# Methods of Computer Algebra for Orthogonal Polynomials

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# Online Demonstrations with Computer Algebra

## Computer Algebra Systems

- I will use the computer algebra system *Maple* to program and demonstrate the considered methods.

## Algorithms

The mostly used algorithms are:

- Algorithms of Linear Algebra with many variables
- Multivariate polynomial factorization
- and the solution of non-linear systems of polynomial equations.

# Orthogonal Polynomials

## Scalar Products

Given: a scalar product

$$\langle f, g\rangle := \int_\alpha^\beta f(x)g(x)\,d\mu(x)$$

with non-negative Borel measure $\mu(x)$ supported in the interval $[\alpha, \beta]$ (or equivalently as a Riemann-Stieltjes integral with nondecreasing $\mu(x)$).

## Special Cases

- absolutely continuous measure $d\mu(x) = \rho(x)\,dx$ with weight function $\rho(x)$,
- discrete measure $\mu(x) = \rho(x)$ supported in $\mathbb{Z}$.

# Orthogonal Polynomials

## Orthogonality

- A system of polynomials $(P_n(x))_{n\geqq 0}$ with $\deg(P_n) = n$

$$P_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \cdots \,, \qquad k_n \neq 0$$

is called orthogonal (OPS) w. r. t. the positive-definite measure $\mu(x)$, if

$$\langle P_m, P_n \rangle = \begin{cases} 0 & \text{if } m \neq n \\ h_n > 0 & \text{if } m = n \end{cases}$$

# General Properties of OPS

## Main Properties

- (**Three-term Recurrence**) Every OPS satisfies

$$x\,P_n(x) = a_n\,P_{n+1}(x) + b_n\,P_n(x) + c_n\,P_{n-1}(x)\ .$$

- (**Zeros**) All zeros of an OPS are simple, lie in the interior of $[\alpha, \beta]$ and have some nice interlacing properties.
- (**Representation by Moments**)

$$P_n(x) = \ C_n \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}\ ,$$

where $\mu_n := \int_\alpha^\beta x^n d\mu(x)$ denote the moments of $d\mu(x)$.

# Classical Families

## Classical "Continuous" Families

The classical OPS $(P_n(x))_{n\geqq 0}$ can be defined as the polynomial solutions of the differential equation:

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) - \lambda_n P_n(x) = 0 \ .$$

## Conclusions

- $n = 1$ yields $\tau(x) = dx + e, d \neq 0$
- $n = 2$ yields $\sigma(x) = ax^2 + bx + c$
- The coefficient of $x^n$ yields $\lambda_n = n(a(n-1) + d)$

# Classical Families

## Classification

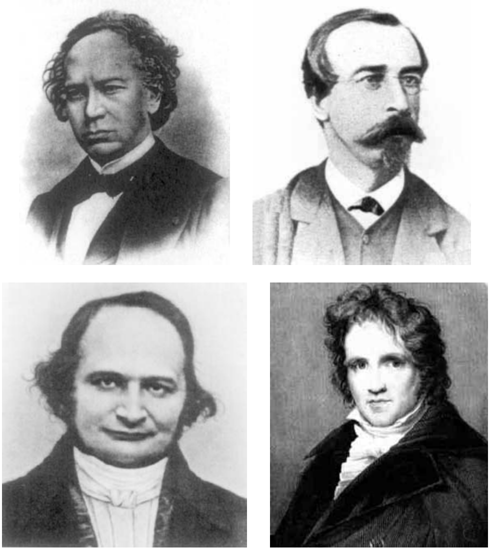
These classical families can be classified (modulo linear transformations) according to the following scheme (Bochner (1929))

- $\sigma(x) = 0$ powers $x^n$
- $\sigma(x) = 1$ Hermite polynomials
- $\sigma(x) = x$ Laguerre polynomials
- $\sigma(x) = 1 - x^2$ Jacobi polynomials
- $\sigma(x) = x^2$ Bessel polynomials

## Ingredients

For the theory one needs

- a representing family $f_n(x)$, here the powers $f_n(x) = x^n$,
- an operator, here the derivative operator $D$, with the simple property $D\, f_n(x) = n\, f_{n-1}(x)$.



Hermite, Laguerre, Jacobi and Bessel

# Classical Families

## Pearson Differential Equation

The corresponding weight function $\rho(x)$ satisfies the Pearson Differential Equation

$$\frac{d}{dx}\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x) \ .$$

## Weight Function

Hence it is given by

$$\rho(x) = \frac{C}{\sigma(x)} e^{\int \frac{\tau(x)}{\sigma(x)} dx} \ .$$

# Classical Continuous Families

The following properties are equivalent, each defining the classical continuous families.

