# Examples for the algorithmic calculation of formal Puiseux, Laurent and power series

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**Abstract:**

Formal Laurent-Puiseux series (LPS) of the form $\sum\limits_{k=k_0}^{\infty} a_k x^{k/n}$ are important in calculus and complex analysis. In some Computer Algebra Systems (CASs) it is possible to define an LPS by direct or recursive definition of its coefficients. Since some operations cannot be directly supported within the LPS domain, some systems generally convert LPS to finite truncated LPS for operations such as addition, multiplication, division, inversion and formal substitution. This results in a substantial loss of information. Since a goal of Computer Algebra is — in contrast to numerical programming — to work with formal objects and preserve such symbolic information, CAS should be able to use LPS when possible.

There is a one-to-one correspondence between formal power series with positive radius of convergence and corresponding analytic functions. It should be possible to automate conversion between these forms. Among CASs only Macsyma [5] provides a procedure `powerseries` to calculate LPS from analytic expressions in certain special cases, but this is rather limited.

In [2]–[4] we gave an algorithmic approach for computing an LPS for a very rich family of functions. It covers e.g. a high percentage of the power series that are listed in the special series dictionary [1]. The algorithm has been implemented by the author and A. Rennoch in the CAS Mathematica [7], and by D. Gruntz in Maple [6].

In this note we present some example results of our Mathematica implementation which give insight in the underlying algorithmic procedure.

## 1 The algorithm

In [2]–[4] three types of functions are covered by an algorithmic procedure for the conversion into their representing Laurent-Puiseux series (LPS) $\sum\limits_{k=k_0}^{\infty} a_k z^{k/n}$ ($k_0 \in \mathbb{Z}$) at the origin: functions of *rational type* which are rational, or have a rational derivative of some order, functions of *exp-like* *type* which satisfy a homogeneous linear differential equation (DE) with constant coefficients, and functions of *hypergeometric type* whose definition is given below.

The most interesting case are the functions of hypergeometric type as almost all transcendental elementary functions like $\exp x$, $\sin x$, $\cos x$, $\arcsin x$, $\arctan x$, and many others are of that type.

An LPS $F = \sum_{k=k_0}^{\infty} a_k x^{k/n}$ $(k_0 \in \mathbb{Z})$ as well as its corresponding function $f$ is called to be of hypergeometric type if it has a positive radius of convergence, and if its coefficients $a_k$ satisfy a recurrence equation (RE) of the form

$$\begin{aligned} a_{k+m} &= R(k)\, a_k \quad \text{for } k \geq k_0 \\ a_k &= A_k \quad\quad\;\; \text{for } k = k_0, k_0+1, \ldots, k_0+m-1 \end{aligned} \tag{1}$$

for some $m \in \mathbb{N}$, $A_k \in \mathbb{C}$ $(k = k_0+1, k_0+2, \ldots, k_0+m-1)$, $A_{k_0} \in \mathbb{C} \setminus \{0\}$, and some rational function $R$. The number $m$ is then called the *symmetry number* of (the given representation of) $F$. A RE of type (1) is also called to be of hypergeometric type.

We want to emphasize that the above terminology of functions of hypergeometric type is definitely more general than the terminology of generalized hypergeometric functions. The function $\sin x$ e.g. is **not** a generalized hypergeometric function as obviously no RE of the type (1) holds for its series coefficients with $m = 1$. So $\sin x$ is not of hypergeometric type with symmetry number 1; it is, however, of hypergeometric type with symmetry number 2. A more difficult example of the same kind is the function $e^{\arcsin x}$ which is neither even nor odd, and nevertheless turns out to be of hypergeometric type with symmetry number 2, too, as we shall see later.

In ([2], Lemma 2.1, and Theorem 8.1) we gave a list of transformations on LPS that preserve the hypergeometric type.

**Lemma** Let $F$ be an LPS of hypergeometric type. Then

| | | |
|---|---|---|
| (a) $x^j F$ $(j \in \mathbb{N})$ , | (b) $F/x^j$ $(j \in \mathbb{N})$ , | (c) $F(Ax)$ $(A \in \mathbb{C})$ , |
| (d) $F\left(x^{\frac{p}{q}}\right)$ $(p, q \in \mathbb{N})$ , | (e) $F(x) \pm F(-x)$ , | (f) $F'$ , |

are of hypergeometric type, too. If $a_{-1} = 0$, then also

(g) $\int F$ is of hypergeometric type.

