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An Asymptotic Version of the Goldbach Conjecture

In the books [5]–[6] many applications of DERIVE to problems in calculus were discussed.
In this article we give an application to number theory. With the use of DERIVE we can easily calculate the number of Goldbach representations of an even integer. So the computer can give the pupil some insight into the distribution of Goldbach representations, supporting the plausibility of the *Goldbach Conjecture* and omitting tedious hand calculations.
Recall that the Goldbach Conjecture states that each positive even integer $g \geq 4$ has at least one representation as a sum $g = p_1 + p_2$ of two prime numbers p_1, p_2. If $A(g)$ denotes the number of such representations of the even integer g, then we will give an argument indicating the asymptotic behaviour

$$A(g) > C\sqrt{g}$$

demonstrating how highly plausible the Goldbach Conjecture is. We develop which even integers have many and which have only few Goldbach representations, and present calculations with DERIVE.

Introduction

The Goldbach Conjecture (see [2]–[4]) is one of those mathematical statements that is easy to formulate and to understand. On the other hand, despite its long history, no proof was found until today, and (especially after the announced proof of *Fermat's Last Theorem* by A. Wiles, 1993, see e. g. [1]) it remains one of the oldest open questions in mathematics. The conjecture stems from correspondence between Goldbach and Euler (see [2], pp. 248–249, [3], pp. 125–129, and [4], pp. 103–107). In his letter to Euler, written 7th June 1742 in Moscow, Goldbach stated the conjecture that any positive integer has a representation as the sum of three primes. In particular, any even positive integer should have a representation as the sum of two primes. (In Goldbach's times the number 1 was considered a prime number. So in our times the "Goldbach conjecture" is a slightly stronger statement.) This letter was first published by Fuss [3] in 1843. For some historical remarks on the Goldbach conjecture, see e. g. [4], p. 106.

In this work we use DERIVE to give pupils some insight into the distribution of Goldbach representations, and present an argument which makes the Goldbach Conjecture plausible.

Table 1 lists the even integers until 40 together with their Goldbach representations as sums of two primes.

g	representations				number $A(g)$
2					0
4	2+2				1
6	3+3				1
8	3+5				1
10	3+7	5+5			2
12	5+7				1
14	3+11	7+7			2
16	3+13	5+11			2
18	5+13	7+11			2
20	3+17	7+13			2
22	3+19	5+17	11+11		3
24	5+19	7+17	11+13		3
26	3+23	7+19	13+13		3
28	5+23	11+17			2
30	7+23	11+19	13+17		3
32	3+29	13+19			2
34	3+31	5+29	11+23	17+17	4
36	5+31	7+29	13+23	17+19	4
38	7+31	19+19			2
40	3+37	11+29	17+23		3

Table 1: A table of the first Goldbach representations

Each pupil should be able to prepare such a table, and therefore to observe that it seems to be likely that the conjecture is valid. On the other hand, by hand calculations we are not able to study the decompositions of "large" even numbers.

Goldbach Representations using DERIVE

Therefore we construct simple DERIVE functions to calculate the number $A(g)$ of Goldbach representations of an even integer g. The function

```
IS_PRIME(x):=IF(NEXT_PRIME(x-1)=x,1,0)
```

obviously yields 1 if x is a prime, and 0 otherwise.

The iterative procedure starting with the vector h_:=[3,0], successively increasing its first element k by two, and in case that both k and $g-k$ are prime its second element by one, until the first element reaches $g/2$, is implemented by the iteration

```
ITERATE([ELEMENT(h_,1)+2,
IF(IS_PRIME(ELEMENT(h_,1))=1 AND IS_PRIME(g-ELEMENT(h_,1))=1,ELEMENT(h_,2)+1,ELEMENT(h_,2))],
h_,[3,0],FLOOR(g/4-1/2))
```

Here FLOOR(g/4-1/2) counts the number of iterations starting at $k = 3$, and ending at $k = g/2$ if $g/2$ is odd, or at $k = g/2 - 1$ if $g/2$ is even.

The second element of the iteration therefore counts the number of Goldbach representations of g. Thus the DERIVE function

```
GOLDBACH_NUMBER(g):=
IF(g=2,0,
   IF(g=4,1,
      ELEMENT(ITERATE([ELEMENT(h_,1)+2,
      IF(IS_PRIME(ELEMENT(h_,1))=1 AND IS_PRIME(g-ELEMENT(h_,1))=1,
      ELEMENT(h_,2)+1,ELEMENT(h_,2))],
      h_,[3,0],FLOOR(g/4-1/2))
      ,2)))
```

calculates the number of Goldbach representations of the positive even integer g.

