1 New Hand-Held Computer Symbolic Algebra Tools in Mathematics Education

An unparalleled opportunity exists today to deliver better mathematics education than we ever thought possible. And it can be delivered to *all* students because of the rapid expansion of inexpensive powerful hand-held computer technology with built-in computer symbolic algebra (CA) software. These amazing products are now available from Casio and Texas Instruments (TI-89, TI-92, CASIO CFX-9970G and CASIO Algebra FX2.0). We fear, however, that our community is not ready to deal with the implications of their use due to misunderstanding, fear, and inexperience.

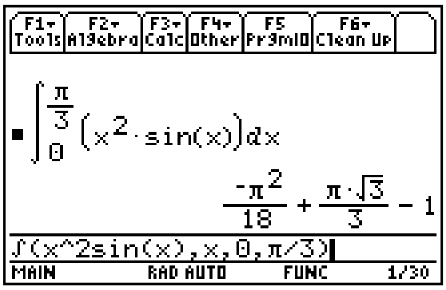
It is a fact that hand-held scientific calculators have *significantly* changed the high school and university mathematics curriculum around the world in the past 25 years. For example, many topics that dealt with paper and pencil "computation" involving transcendental functions have been deleted. Many sections and even some chapters in textbooks dealing with paper-and-pencil computation methods became obsolete and disappeared from the curriculum.

Why? Because hand-held scientific calculators provided better ways to "compute" than paper-and-pencil methods. *The same thing (obsolescence) will soon happen with paper-and-pencil symbolic algebraic manipulations common today because of student use of inexpensive hand-held CA systems that now exist and soon will proliferate.*

It is important to note that *less time is now spent* on certain topics (ones made obsolete by scientific calculators) but we still "do" the same things. For example, we still "compute" the sine of 14.25 degrees but not by the time consuming method of paper-and-pencil linear interpolation. What changed was *not* the "to do's" but the "how to" do the "to do's." It is also equally important to note that many educators found pedagogical ways to use scientific calculators that enhanced the teaching and learning of mathematics.

We should continue to teach the same *content* topics, but we should expect the methods we will use "to do" or "to apply" the topics will change (and likely be much faster) because of advancing technology. For example, some reformers have said it is no longer necessary or desirable to teach factoring. We believe they are wrong. The mathematical topic of factoring *is a major and important topic.* It *must* remain in the curriculum. However, in the past factoring was a mental or tedious paper-and-pencil exercise that often hid the really beautiful underlying mathematics. Recall using the "rational zeros theorem" to factor $2x^3 - 5x^2 - 9x + 18$? What a painful experience for students—and it took a good deal of time to do just one example! With CA this polynomial can be factored instantly. What is important and was often lost in the fog of tedious computations was recognizing what the factors can tell us about the behavior of the expression. The *concept* topic of factoring *is* important! Integrating CA into the curriculum means the same topics can be taught in less time so more time can be devoted to new mathematics, better mathematics, understanding, proof, problem solving and so forth.

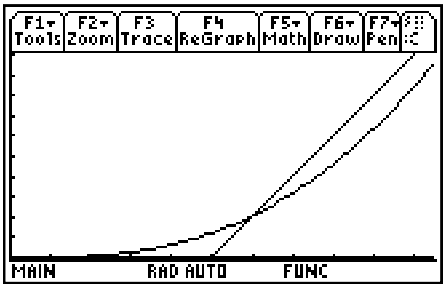
Consider the "exercise" of evaluating the definite integral given in the example below. We use the TI-89 "*integrate*" command to do the computation.

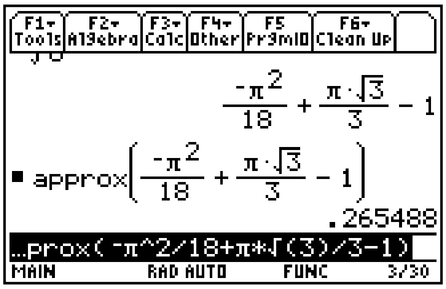


The answer shown is "exact" but what "*is*" $\frac{-\pi^2}{18} + \frac{\pi\sqrt{3}}{3} - 1$ really? How do we know it is correct? Rather than asking students to do the tedious "paper-and-pencil" manipulations, with no real understanding of the integral concept necessary to find the "exact" answer, a much better series of questions can now be asked.

a. How do you know this definite integral exists?
b. Describe a "problem" for which the integral is the "answer."
c. *Estimate* the answer without using CA and compare your estimate with the CA solution.