It is essential for the development of our algorithm that functions of hypergeometric type satisfy a simple differential equation, see ([2], Theorems 2.1, and 8.1).

**Theorem** Each LPS of hypergeometric type satisfies a homogeneous linear differential equation with polynomial coefficients.

The following algorithm for a function call `PowerSeries[f,x,0]` corresponding to the conversion of the function $f$ into its representing LPS with respect to the variable $x$ was introduced in ([2], Algorithm 3.1).

**Algorithm**

(1) **Rational functions**
If $f$ is rational in $x$, then use the **rational algorithm** (see e.g. [2], Section 4).

(2) **Find a homogeneous linear differential equation for $f$ with polynomial coefficients** (for details, see [2], Section 5).

(a) Fix a number $N_{\max} \in \mathbb{N}$, the maximal order of the differential equation searched for; a suitable value should be $N_{\max} := 4$.

(b) Set $N := 1$.

(c) Calculate $f^{(N)}$; **either**, if the derivative $f^{(N)}$ is rational, apply the **rational algorithm**, and integrate;

(d) **or** find a homogeneous linear differential equation with polynomial coefficients for $f$ of order $N$

$$\sum_{j=0}^{N} p_j f^{(j)} = 0$$

with polynomials $p_j$ $(j = 1, \ldots, N)$ in the variable $x$.

(e) If (d) was not successful, then increase $N$ by one, and go back to (c), until $N = N_{\max}$.

(3) **Find the corresponding recurrence equation** (see [2], Section 6).
Suppose you found a homogeneous linear differential equation with polynomial coefficients for $f$ in step (2), then transfer it into a recurrence equation for the coefficients $a_n$. The recurrence equation is then of the special type

$$\sum_{j=0}^{M} P_j a_{n+j} = 0 \tag{2}$$

with polynomials $P_j$ $(j = 0, \ldots, M)$ in the variable $n$, and $M \in \mathbb{N}$.

(4) **Type of recurrence equation** (see [2], Section 7).
Determine the type of the recurrence equation according to the following list.

(a) If the RE (2) contains only two summands then $f$ is of hypergeometric type, and an explicit formula for the coefficients can be found by the hypergeometric coefficient formula (see e.g. [2], equation (2.2)), and some initial conditions.

(b) If the DE has constant coefficients ($c_j \in \mathbb{C}$ $(j = 0, \ldots, Q)$)

$$\sum_{j=0}^{Q} c_j f^{(j)} = 0 ,$$

then $f$ is of *exp-like type*. In this case the substitution $b_n := n! \cdot a_n$ leads to the recurrence equation

$$\sum_{j=0}^{Q} c_j b_{n+j} = 0 ,$$

which has the same constant coefficients as the differential equation, and is solved by a known algebraic scheme using the first $Q$ initial coefficients, (see e.g. [2], Section 7).

(c) If the recurrence equation is none of the above types, try to solve it by other known recurrence equation solvers.

The above algorithm has been implemented by the author and A. Rennoch in MATHEMATICA [7], and by D. Gruntz in MAPLE [6]. In the preceding sections we present some example results of our MATHEMATICA implementation [3] which give insight in the underlying algorithmic procedure.

## 2 Examples of formal Laurent series

In this section we present some of the results in the form of direct MATHEMATICA output. For didactical purposes, our MATHEMATICA implementation informs about the intermediate calculations.

First we consider power series of hypergeometric type. The exponential function is a very easy example.

```
In[1]:= PowerSeries[E^x,x,0]
ps-info: 1 step(s) for DE:
         -f[x] + f'[x] = 0
ps-info: RE for all n >= 0:
                     a[n]
         a[1 + n] = -----
                     1 + n
ps-info: function of hypergeometric type
ps-info: for all n <= -1: a[n]=0
ps-info: a[0] = 1
             k
            x
Out[1]= Sum[--, {k, 0, Infinity}]
            k!
```

Note that the division by $x^m$ ($m \in \mathbb{N}$) leads to a different DE, but the corresponding RE is similar.

```
In[2]:= PowerSeries[E^x/x^3,x,0]
ps-info: 1 step(s) for DE:
         (3 - x) f[x] + x f'[x] = 0
ps-info: RE for all n >= -3:
                     a[n]
         a[1 + n] = -----
                     4 + n
ps-info: function of hypergeometric type
ps-info: for all n <= -4: a[n]=0
ps-info: a[0] = 1
             -3 + k
            x
Out[2]= Sum[-------, {k, 0, Infinity}]
              k!
```