DERIVE generates a table of the numbers of Goldbach representations for $g = 2, 4, \ldots, 40$ by the command

```
VECTOR(GOLDBACH_NUMBER(g),g,2,40,2)
```

with the result

$$[0, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 3, 3, 2, 3, 2, 4, 4, 2, 3] .$$

We can use DERIVE directly to produce Table 1 using the similar DERIVE function

```
GOLDBACH_REPRESENTATIONS(g):=
IF(g=2,[],
   IF(g=4,[[2,2]],
         ELEMENT(ITERATE([ELEMENT(h_,1)+2,
         IF(IS_PRIME(ELEMENT(h_,1))=1 AND IS_PRIME(g-ELEMENT(h_,1))=1,
            APPEND(ELEMENT(h_,2),[[ELEMENT(h_,1),g-ELEMENT(h_,1)]]),
            ELEMENT(h_,2)
           )],h_,[3,[]],FLOOR(g/4-1/2)),2)))
```

that generates the pair $[g, A(g)]$ for an even integer g. Therefore the call

```
VECTOR([g,GOLDBACH_REPRESENTATIONS(g),DIMENSION(GOLDBACH_REPRESENTATIONS(g))],g,2,40,2)
```

produces Table 1.

No particular structure is obvious from Table 1 besides the hope that the expected value for $A(g)$ seems to increase with increasing g. We encourage pupils to use these DERIVE functions for larger values of g to convince themselves about the plausibility of the Goldbach Conjecture.

Asymptotic Analysis

We will now give a theoretical argument for this purpose.

Let an even number g be given. We wish to estimate the number $A(g)$ of Goldbach representations of $g = p_1 + p_2$ as sum of two primes p_1 and p_2. We note that all positive integers smaller than or equal to g are either prime or have a representation $n = p\,m$ with a prime number $p \leq \sqrt{g}$ and an integer m (easy exercise).

Let P denote the set of all prime numbers, p_k denote the kth prime number, and let

$$P_g := \{p \in P \mid p \leq \sqrt{g}\}$$

denote the set of prime numbers corresponding to g having m_g elements, say.

We look for positive integers x_g that generate a Goldbach representation for g as sum $g = x_g + (g - x_g)$ with $g - x_g \in P_g$, and we consider only those representations for which neither x_g nor $g - x_g$ lie in P_g, i. e. both summands are greater than $\sqrt{g}$, and we call such a representation a *strict Goldbach representation*.

For all prime numbers $p_k \in P_g$ we can now make the following observations. The numbers of the arithmetic sequence $n\,p_k$ $(n \in \mathbb{N}, n \geq 2)$ are no candidates to be an x_g for a strict Goldbach representation as they are not prime. Moreover one more arithmetic sequence of the form $n\,p_k + \hat{g}_k$ $(n \in \mathbb{N}, n \geq 2)$ is excluded from consideration as in this case the numbers $g - x_g$ are divisible by p_k: We have $(m \in \mathbb{N})$

$$g - x_g = m\,p_k$$

or

$$x_g = g - m\,p_k = n\,p_k + \hat{g}_k$$

with $\hat{g}_k = g$ modulo p_k, i. e., $\hat{g}_k$ is the remainder for the division of g by p_k. We note, in particular, that if g is divisible by p_k, then these two sequences agree.

So at most the elements of two of the p_k arithmetic sequences

$$\{x = n\,p_k + m \in \mathbb{N} \mid x \leq g,\, n \in \mathbb{N}\} \qquad (m = 0, \ldots, p_k - 1) \tag{1}$$

are excluded as summands x_g of a strict Goldbach representation by reason of divisibility by p_k. The remaining $p_k - 2$ arithmetic sequences (1) contain only numbers x for which neither x nor $g - x$ is divisible by p_k.