This solution is easy and involves a computer or graphing calculator graph in the interval $[0, \pi/3]$ by $[0, 1]$ together with the observation that the area under the curve is "about" the same as that under the line shown $(1/2) \cdot \text{base} \cdot \text{height} \approx 0.5 \cdot 0.5 \cdot 1 = 0.25$.





To conclude that the CA answer *must* be near 0.25 requires real understanding of calculus concepts (not low order manipulative skills).

Students will demand the use of CA because it provides a "better" tool to do the tedious algebraic manipulations common in "mathematics" today. To do otherwise is a waste of valuable teaching time and learning opportunities. We have wonderful examples of innovative curricula from Austria "better mathematics better" using CA [34]. Use of hand-held CA *together with a recognition that some of what we once did is now obsolete* can provide the time to spend in the classroom on more worthwhile topics!

What is needed today and in the future is a school and university mathematics curriculum that takes advantage of computer algebra technology to assist students in gaining mathematical understanding, in becoming powerful and thoughtful "thinkers," communicators, and problem solvers. There should be a *balanced* approach to the use of computer algebra technology in mathematics teaching and learning. Some of us in mathematics education have an appreciation of the deeper and richer understanding of mathematics that is possible when technology is used effectively. The great challenge for mathematics educators in the future is to make clear to the "public" that such good mathematics is both possible and desirable [17].

Bert K. Waits (Columbus)

3.8.2 The Dutch Perspective

Computer algebra has been an issue in mathematics education in the Netherlands for some years already. This does not imply that the discussion on this phenomenon has resulted in an agreement; no consensus on the role of computer algebra in the mathematics classroom has emerged so far.

Below, I describe some recent developments in my country from a personal perspective. I confine myself to mathematics education at upper secondary, pre-university level. As far as the university level is concerned, I refer to the Internet site of Computer Algebra Netherlands (`www.can.nl`), that provides information on the use of computer algebra in academic education.

Firstly, I describe the Dutch situation concerning curriculum and assessment. Secondly, I briefly consider the first educational experiments with computer algebra. Thirdly, the rise of the graphics calculator is discussed. In the end, computer algebra comes into the picture again, but now in a hand-held format. I conclude with an imaginary jump into future.

Understanding the developments in my country requires some knowledge of the organisation of the curriculum and assessment.

As far as the curriculum is concerned, it is important to notice that there is no detailed curriculum that prescribes which topic should be taught when. The curriculum is defined by a description of skills and concepts that will be assessed by the end of secondary school; the schools are free to choose how they get there. They can also decide on the textbooks they want their students to use. It is because of this relative freedom that the final assessment is so important.

The final assessment at upper secondary level consists of two parts: a school set assessment and a final national examination that is externally set and internally graded. This national examination is very important for the implementation of technology; if a certain technology device is not allowed at the final examination, it will not easily become popular in the classroom. The current regulation is that the graphics calculator is required at the final examination, whereas computer algebra is excluded. Two arguments guided this legislation. Firstly, it would be hard to organize a national examination throughout the country with computer access for all the candidates; hand-held computer algebra was not yet available. Secondly, the financial aspect was important. If computer access was required at the final examination, schools would need money to buy them, whereas hand-held technology devices are supplied by the students themselves.

For the school assessment, the authorities recommend the use or partial use of a computer, but again, the schools are free to decide. I have the impression that the number of schools that use a computer in their examination is increasing. The computer is often used in combination with investigation tasks where a written or oral report forms the assessment.

Obviously, the Dutch policy on technology is a careful one. Information about the different strategies concerning technology use and assessment in other countries can be found in [23].

The first project on computer algebra at upper secondary level started in 1990. The idea of this two-year project was to develop short instructional units that were tested in pilot schools. Although the production of these materials was useful, the project as a whole was not very successful. This was caused by the lack of computer facilities at schools and by the difficulties students had with the user friendliness: they had little 'computer literacy' and a windows interface was not available. Obviously, the time was not yet ripe for the implementation of computer algebra at this level.

By the end of this project, a group of volunteering teachers decided to continue the work. This group, called CAVO, existed until 1998 and was a lively and important platform for further development and discussion (see [25]). In the mean time, however, the graphics calculator came on the market, and attracted much attention.