Also, if we substitute $x$ by $x^4$, e.g., a totally different DE is found, the symmetry number increases to 4, and we get the output

```
In[3]:= PowerSeries[x*Exp[x^4],x,0]
ps-info: 1 step(s) for DE:
                   4
         (-1 - 4 x ) f[x] + x f'[x] = 0
ps-info: RE for all n >= -2:
                     4 a[n]
         a[4 + n] = ------
                     3 + n
ps-info: function of hypergeometric type
```

```
ps-info: a[0] = 0, a[1] = 1, a[2] = 0, a[3] = 0, a[4] = 0
             1 + 4 k
            x
Out[3]= Sum[--------, {k, 0, Infinity}]
               k!
```

Note that if the RE has symmetry number $m$ then the resulting LPS is decomposed of $m$ shifted series. In the given case three of these four summands vanish by the calculated initial values. This is not the case for the LPS of $\sin(x+y)$, whose resulting RE has symmetry number 2, and we get a decomposition in two shifted series

```
In[4]:= PowerSeries[Sin[x+y],x,0]
ps-info: 2 step(s) for DE:
         f[x] + f''[x] = 0
ps-info: RE for all n >= 0:
                              a[n]
         a[2 + n] = -(---------------)
                       (1 + n) (2 + n)
ps-info: function of hypergeometric type
ps-info: a[0] = Sin[y], a[1] = Cos[y]
                k   1 + 2 k                                          k   2k
            (-1)   x        Cos[y]                               (-1)   x    Sin[y]
Out[4]= Sum[---------------------, {k, 0, Infinity}] + Sum[-----------------, {k, 0, Infinity}]
                 (1 + 2 k)!                                         (2 k)!
```

obviously corresponding to the addition formula $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

The following are some functions of hypergeometric type some of which may be unexpected.

```
In[5]:= PowerSeries[ArcSin[x],x,0]
ps-info: 2 step(s) for DE:
                             2
         x f'[x] + (-1 + x ) f''[x] = 0
ps-info: RE for all n >= 0:
                            2
                           n  a[n]
         a[2 + n] = ---------------
                     (1 + n) (2 + n)
ps-info: function of hypergeometric type
ps-info: a[0] = 0, a[1] = 1, a[2] = 0
             1 k  1 + 2 k      2
            (-)  x        (2 k)!
             4
Out[5]= Sum[---------------------, {k, 0, Infinity}]
                  2
               k!  (1 + 2 k)!
```

The inverse sine function is an example of a function of hypergeometric type whoses square is of hypergeometric type, too.

```
In[6]:= PowerSeries[ArcSin[x]^2,x,0]
ps-info: 3 step(s) for DE:
```

```
                                          2   (3)
         f'[x] + 3 x f''[x] + (-1 + x ) f    [x] = 0
ps-info: RE for all n >= 1:
                            2
                           n  a[n]
         a[2 + n] = ---------------
                    (1 + n) (2 + n)
ps-info: function of hypergeometric type
ps-info: a[1] = 0, a[2] = 1, a[3] = 0
              k  2 + 2 k   2
             4  x        k!
Out[6]= Sum[------------------, {k, 0, Infinity}]
            (1 + k) (1 + 2 k)!
```

Other example functions of hypergeometric type with squares of hypergeometric type are $e^x$, $\sin x$, $\cos x$, $\sinh x$ (whose square has symmetry number 4), $\cosh x$, Bessel J $x$, Bessel I $x$, as well as the next two examples one of which was mentioned before.

```
In[7]:= PowerSeries[Exp[ArcSin[x]],x,0]
ps-info: 2 step(s) for DE:
                                    2
         f[x] + x f'[x] + (-1 + x ) f''[x] = 0
ps-info: RE for all n >= 0:
                              2
                     (1 + n ) a[n]
         a[2 + n] = ---------------
                    (1 + n) (2 + n)
ps-info: function of hypergeometric type
ps-info: a[0] = 1, a[1] = 1
             k  2 k        5          2
            4  x    Product[- - 2 j + j , {j, k}]
                              4
Out[7]= Sum[-------------------------------------, {k, 0, Infinity}] +
                              (2 k)!

          k  1 + 2 k         1        2
         4  x        Product[- - j + j , {j, k}]
                             2
>    Sum[---------------------------------------, {k, 0, Infinity}]
                          (1 + 2 k)!

In[8]:= PowerSeries[Exp[ArcSinh[x]],x,0]
ps-info: 2 step(s) for DE:
                                    2
         -f[x] + x f'[x] + (1 + x ) f''[x] = 0
ps-info: RE for all n >= 0:
                    (1 - n) a[n]
         a[2 + n] = ------------
                       2 + n
ps-info: function of hypergeometric type
ps-info: a[0] = 1, a[1] = 1
```

```
                  1  k  2 k
               (-(-))  x    (2 k)!
                  4
Out[8]= x + Sum[------------------, {k, 0, Infinity}]
                                2
                  (1 - 2 k) k!
```