Possible representation numbers for a strict Goldbach representations of g are the numbers $x \in M_0 := [\sqrt{g}, g/2]$. This gives $A_0 = \left[\frac{g}{2} - \sqrt{g}\right]$ possible strict Goldbach summands, where $[x]$ denotes the largest integer that is smaller than or equal to x. In a first step we eliminate all numbers divisible by 2, and we have remaining $A_1 = A_0/2$ possible Goldbach summands. We now iteratively pass through all prime numbers in P_g, and as those generate all composite numbers in M_0, this procedure eliminates all composite numbers, and only prime Goldbach summands remain. This procedure gives Table 2.

k	p_k	elimination steps	A_k
1	2	only odd numbers (or else 2 is divisor)	$\frac{A_0}{2}$
2	3	only each third number (or else 3 is divisor)	$\frac{1}{3}\frac{A_0}{2}$
3	5	only 3/5 of the numbers (or else 5 is divisor)	$\frac{3}{5}\frac{1}{3}\frac{A_0}{2}$
$\vdots$	$\vdots$	$\cdots$	$\cdots$
m_g	p_{m_g}	only $(p_{m_g}-2)/p_{m_g}$ of the numbers (or else p_{m_g} is divisor)	$\frac{p_{m_g}-2}{p_{m_g}} \cdots \frac{3}{5}\frac{1}{3}\frac{A_0}{2}$

Table 2: Estimation of the number of strict Goldbach representations

Unfortunately the table is only "almost correct" (otherwise we would have proved the Goldbach conjecture, as we shall see soon), as we did not consider the fact that the intermediately remaining sets of possible Goldbach numbers in general form quite irregular sets. So it is only *in the average* that the $(p_{k+1}-2)/p_{k+1}$th part of the representations in the kth step survive the elimination.

However, if g is large, it is highly plausible that our estimates are quite accurate.

Using the data of the table we get the following (asymptotic) estimate

$$\begin{aligned}
A(g) &\geq \frac{p_{m_g}-2}{p_{m_g}} \cdot \frac{p_{m_g-1}-2}{p_{m_g-1}} \cdot \frac{p_{m_g-2}-2}{p_{m_g-2}} \cdots \frac{3}{5}\,\frac{1}{3}\,\frac{A_0}{2} \\
&> \frac{p_{m_g}-2}{p_{m_g}} \cdot \frac{p_{m_g}-4}{p_{m_g}-2} \cdot \frac{p_{m_g}-6}{p_{m_g}-4} \cdots \frac{3}{5}\,\frac{1}{3}\,\frac{A_0}{2} \\
&= \frac{A_0}{2\,p_{m_g}} \geq \frac{A_0}{2\sqrt{g}} = \frac{\left[\frac{g}{2} - \sqrt{g}\right]}{2\sqrt{g}} \approx \frac{\sqrt{g}}{4} - \frac{1}{2}\,.
\end{aligned}$$

In particular, this estimate suggests that the number of Goldbach representations is not only greater than zero for all positive even integers g, but increases with increasing g.

Calculations for Large Even Integers

In practice, the given estimate is much too low: To obtain it, we took the product over all odd numbers rather than the primes only which obviously makes the bound too low. Furthermore, if g possesses many prime divisors, then—as we remarked—only one rather than two of the corresponding arithmetic sequences is eliminated further enlarging the number of Goldbach representations. The *prime factorials* $2 \cdot 3 \cdot 5 \cdots p_k$ possess particularly many prime divisors.

On the other hand, if g possesses only few prime divisors, then there are few Goldbach summands. This is the case, in particular, for the powers of two that possess only the divisor two.

Tables 3 and 4 show the numbers of Goldbach representations of the first prime factorials, and powers of two, respectively, demonstrating the above remarks.

g	$2 \cdot 3 \cdot 5 = 30$	$2 \cdot 3 \cdot 5 \cdot 7 = 210$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$
number	3	19	114	905
$\left[\frac{\sqrt{g}}{4} - \frac{1}{2}\right]$	0	3	11	42

Table 3: Goldbach representations of prime factorials

g	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536
number	2	2	5	3	8	11	22	25	53	76	151	244	435
$\left[\frac{\sqrt{g}}{4} - \frac{1}{2}\right]$	0	0	1	2	3	5	7	10	15	22	31	44	63

Table 4: Goldbach representations of powers of two

DERIVE produces these lists with the commands

```
PRIME(k):=IF(k=1,2,NEXT_PRIME(PRIME(k-1)))

PRIME_FAC(g):=PRODUCT(PRIME(k_),k_,1,g)

VECTOR([PRIME_FAC(k),GOLDBACH_NUMBER(PRIME_FAC(k)),FLOOR(SQRT(PRIME_FAC(k))/4-1/2)],k,3,6)
```

and

```
VECTOR([2^k,GOLDBACH_NUMBER(2^k),FLOOR(SQRT(2^k)/4-1/2)],k,4,16)
```

The example $g = 128$ shows that in particular cases (for small g) our estimate is not much too low. On the other hand, we considered only very small numbers, and it seems as if for increasing g our estimate is much too low.

References

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