The development of the graphics calculator elicited discussion on which technology platform should be used in secondary education (see [20]). The Dutch authorities decided that the implementation of the graphics calculator would be the first step to take. Therefore, they supported a research project on this issue in 1992. This project was carried out by the Freudenthal Institute, a research group on mathematics education. Later it became an integrated part of a larger curriculum development project called Profi. Results of the Profi-project included student textbooks that integrated the use of the graphics calculator, and experimental examinations that required the availability of a hand-held graphing device. The role of the graphics calculator in this project is summarized in [24].

Some educators and teachers were, however, opposed to the implementation of the graphics calculator. Their arguments were that computer algebra is a much more sophisticated mathematical tool, and that a graphics calculator is only a temporary step backwards compared to the possibilities that PC's offer. In 1996, however, a questionnaire revealed that PC's were hardly ever used during mathematics lessons, although they were available in schools. This supports the idea that real implementation of technology requires that the student has direct access to the device. The limited mathematical power of the graphics calculator is not an important disadvantage: it allows teachers, textbook authors and examination boards to have sufficient time to carefully integrate a graphical and numerical tool without having to cope with computer algebra in the mean time.

The choice of the graphics calculator may be a temporary preference indeed: the symbolic calculator raises the issue again. Nowadays, computer algebra is also available in a hand-held format. A first pilot experiment using the TI-92 revealed that the students appreciated this machine as an 'algebraic calculator', but not so much as a dynamic geometry tool [22].

When the Dutch Association of Mathematics Teachers became aware of the possible impact of symbolic calculators on secondary mathematics education, an Advisory Board on Computer Algebra and Symbolic Calculator was formed. In May, 1998, this Board concluded that:

- computer algebra should be implemented in upper secondary education;
- research was needed in order to find answers to the pedagogical and curriculum issues that will be raised by this;
- as a computer algebra platform, the PC would be preferred to the symbolic calculator, at least in the long term.

For the full report of the Board (in Dutch!) I refer to the Dutch Association of Mathematics Teachers site: `www.nvvw.nl`.

In the fall of 1998, the Freudenthal Institute conducted an explorative case study using the symbolic calculator. This machine turned out to be quite useful in investigation tasks. The sophisticated use of variables and parameters, however, was not always clear to the students. Furthermore, some students were reluctant to use computer algebra for the application of techniques that they had not yet mastered manually.

At present, many research questions concerning the role of computer algebra in secondary education are still unanswered (see [21]). No decisions on its implementation in the Netherlands have been made so far. In the next few years, I expect three developments to take place.

Firstly, teachers, examination boards and school book authors will get used to the graphics calculator and will take advantage of the pedagogical possibilities that these devices offer.

Secondly, research will be carried out concerning the role of computer algebra in the learning of mathematics and, more specifically, in the learning of algebraic concepts. Such a study was started recently by the Freudenthal Institute.

Thirdly, research will be carried out on the possibilities of computer algebra as a wide-range technology tool. A project that focuses on the use of a computer algebra environment in combination with a text editor (to write mathematical reports) and an Internet browser has been started at the Algemeen Pedagogisch Studiecentrum, an institute for improvement of (mathematics) education.

It is my hope that these developments will lead to a carefully considered implementation of computer algebra in secondary education.

Paul Drijvers (Utrecht)

3.8.3 Computer Algebra in Teaching and Learning Mathematics: Experiences at the University of Plymouth, UK

There are many ways of using computer algebra systems in the teaching and learning of mathematics. Research in the literature shows that student learning can be enhanced when CAS is used as an add-on through 'laboratory activities' (see for example the work of Mayes [43] and Heid et al. [33]) and when more fully integrated into the curriculum (see for example the report of the Austrian Experiment [4]).

Our experience of using the computer algebra system *DERIVE* in some of our courses in Plymouth began in the late 1980s with some experimental research work with engineering students. Encouraged by the outcomes of this research we have increased the use of CAS to the mathematics degree programmes. Inevitably we have experienced resistance along the way from many of our colleagues. Their concern is that students may become too dependent on CAS and will not develop appropriate mental mathematics skills. These concerns have encouraged the team of CAS enthusiasts at Plymouth to develop ways of using CAS to enhance learning through investigational activities that develop concepts and understanding.