Both are special cases of ($A = 1$, and $A = -i$, $x \mapsto ix$)

```
In[9]:= PowerSeries[(x+Sqrt[1+x^2])^A,x,0]
ps-info: 2 step(s) for DE:
            2                          2
         -(A  f[x]) + x f'[x] + (1 + x ) f''[x] = 0
ps-info: RE for all n >= 0:
                    (A - n) (A + n) a[n]
         a[2 + n] = --------------------
                      (1 + n) (2 + n)
ps-info: function of hypergeometric type
ps-info: a[0] = 1, a[1] = A
                k  2 k           -A                A
            (-4)  x    Pochhammer[--, k] Pochhammer[-, k]
                                  2                2
Out[9]= Sum[---------------------------------------------, {k, 0, Infinity}] +
                               (2 k)!

               k    1 + 2 k           1 - A                1 + A
           (-4)  A x        Pochhammer[-----, k] Pochhammer[-----, k]
                                         2                    2
>       Sum[----------------------------------------------------------, {k, 0, Infinity}]
                                   (1 + 2 k)!
```

Here `Pochhammer[a,k]` denotes the *Pochhammer symbol* (or *shifted factorial*) defined by

$$\texttt{Pochhammer}[\mathrm{a},\mathrm{k}] = (a)_k := \begin{cases} 1 & \text{if } k = 0 \\ a \cdot (a+1) \cdots (a+k-1) & \text{if } k \in \mathbb{N} \end{cases} .$$

Note that $\frac{(a)_k}{k!} = \binom{a+k-1}{k}$. Here are more examples.

```
In[10]:= PowerSeries[E^(x^2)*Erf[x],x,0]
ps-info: 2 step(s) for DE:
         -2 f[x] - 2 x f'[x] + f''[x] = 0
ps-info: RE for all n >= 0:
                    2 a[n]
         a[2 + n] = ------
                    2 + n
ps-info: function of hypergeometric type
                                2
ps-info: a[0] = 0, a[1] = --------, a[2] = 0
                          Sqrt[Pi]
```

```
                 k  1 + 2 k
              2 4  x        k!
Out[10]= Sum[-------------------, {k, 0, Infinity}]
             Sqrt[Pi] (1 + 2 k)!

In[11]:= PowerSeries[E^x-2 E^(-x/2) Cos[Sqrt[3]x/2-Pi/3],x,0]
ps-info: 3 step(s) for DE:
                  (3)
         -f[x] + f   [x] = 0
ps-info: RE for all n >= 0:
                             a[n]
         a[3 + n] = ----------------------
                    (1 + n) (2 + n) (3 + n)
ps-info: function of hypergeometric type
                                    3
ps-info: a[0] = 0, a[1] = 0, a[2] = -, a[3] = 0, a[4] = 0
                                    2
                        2 + 3 k
             9 (1 + k) x
Out[11]= Sum[------------------, {k, 0, Infinity}]
                 (3 + 3 k)!

In[12]:= PowerSeries[Integrate[Exp[-A^2 t^2] Cos[2 x t],{t,0,Infinity}],x,0]
ps-info: 1 step(s) for DE:
                     2
         2 x f[x] + A  f'[x] = 0
ps-info: RE for all n >= -1:
                     -2 a[n]
         a[2 + n] = ----------
                     2
                    A  (2 + n)
ps-info: function of hypergeometric type
                Sqrt[Pi] Abs[A]
ps-info: a[0] = ---------------
                        2
                     2 A
ps-info: a[1] = 0
                -2 k           2 k
             (-A   )  Sqrt[Pi] x    Abs[A]
Out[12]= Sum[----------------------------, {k, 0, Infinity}]
                          2
                       2 A  k!
```

