1 ORTHOGONAL POLYNOMIALS AND COMPUTER ALGEBRA

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Abstract: Orthogonal polynomials have a long history, and are still important objects of consideration in mathematical research as well as in applications in Mathematical Physics, Chemistry, and Engineering. Quite a lot is known about them. Particularly well-known are differential equations, recurrence equations, Rodrigues formulas, generating functions and hypergeometric representations for the classical systems of Jacobi, Laguerre and Hermite which can be found in mathematical dictionaries. Less well-known are the corresponding representations for the classical discrete systems of Hahn, Krawtchouk, Meixner and Charlier, as well as addition theorems, connection relations between different systems and other identities for these and other systems of orthogonal polynomials. The ongoing research in this still very active subject of mathematics expands the knowledge database about orthogonal polynomials continuously. In the last few decades the classical families have been extended to a rather large collection of polynomial systems, the so-called Askey-Wilson scheme, and they have been generalized in other ways as well.

Recently new algorithmic approaches have been discovered to compute differential, recurrence and similar equations from series or integral representations. These methods turn out to be quite useful to prove or detect identities for orthogonal polynomial systems. Further algorithms to detect connection coef-

ficients or to identify polynomial systems from given recurrence equations have been developed. Although some algorithmic methods had been known already in the last century, their use was rather limited due to the immense amount of calculations. Only the existence and distribution of computer algebra systems makes their use simple and useful for everybody.

In this plenary lecture an overview is given of how algorithmic methods implemented in computer algebra systems can be used to prove identities about and to detect new knowledge for orthogonal polynomials and other hypergeometric type special functions. Implementations for this type of algorithms exist in Maple, Mathematica and REDUCE, and maybe also in other computer algebra systems. Online demonstrations will be given using Maple V.5.

COMPUTER ALGEBRA

What is Computer Algebra?

In the work with programming languages like Pascal or C any variable used has to be *declared* to connect the variable name with a fixed amount of *memory*. Hence all numbers are *static* in size. As a result there is a maximal integer that can be represented, and decimal numbers have a fixed degree of precision.

The situation is quite different in computer algebra systems which constitute high level programming languages. In this talk we speak about *general purpose systems* like Axiom [14], DERIVE [27], Macsyma [23], Maple [6], Mathematica [32], MuPAD [8] and REDUCE [13]. We will present examples in Maple V.5.

In computer algebra systems numbers are *dynamical* objects whose size, i.e. the number of memory cells allocated, depends on their actual size. As a result there is no maximal integer, one can deal with integers with arbitrary many digits, and their use is restricted only by time and space limitations of the memory available. Moreover one can work with decimal numbers of arbitrary precision.[1]

As an example, entering the line

```
> factorial(100);
```

Maple computes

$$
\begin{aligned}
&933262154439441526816992388562667004907159682643816214\backslash\\
&\quad 68592963895217599993229915608941463976156518286253697\backslash\\
&\quad 920827223758251185210916864000000000000000000000000
\end{aligned}
$$

We can factorize this number by the command[2]

```
> ifactor(%);
```

$$
\begin{aligned}
&(2)^{97}\,(3)^{48}\,(5)^{24}\,(7)^{16}\,(11)^{9}\,(13)^{7}\,(17)^{5}\,(19)^{5}\,(23)^{4}\,(29)^{3}\,(31)^{3}\,(37)^{2}\,(41)^{2}\\
&\quad (43)^{2}\,(47)^{2}\,(53)\,(59)\,(61)\,(67)\,(71)\,(73)\,(79)\,(83)\,(89)\,(97)
\end{aligned}
$$

The following computes a binomial coefficient

```
> binomial(500,50);
```

231442282798430046901775687166104881254565781906279 25\
22329327913362690

and in the next line 1,001 binomial coefficients are computed and added together:

```
> add(binomial(1000,k),k=0..1000);
```