Recruitment of students to engineering degrees in the United Kingdom has been falling steadily over the past decade. Among the many reasons for this phenomenon is the perception among school students that mathematics and science, physics in particular, are hard and unglamorous subjects of study, and that to be an engineer is a low status occupation. Attempts to redress this perception through the National Curriculum in secondary schools have led to a widening gap between a student's knowledge of mathematics and physics at the age of eighteen and the traditional starting points of degrees in engineering subjects.

At the same time access to higher education in the United Kingdom has widened considerably to admit far more mature students (aged 21 or over) than hitherto. This widening of access has enriched the undergraduate population in many ways but has usually meant that the mathematical and scientific background of such students is weak.

An alternative approach is to broaden the curriculum by developing undergraduate engineering degree courses that are focused more on Design, Communications or Technology Management rather than the more traditional and mathematically more demanding areas of Mechanical or Electrical Engineering. As a consequence the customary entry requirements of Advanced level Mathematics (or equivalent) (Advanced level is a school leaving public examination in England and Wales taken at 18 years of age) has been removed and about one third of the intake to these new courses has not studied any mathematics beyond the age of 16 years. Consequently they begin their undergraduate studies lacking much of the mathematical ability, thinking and confidence which earlier cohorts have displayed.

How then can such students be taught an appropriate mathematics course given their weak knowledge base? Fortunately the great majority of today's new undergraduates possess significant IT skills. It was therefore decided to exploit the fact that they are comfortable with IT and use it in a central role to support the teaching and learning of their mathematics.

We encourage the use of graphics calculators, particularly the TI-83, and the computer algebra package *DERIVE*. Once their prices have become competitive, we intend to use the TI-92 and TI-89 as well. We have a dedicated computer laboratory with 16 PCs and students will usually spend two hours a week in the laboratory for the mathematical methods modules. In this time they normally follow guided investigations which either follow up some work introduced in a lecture or prepare for work which will then be followed up in a subsequent lecture. We regard this approach as a modification of Buchberger's white-box/black-box principle [12]. Our work with *DERIVE* has been ongoing for some years now and may be read about elsewhere ([51], [6], [52]). In lectures we may often use a laptop computer and overhead viewscreen for demonstrations with *DERIVE* and likewise with the graphics calculators.

The various topics of the module syllabus in the Design, Communications and Technology Programme are introduced via integrated case studies whose origins are based either in the student's previous experience or engineering knowledge. The case studies are integrated in the sense that a scenario is presented, requiring a model and/or solution, and the students identify mathematics that they feel appropriate. The solution is then progressed until further support is required.

Given the academic background described above this often means some sort of support with the implementation of a piece of mathematics (e.g. solution of an equation, differentiation of an expression). This is where the IT comes in—at this stage the students make use of a CAS (currently *DERIVE*) to provide the support and help them to progress their solution. The strategy of 'identify the relevant mathematics, progress the solution, seek CAS assistance, progress the solution,...' is continued until a satisfactory solution is obtained. En route the tutor notes any pieces of mathematics which will need to be revisited for expansion in order that the overall learning experience is mathematically coherent and not just a jumble of techniques.

The use of *DERIVE* with a group of students enrolled on building courses at the University of Plymouth is discussed now. As with the students discussed earlier these students have also rarely studied mathematics beyond the age of 16 and often have weak mathematical backgrounds. The mathematics module that they take covers mathematics and some statistics and is designed to prepare them for the mathematics that they need in their science and surveying modules, as well as introducing them to the statistics that they need for subjects like economics.

DERIVE is employed as a tool to help them develop a graphical understanding of some of the mathematics that they meet in the module. The ideas of transformations of graphs provide a theme that the sessions use in the context of different functions. The students have six computer lab sessions (once every two weeks), of which four make use of *DERIVE* and the other two a statistics package.

Computer Algebra has been less readily accepted into the mathematics degree programme because of the need to develop a different view of mathematics and the associated knowledge and skills and the more traditional curriculum. We would also suggest that for engineers, mathematics is seen as more of a tool for solving problems than in a mathematics programme in which understanding the concepts is a more important outcome. However we are encouraged that the situation is changing. Our experiences of using *DERIVE* and the move to a more student centred learning culture is encouraging colleagues in the Department to investigate the use of Maple across the programmes.

The first year calculus course is designed to explore the fundamental concepts of calculus and to introduce some of the applications. Students on the course are in the first semester of their first year at university and so have not usually met any type of computer algebra system before. DERIVE is used extensively as a problem solving tool and in investigations to introduce new mathematical concepts to the students.