By the use of the algorithm we discovered a misprint in [1]. We have

```
In[13]:= PowerSeries[4/x*Integrate[Exp[t^2]*Erf[t],{t,0,Sqrt[x]/2}],x,0]
ps-info: 3 step(s) for DE:
                                          2              2
         -f + (12 - 3 x) f'[x] + (18 x - x ) f''[x] + 4 x  f'''[x] = 0
ps-info: RE for all n >= 0:
```

```
                   (1 + n) a[n]
         a[1 + n] = -------------------
                    2 (2 + n) (3 + 2 n)
                             n
ps-info: RE modified to (n -> -)
                             2
ps-info: RE for all n >= -1:
                  (2 + n) a[n]
         a[2 + n] = -----------------
                    2 (3 + n) (4 + n)
ps-info: function of hypergeometric type
                  1
ps-info: a[0] = --------, a[1] = 0
                Sqrt[Pi]
                            k
                           x  k!
Out[13]= Sum[---------------------------, {k, 0, Infinity}]
             Sqrt[Pi] (1 + k) (1 + 2 k)!
```

so that in formula (5.18.3) of [1] a factorial sign is missing, and it should read as

$$\sum_{k=0}^{\infty} \frac{k!}{(2k+2)!} x^k = \frac{1}{x} \sum_{k=1}^{\infty} \frac{1}{k \left(\frac{1}{2}\right)_k} \left(\frac{x}{4}\right)^k = \frac{4}{x} \int_0^{\frac{\sqrt{x}}{2}} e^{t^2} \operatorname{erf} t \, dt$$

(in [1] the error function is defined by $\operatorname{erf} x = \int_0^x e^{-t^2} dt$, and is not normalized by a factor $2/\sqrt{\pi}$). Next are two examples of exp-like type

```
In[14]:= PowerSeries[Sin[x]*Exp[x],x,0]
ps-info: 2 step(s) for DE:
         2 f[x] - 2 f'[x] + f''[x] = 0
ps-info: RE for all n >= 0:
                    2 (-a[n] + a[1 + n] + n a[1 + n])
         a[2 + n] = --------------------------------
                             (1 + n) (2 + n)
ps-info: DE has constant coefficients
ps-info: modified RE (b[n] = n! a[n]):
         2 b[n] - 2 b[1 + n] + b[2 + n] = 0
             k/2  k     k Pi
            2    x  Sin[----]
                          4
Out[14]= Sum[-----------------, {k, 0, Infinity}]
                    k!

In[15]:= PowerSeries[Cos[x]*Exp[2x],x,0]
ps-info: 2 step(s) for DE:
         5 f[x] - 4 f'[x] + f''[x] = 0
ps-info: RE for all n >= 0:
                    -5 a[n] + 4 a[1 + n] + 4 n a[1 + n]
         a[2 + n] = -----------------------------------
                              (1 + n) (2 + n)
```

```
ps-info: DE has constant coefficients
ps-info: modified RE (b[n] = n! a[n]):
         5 b[n] - 4 b[1 + n] + b[2 + n] = 0
              k/2  k              1
             5    x  Cos[k ArcTan[-]]
                                  2
Out[15]= Sum[------------------------, {k, 0, Infinity}]
                       k!
```

and finally we give some examples of rational type

```
In[16]:= PowerSeries[1/(x-x^3),x,0]
                                           1          -1        1       1
ps-info: rational algorithm applied (------ = ---------- + - - ---------)
                                              3   2 (-1 + x)    x   2 (1 + x)
                                         x - x
                          k   k
         1       (1 - (-1) ) x
Out[16]= - + Sum[--------------, {k, 0, Infinity}]
         x             2

In[17]:= PowerSeries[1/(x^2+3x+2),x,0]
                                              1            1       1
ps-info: rational algorithm applied (------------ = ----- - -----)
                                                    2   1 + x   2 + x
                                         2 + 3 x + x
                         1  k
                       (-(-))
                  k      2      k
Out[17]= Sum[((-1)  - -------) x , {k, 0, Infinity}]
                         2
```