- Differential equation for $(P_n(x))_{n \geqq 0}$.
- Pearson Differential Equation $(\sigma\, \rho)' = \tau\, \rho$ for weight $\rho(x)$.
- With $(P_n(x))_{n \geqq 0}$ also $(P'_{n+1}(x))_{n \geqq 0}$ is an OPS.
- Derivative Rule:

$$\sigma(x)\, P'_n(x) = \alpha_n\, P_{n+1}(x) + \beta_n\, P_n(x) + \gamma_n\, P_{n-1}(x)\ .$$

- Structure Relation: $P_n(x)$ satisfies

$$P_n(x) = \widehat{a}_n\, P'_{n+1}(x) + \widehat{b}_n\, P'_n(x) + \widehat{c}_n\, P'_{n-1}(x)\ .$$

- Rodrigues Formula: $P_n(x)$ is given as

$$P_n(x) = \frac{E_n}{\rho(x)} \frac{d^n}{dx^n} \Big( \rho(x)\, \sigma(x)^n \Big)\ .$$

# Classical Discrete Families

## Classical Discrete OPS

- By replacing the differential operator $D$ by the difference operator $\Delta f(x) = f(x+1) - f(x)$ one gets a rather similar theory for the classical discrete OPS that are solutions of certain difference equations.
- In this case the representing family are the falling factorials $f_n(x) = x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ with the property $\Delta\, f_n(x) = n\, f_{n-1}(x)$.
- The classification leads to the Charlier, Meixner, Krawtchouk and Hahn polynomials.
- Both the continuous and the discrete classical OPS can be represented by hypergeometric functions.

# Hypergeometric Functions

## Generalized Hypergeometric Function

The formal power series

$$ {}_pF_q\left( \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, z \right) = \sum_{k=0}^{\infty} A_k \, z^k \, , $$

whose summands $\alpha_k = A_k z^k$ have a rational term ratio

$$ \frac{\alpha_{k+1}}{\alpha_k} = \frac{A_{k+1} \, z^{k+1}}{A_k \, z^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \frac{z}{(k+1)} \, , $$

is called the generalized hypergeometric function.

# Hypergeometric Functions

## Hypergeometric Terms

The summand $\alpha_k = A_k z^k$ of a hypergeometric series is called a hypergeometric term.

## Formula for Hypergeometric Terms

For the coefficients of the generalized hypergeometric function one gets the following formula using the shifted factorial (Pochhammer symbol) $(a)_k = a(a+1)\cdots(a+k-1)$

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \, . $$

# Hypergeometric Functions

## OPS as Hypergeometric Functions

- Substituting the power series $f(x) = \sum_{k=0}^{\infty} A_k\, x^k$ into the differential equation and equating coefficients yields a recurrence equation for $A_k$.
- Using this recurrence one gets for example for the Laguerre polynomials

$$L_n^{\alpha}(x) = \binom{n+\alpha}{n} {}_1F_1\left(\left.\begin{matrix} -n \\ \alpha+1 \end{matrix}\,\right|\, x\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k,$$

- and the Hahn polynomials are given by

$$Q_n^{(\alpha,\beta)}(x,N) = {}_3F_2\left(\left.\begin{matrix} -n, -x, n+1+\alpha+\beta \\ \alpha+1, -N \end{matrix}\,\right|\, 1\right).$$

# Properties of Classical OPS

## Relations Between Classical OPS

Using linear algebra one can compute the coefficients of the following identities – expressed through the parameters $a, b, c, d$ und $e$ – (Lesky (1985), K./Schmersau (1998)):

**(RE)** $$x\,P_n(x) = a_n\,P_{n+1}(x) + b_n\,P_n(x) + c_n\,P_{n-1}(x)$$

**(DR)** $$\sigma(x)\,P_n'(x) = \alpha_n\,P_{n+1}(x) + \beta_n\,P_n(x) + \gamma_n\,P_{n-1}(x)$$

**(SR)** $$P_n(x) = \widehat{a}_n\,P_{n+1}'(x) + \widehat{b}_n\,P_n'(x) + \widehat{c}_n\,P_{n-1}'(x)$$

*Maple*

# Classical OPS Solutions of Holonomic Recurrence Equations

## Inverse Algorithm

- We showed that the coefficients of the recurrence equation of the classical systems can be written in terms of the coefficients $a$, $b$, $c$, $d$, and $e$ of the differential / difference equation.
- If one uses these formulas in the backward direction, then one can determine the possible differential / difference equations from a given recurrence.
- For this purpose one must solve a non-linear system.