The topic on boundary layers is part of an introductory course on non-linear systems. This course covers three broad topics: the use of geometric methods to study the solution of first and second order differential equations; the study of discrete systems (recurrence relations) including ideas of bifurcations and chaos; and asymptotic methods of solving differential equations. Computer algebra in the form of *DERIVE* or the TI-92 is used to draw direction fields for the geometric approach and to explore iterations on the discrete systems. The asymptotic methods part of the course is more algebraic in approach.

In describing these examples there has been an implicit assumption that the students still need to learn to do the same mathematics that was required in a pre-computer algebra age. In the short term computer algebra has revolutionised the way that we teach mathematics but not what is taught, with the assumption that computer algebra is a desirable but not an essential ingredient. In the longer term there is the need to change the mathematics that is taught as well as the way that we teach it. If this does happen then computer algebra will be an essential ingredient of any new style course.

To return to the present situation, where it is possible to enhance our teaching with computer algebra, students who have followed such courses soon realise what an asset computer algebra can be to their learning. They also expect to see it forming part of their other mathematical studies and are clearly disappointed if more traditional approaches are taken. Students once exposed to Maple, *DERIVE* or the TI-92 will place pressure on their teachers, now and in the future, to make full use of such technology in their teaching.

For further details of these examples visit `www.tech.plym.ac.uk/maths/ctmhome/ctm.html`

John Berry, Ted Graham, Jenny Sharp, Stewart Townend, Anthony Watkins
(Plymouth)

3.8.4 The Educational Use of Computer Algebra Systems at the University of Illinois

This is a report on the development of computer algebra courses at the University of Illinois at Urbana-Champaign. It describes the changes in the curriculum that have been taking place and the reasons for these.

In the fall of 1991, I offered a course on computational group theory to graduate students. This was a hands-on course, with students sitting at Sun workstations attempting problems and me roaming around giving them mathematical and computational suggestions as the need arose. The students were highly motivated to learn the material, often stayed on long after class finished, and ultimately solved one of the unsolved problems resulting in [9] being published. Further details on this course are given in [8].

No further computer algebra course was offered until the fall of 1994, when I gave a course on elliptic curves by computer for graduate students. This did not lead as before to a publication by a large proportion of the class. Instead, some of the students came out with individual papers. It was in fact becoming increasingly standard for graduate students in algebra at UIUC to use software packages in their learning and research. Computers allowed a much wider class of examples to be investigated, giving students a more solid grounding in their area and sometimes allowing them to make discoveries that earlier researchers had overlooked. I had to be careful to avoid them using these systems as crutches. The right attitude to cultivate seemed to be one of partnership between human and computer.

The next step was to institutionalize these sporadic topics courses. We applied for a course number and Math 420, Computer Algebra Systems, was created. Together with Math 321 (on Groebner bases) and four other courses this formed the mathematics department's part of the campus-wide Computational Science and Engineering Option. We set up a web page (`http://www.math.uiuc.edu/~boston/math420.html`) with links to documentation on many software packages and the students learned how to pick the right system for the problem at hand, obtain on-line help or web help, and avoid various pitfalls. It was a very practical hands-on training with students learning how to use many advanced computer algebra systems, such as MAGMA, PARI-GP, KASH, Macaulay2, ...

Math 420 has been offered twice so far, in the spring of 1997 and of 1998. It will be given again in spring 2000. Enrolment suggests that it should be offered once every two years. Computer algebra seems to be very popular with graduate students, who use it routinely in their work. Many other graduate courses now include short visits to the computer lab to supplement with examples the more theoretical approach usually used in lecture/discussion. Math 321, mentioned above, is an undergraduate course, intended for bright math majors. All indications are that the trends described above will continue and that in future computer algebra courses will be required parts of both the graduate and undergraduate curriculum.

Nigel Boston (Urbana-Champaign)

3.8.5 Mathematics Education from a *Mathematica* Perspective

aindexFowler@ David Fowler (Lincoln)

In this section I would like to speak about changes in educational practice resulting from the availability of computer algebra systems, using *Mathematica* as an exemplar of such systems. Clearly, the practice of mathematics itself has changed radically during the past twenty years, as we exploit our increasing ability to shift the burden of algorithmic processing from humans to machines. The expanding capacity of computer systems for graphically representing complex mathematical concepts and processes, and the possibilities offered by electronic communication have significantly altered traditional mathematical discourse. In the words of one mathematician active in the "calculus reform" movement, "The most visible force for change in the mathematics curriculum is the computer, a mathematics-speaking device that has totally transformed science and society" [48].