The following is the generating function $f(x) = \sum_{k=0}^{\infty} a_k x^k$ of the Fibonacci numbers $a_n$ that are defined by the recurrence

$$a_{n+1} = a_n + a_{n-1} \;, \qquad a_0 = 0 \;, \quad a_1 = 1 \;.$$

The call

```
In[18]:= PowerSeries[x/(1-x-x^2),x,0]
                                                x
ps-info: rational algorithm applied (---------- =
                                                     2
                                         1 - x - x

          -5 + Sqrt[5]              5 + Sqrt[5]
>   ------------------------ - --------------------)
        -(-1 + Sqrt[5])          1 + Sqrt[5]
    10 (--------------- + x)   10 (----------- + x)
               2                        2
```

```
                   -2          k        -10     k   k
            (-(-----------)  + (--------) ) x
               1 + Sqrt[5]            3/2
                                 5 - 5
Out[18]= Sum[---------------------------------, {k, 0, Infinity}]
                           Sqrt[5]
```

produces a well-known closed formula for the Fibonacci numbers.

## 3 Logarithmic singularities and Puiseux series

The algorithm covers functions which correspond to hypergeometric type Laurent-Puiseux series rather than just Laurent series. We give some examples.

```
In[19]:= PowerSeries[Sin[Sqrt[x]],x,0]
ps-info: 2 step(s) for DE:
         f[x] + 2 f'[x] + 4 x f''[x] = 0
                         1
ps-info: RE for all n >= -:
                         2
                           -a[n]
         a[1 + n] = -------------------
                    2 (1 + n) (1 + 2 n)
                              n
ps-info: RE modified to (n -> -)
                              2
ps-info: RE for all n >= 0:
                           a[n]
         a[2 + n] = -(---------------)
                      (1 + n) (2 + n)
ps-info: function of hypergeometric type
ps-info: a[0] = 0, a[1] = 1, a[2] = 0
                 k  1/2 + k
             (-1)  x
Out[19]= Sum[--------------, {k, 0, Infinity}]
               (1 + 2 k)!
```

Note that, again, the DE for $\sin\sqrt{x}$ is rather different from that for $\sin x$.

```
In[20]:= PowerSeries[((1+Sqrt[x])/x)^(1/3),x,0]
ps-info: 2 step(s) for DE:
                                            2                      2
         (1 + 2 x) f[x] + (-21 x + 33 x ) f'[x] + 18 (-1 + x) x  f''[x] = 0
                         1
ps-info: RE for all n >= -:
                         6
                    (2 + 3 n) (1 + 6 n) a[n]
         a[1 + n] = ------------------------
                      (4 + 3 n) (5 + 6 n)
```

```
                                n
ps-info: RE modified to (n -> -)
                                6
ps-info: RE for all n >= -4:
                    (1 + n) (4 + n) a[n]
         a[6 + n] = --------------------
                      (5 + n) (8 + n)
ps-info: function of hypergeometric type
                                                1
ps-info: a[0] = 1, a[1] = 0, a[2] = 0, a[3] = -, a[4] = 0, a[5] = 0
                                                3

              -(1/3) + k             1
             x           Pochhammer[-(-), 2 k]
                                     3
Out[20]= Sum[--------------------------------, {k, 0, Infinity}] +
                          (2 k)!

          1/6 + k           2
         x        Pochhammer[-, 2 k]
                            3
>    Sum[---------------------------, {k, 0, Infinity}]
               3 (1 + 2 k)!
```

The algorithm covers moreover the case of logarithmic singularities which occur if the derivative of some order of $f$ corresponds to a series of hypergeometric type. An example of that kind is

```
In[21]:= PowerSeries[ArcSech[x],x,0]
ps-info: 2 step(s) for DE:
                  2                  3
         (-1 + 2 x ) f'[x] + (-x + x ) f''[x] = 0
ps-info: RE for all n >= -1:
                    n (1 + n) a[n]
         a[2 + n] = --------------
                                2
                       (2 + n)
ps-info: function of hypergeometric type
                                             1
ps-info: working with f' = -(-----------------)
                                                -2   2
                             Sqrt[-1 + x  ] x
ps-info: RE for all n >= -2:
                    (2 + n) a[n]
         a[2 + n] = ------------
                       3 + n
ps-info: a[0] = -1, a[1] = 0
                                 1 k   2 k
                              -((-)  x    (2 k)!)
                                 4
Out[21]= Log[2] - Log[x] + Sum[------------------, {k, 1, Infinity}]
                                           2
                                    2 k k!
```

## References


[1] E. R. Hansen: A table of series and products. Prentice-Hall, Englewood Cliffs, NJ, 1975.

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