# Classical OPS Solutions of Holonomic Recurrence Equations

## Example

- Given: the recurrence equation

$$P_{n+2}(x) - (x - n - 1)\, P_{n+1}(x) + \alpha(n+1)^2 P_n(x) = 0$$

  Does this equation have classical OPS solutions? *Maple*

- We find out that the solutions of this equation are shifted Laguerre polynomials for $\alpha = 1/4$. For $\alpha < 1/4$ the recurrence has Meixner and Krawtchouk polynomial solutions.

# Zeilberger Algorithm

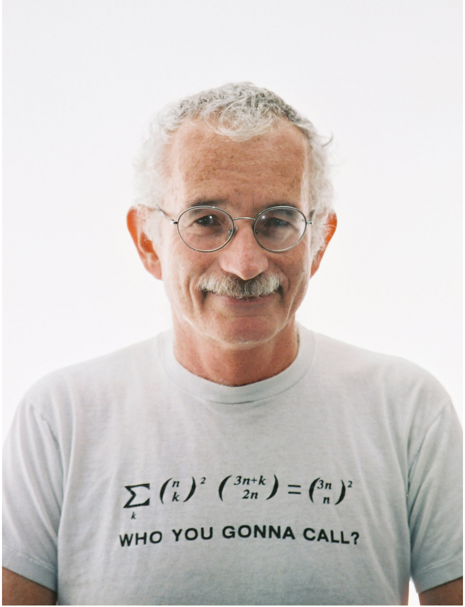
## Recurrence Equations for Hypergeometric Series

Doron Zeilberger (1990) designed an algorithm to compute recurrence equations for hypergeometric sums of the type

$$s_n = \sum_{k=-\infty}^{\infty} F(n,k) \, .$$

## Holonomic Recurrence Equations

His algorithm results in a holonomic recurrence equation for $s_n$. A recurrence equation is called holonomic if it is linear, homogeneous, and has polynomial coefficients.




Doron Zeilberger

# Zeilberger Algorithm

## Differential Equations for Hypergeometric Series

A similar algorithm yields a holonomic differential equation for series of the form

$$s(x) = \sum_{k=-\infty}^{\infty} F(x,k) \; .$$

## Algebra of Holonomic Differential resp. Recurrence Equations

- Holonomic functions form an algebra, i. e. sum and product of holonomic functions are again holonomic, and there are linear algebra algorithms to determine the resulting differential and recurrence equations.

# Zeilberger Algorithm

## Application to Orthogonal Polynomials

- As an example, we will apply Zeilberger's algorithm to the Laguerre polynomials

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k \; .$$

- With the above algorithms one can also compute recurrence and differential equations for the square $L_n^\alpha(x)^2$, or for the difference $L_{n+1}^\alpha(x) - L_n^\alpha(x)$.

*Maple*

# Power Series Algorithm

## Inverse Algorithm

- Whereas Zeilberger's algorithm uses the right hand side of

$$f(x) = \sum_{k=-\infty}^{\infty} A_k\, x^k$$

  to detect its left hand side (or a differential equation for it), my FPS algorithm (= Formal Power Series) starts from the left hand side to detect the right hand side.
- This algorithm (K. (1992)) has been embedded in Maple by Torsten Sprenger as `convert(...,FormalPowerSeries)`.

*Maple*

# Online Orthogonal Polynomials

## CAOP = Computer Algebra and Orthogonal Polynomials

- CAOP is a web tool for calculating formulas for orthogonal polynomials belonging to the Askey-Wilson scheme using Maple.
- The implementation of CAOP was originally done by René Swarttouw as part of the Askey-Wilson Scheme Project performed at RIACA in Eindhoven in 2004.
- The present site is a completely revised version of this project which has been done by Torsten Sprenger under my supervision in 2012 and is maintained at the University of Kassel.
- http://www.caop.org/

# Askey-Wilson Scheme

## Askey-Wilson Scheme

- In CAOP you saw all the families of the Askey-Wilson Scheme.
- Besides the already mentioned cases there are the
  - discrete measure supported in $q^{\mathbb{Z}}$ (Hahn tableau);
  - discrete measure supported on a quadratic lattice (Wilson tableau);
  - discrete measure supported on a $q$-quadratic lattice (Askey-Wilson tableau).
- It turns out that the three above classes can be treated in a similar way as the continuous and the discrete cases leading to similar theories.
- However, these computations are very tedious and can be done much easier with the use of computer algebra. This research is ongoing and not yet finished.

I would like to thank you very much for your interest!