All of the above capabilities are embodied in *Mathematica*, a fully integrated environment for technical computing. Definitive quantitative studies showing the effects of using *Mathematica* in education are relatively sparse. Qualitative information, on the other hand, is abundant, and one can infer that there is considerable impact on educational practice resulting from the availability of *Mathematica* and other computer algebra applications.

An early and reportedly successful application of computer algebra systems to calculus instruction is the Calculus&*Mathematica* project [50], developed at the University of Illinois at Urbana-Champaign and the Ohio State University and tested at thirty other sites for about six years. The project has received high praise in a US National Research Council report [37]. The mathematics department at the University of Missouri-Columbia has adopted a variation of Calculus&*Mathematica*, in which all undergraduate calculus is taught through *Mathematica*-based instruction, and they report equally successful results [46].

The journal *Mathematica in Education and Research*, begun as a newsletter in 1991, just three years after the introduction of *Mathematica*, provides a useful guide to the evolving uses of computer algebra in education. In an editorial introduction to Volume 1, Number 1, Wellin [53] described four components of *Mathematica*'s potential for changing education:

1. Active involvement of students in learning;
2. Experimentation as a means of understanding mathematical concepts;
3. Visualization of mathematical processes;
4. Access of students to real-world problems.

These themes have continued as the principle set of arguments for using computer algebra systems, particularly in the teaching of calculus. Wellin also raised the salient issues regarding the uses of computer algebra-based instruction; namely, the establishment of labs, the role of an instructor in a lab rather than a lecture setting, and the articulation of *Mathematica*-taught courses with conventionally-taught courses in the university curriculum. Again, these issues have continued to be raised in any analysis of computer algebra-based instruction. In subsequent issues of the journal, authors describe specific calculus concepts that can be effectively taught with *Mathematica*. In some cases, these authors temper their enthusiasm for computer algebra with cautionary advice. Cohen [14] provides suggestions for effectively using *Mathematica* in a "new calculus" course, using arclength as a prototypical example. Among Cohen's suggestions for successful instruction is the need to provide coding templates, since students have problems with accurately entering correct *Mathematica* syntax. A related difficulty is described by DeJong [16] in an article on symbolic algebra computer laboratories. DeJong describes an "empowerment problem": Students do not easily acquire confidence and ability to use *Mathematica*.

Other authors concentrate on the advantages of using *Mathematica* without mention of lab-based difficulties. For example, Prevost [45] extols the virtues of using *Mathematica* graphics to reinforce the limit concept. From this and other articles in later volumes of the journal, one might infer that the pedagogical questions related to using *Mathematica* in calculus instruction have been solved. However, Holdener [35], in 1997, reported that "continuing controversies" still existed concerning the use of Calculus&*Mathematica*. She cited "Gadgetry over Intellect," "Proof-abuse," and "Lack of Necessary Hand Skills" as concerns raised by opponents of computer-based calculus courses in general and Calculus&*Mathematica* in particular. Although there were rebuttals to each of these concerns by proponents of computer-based calculus, one can see that in the last third of the 1990s, computer algebra systems are still far from being in universal use for calculus instruction.

The range of educational articles in *Mathematica in Education and Research* extends far beyond calculus instruction. In physics, for example, Gilfoyle [29] describes the use of the transfer matrix method to present an approach to quantum tunneling for undergraduate physics students. In engineering, Sipcic [47] argues for a large-scale revision of the traditional approach to teaching mechanics, and offers numerous *Mathematica* examples for instruction. Benninga and Wiener [5] present a series of six articles from a graduate course in financial engineering. Akritas and Bavel [1] describe the use of *Mathematica* to teach historical topics in a college liberal arts course. Although *Mathematica* was designed primarily for advanced technical computing, innovative teachers have explored its use in pre-college mathematics instruction. A comprehensive collection of visualizations for classroom use on CD-ROM is described by Gloor in the next subsection 3.8.6. Mathews and McCallister [42] present a study that found a statistically significant difference in the problem-solving performance between groups of algebra students that did not use *Mathematica* and those that did. Peckman [44] suggests that *Mathematica* could be used to give high school students in the USA a deeper understanding of the concept of function and provides numerous examples. Holzinger [36] discusses research to test the motivation of students using *Mathematica* at Handelsakademie, a high school in Graz-Austria, using 35 different *Mathematica* 3.0 notebooks. Holzinger found higher motivation and greater interest in problem-solving among students who used *Mathematica*. He also found that the solution of mathematical problems did not become easier for the student after exposure to computer-aided math instruction.

Computer algebra systems are beginning to be used for introducing students to a new set of mathematical concepts. For example, a set of notebooks by this author includes novel visualization of very large and very small numbers, fractal dimension, cellular automata-generated music and other topics formerly outside the mainstream pre-college curriculum [28].

During the first decade of use, *Mathematica* has had a steadily growing influence on education. Some university instructors continue to express doubts about using symbolic algebra systems to teach basic subjects such as calculus and linear algebra. Consider, however, that an emerging group of students who learned these subjects using computers and advanced symbolic calculators are now becoming university professors. These instructors will organize their courses with the computer as an implicit tool, rather than as an add-on device for displaying an isolated animation or performing a few specialized computations. Simultaneously, computing machinery that can easily handle *Mathematica* continues to become more widely available, so that students need not be limited to labs for their instruction.

Finally, the emergence of a web-based language, most probably MathML, will further blur the division between journal, textbook and universal electronic communication [27]. It is highly likely that symbolic algebra will provide computational power, and eventually inferential power—on-line proof-generating applications—that will be a standard resource for mathematical cognition.

David Fowler (Lincoln)

3.8.6 Visualization: Courseware for Mathematics Education

We report on the projects *Analysis Alive* and *Illustrated Mathematics*, which both deal with the application of computer algebra systems on mathematics education. In both projects, computer algebra is not the object to be taught but serves as an aid in the process of teaching and learning mathematics. Thus, the focus entirely lies on mathematics, and the computer algebra systems, on which our software relies, are mainly a tool.

Visualization in Mathematics

The importance of visualization can hardly be overestimated in general cognitive skill acquisition and problem solving processes (see [3, 18]). Pictures activate mental processes such as the perception of spatial relationships, intuitive comprehension of complex processes, or the observation of patterns and, therefore, aid the process of understanding. Looking at a picture, we use it as a vehicle of thinking, but intend to understand processes and behaviors of the real world.

Learning can be achieved through the translation between representations at different levels of abstraction. Visualization can be seen as providing the relevant representations to assist the learner in carrying out this cognitive process. The useful aspects of visualization are the translation from representations which are more abstract to those which are less abstract. Therefore, current techniques of scientific visualization can bring invaluable insight to students.

In particular in mathematics we deal with abstract structures, which visual representation helps to enlighten. This is important, particularly for those students who have difficulty understanding abstract mathematical objects. The objects we have in mind are not primarily geometric figures, but arbitrary mathematical objects such as infinite sequences, complex functions, or conformal mappings. For beginners, these terms are most difficult to grasp. Therefore, their visualization is a key to understanding these complex topics.

Computer Algebra Systems in Mathematics Education

Computer algebra systems (CAS) provide the necessary algorithms needed to compute mathematics visualizations. CAS give teachers and students also another and more direct approach to using the computer. Applying a CAS, much less effort to treat a simple practical problem with the computer is needed than is with the classical approach, learning a full programming language first. So the focus moves from computer handling to the application. This enables the possibility of applying the computer in education not as a teaching object but as a tool to solve problems in other disciplines. Therefore, an introduction to CAS belongs to a modern curriculum in the education of scientists and engineers.

These two reasons—using CAS for visualizations and introducing CAS in education—make it natural to choose visualizations for the first contact of students with CAS.

Therefore, it is not surprising that many mathematicians have already combined the teaching of mathematics with a course on a computer algebra system (see e.g. [32, 26, 10, 19, 41, 15, 11, 7, 49]). However, this approach leads to additional difficulties for the students as they have to acquire not only one but two skills: the understanding of mathematics together with the understanding of the special computer algebra system which has a priori nothing to do with mathematics itself. Thus, this approach might even produce negative interferences.

The Grey Box Approach

Both in *Illustrated Mathematics* and in *Analysis Alive*, we tried to avoid overloading students (and teachers) by the issues to be taught (mathematics) and the technology (CAS). For that, we shielded the user as much as possible from the intricacies of both operating system and computer algebra system, by providing a grey box[1] consisting of the following parts.

- Electronic documents containing the ready-made visualizations (graphics and animations) of mathematical objects.
- Programs providing commands for the creation of new visualizations according to user-specified parameters.

As the documents already contain the commands to compute the graphics contained therein, the user only has to change the parameters and to process the command in order to obtain his own visualizations. In particular, users do not need to learn the subtleties of the input syntax let alone to write programs in a CAS. They merely need to handle the basic functionalities of its user interface such as opening and browsing documents and evaluating a command. (For a more detailed discussion of the advantages of the use of a CAS for such a grey box, see [2].)

Numerous authors chose similar approaches and provide programs which allow the user to perform experiments. The resulting packages usually treat only a few issues and do not cover an entire course. A notable exception is [13], which consists of a black box built on a limited version of *Mathematica*.

Illustrated Mathematics

The goal of this project (cf. [30, 31]) was to provide a comprehensive collection of graphics and animations for topics in mathematics at the high-school and undergraduate and graduate college level. The visualizations are inteded for classroom use and can be used for demonstration during class, printed as hardcopy, or included in other documents.

The collection (provided as *Mathematica* notebooks) is organized by mathematical topic and is not intended to replace textbooks. Teachers can select the examples that fit their syllabus and incorporate them into their lectures and class notes.

The programs (written in *Mathematica*'s own programming language) allow users to experiment by seeing the effects of changing parameters on the objects they are studying.

The topics range from basics such as sequences and series and end up with complex functions and minimal surfaces.

[1] We use the term *grey box* like *black box* for indicating that it is not necessary to know the inside process but merely the functionality. However, the entire software is user readable, but unlike a *white box* its internals are not discussed.

For further information on *Illustrated Mathematics* please consult the world wide web at `http://www.amrhein.ch/IM`.

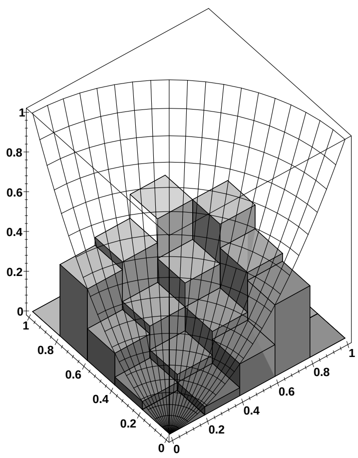
Analysis Alive

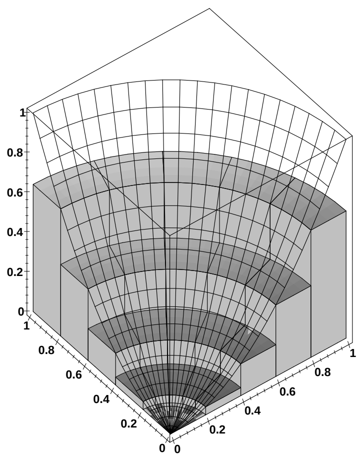
This project (cf. [54]) addresses students of mathematics as well as of sciences and engineers. It offers a new complete interactive approach to "Analysis" as it is taught at German universities.[2] It connects the concept of a modern textbook on this field and the opportunities of a modern computer algebra system in such a way that the user can profit from the advantages of both systems for knowledge acquisition without being obliged to master secondary skills.

Analysis Alive comprises a textbook and a CD–ROM jointly forming a complex unit. First of all, the book itself can be used as a common modern textbook combining the representation of the material, hundreds of illustrated examples and exercises presented directly within the current text. The text, however, is tightly linked to the electronic documents on the CD–ROM. For almost all relevant issues, the user can find visualizations in electronic form. These graphics and animations are presented in a similar way as in *Illustrated Mathematics*. They are presented in the form of *Maple* worksheets, and the software providing the commands for the creation of user-chosen examples relies only on *Maple*.

The text, however, is tightly linked to the electronic documents on the CD–ROM. Icons and background shading show which parts of the book are represented as visualizations on the CD–ROM. This direct relation offers now easy creation of visualization examples for the hundreds of examples listed in the text. Moreover, and much more important, it provides a great opportunity for experiments.

The following graphics show the effect of the transformation from cartesian to polar coordinates for the integral approximation.





[2] This course roughly corresponds to a course in higher calculus.

For further information on *Analysis Alive* please consult the world wide web at `http://WWW.amrhein.ch/AA`.

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