10715086071862673209484250490600018105614048117055336\
07443750388370351051124936122493198378815695858127594\
67291755314682518714528569231404359845775746985748039\
34567774824230985421074605062371141877954182153046474\
98358194126739876755916554394607706291457119647768654\
216766042983165262438683720566806 9376

As you know, the integer factorization of this number is very simple.

```
> ifactor(%);
```

$$(2)^{1000}$$

More decisively, in computer algebra systems the work is not restricted to numbers, but one can easily deal with other mathematical objects. Major objects of consideration are multivariate polynomials. By the Euclidean algorithm gcd-computations for polynomials can be carried out similarly as for numbers. Those algorithms are implemented in general purpose computer algebra systems. Hence rational functions can be put in lowest terms, etc.

As an example, the following computation puts the rational function

$$\frac{1-x^{10}}{1-x^4}$$

in lowest terms:

```
> normal((1-x^10)/(1-x^4));
```

$$\frac{x^8+x^6+x^4+x^2+1}{x^2+1}$$

One of the highlights of computer algebra systems is rational factorization which can be handled algorithmically (see e.g. [11], Chapter 8). Let's define a two-variate polynomial p:

```
> p:=product(((x+y)^j-1/j^2),j=1..5);
```

$$p := (x+y-1)\,((x+y)^2-\frac{1}{4})\,((x+y)^3-\frac{1}{9})\,((x+y)^4-\frac{1}{16})\,((x+y)^5-\frac{1}{25})$$

and let's expand it

```
> p:=expand(p);
```

$$
\begin{aligned}
p := {} & \frac{1}{14400}\,x + \frac{1}{14400}\,y + \frac{1}{1800}\,x\,y + \frac{1}{3600}\,y^2 + \frac{1}{960}\,x^2\,y + \frac{1}{960}\,x\,y^2 \\
& - \frac{53}{2400}\,x^5\,y - \frac{53}{960}\,x^4\,y^2 - \frac{53}{720}\,x^3\,y^3 - \frac{53}{960}\,x^2\,y^4 + \frac{5}{36}\,y^{12} + y^{15} - \frac{1}{4}\,y^{13} - y^{14} \\
& + \frac{77}{144}\,x^{10}\,y + \frac{385}{144}\,x^9\,y^2 + \frac{385}{48}\,x^8\,y^3 + \frac{385}{24}\,x^7\,y^4 + \frac{539}{24}\,x^6\,y^5 + \frac{539}{24}\,x^5\,y^6 \\
& + \frac{385}{24}\,x^4\,y^7 + \frac{385}{48}\,x^3\,y^8 + \frac{385}{144}\,x^2\,y^9 + 6435\,x^8\,y^7 + 6435\,x^7\,y^8 + 5005\,x^6\,y^9 \\
& + 3003\,x^5\,y^{10} + 1365\,x^4\,y^{11} + 455\,x^3\,y^{12} + 105\,x^2\,y^{13} - 2002\,y^5\,x^9 \\
& - 3003\,y^6\,x^8 - 3432\,y^7\,x^7 - 3003\,y^8\,x^6 - 2002\,y^9\,x^5 - 1001\,y^{10}\,x^4 \\
& - 364\,y^{11}\,x^3 - 91\,y^{12}\,x^2 - 14\,y^{13}\,x + 15\,x\,y^{14} - \frac{13}{4}\,x^{12}\,y - \frac{39}{2}\,x^{11}\,y^2 \\
& - \frac{143}{2}\,x^{10}\,y^3 + \frac{181}{30}\,x^3\,y^7 + \frac{181}{80}\,x^2\,y^8 + \frac{2807}{800}\,y^5\,x^4 + \frac{2807}{1200}\,y^6\,x^3 + \frac{401}{400}\,y^7\,x^2 \\
& + \frac{401}{1600}\,y^8\,x + \frac{181}{360}\,x\,y^9 + \frac{19}{1800}\,x^7\,y + \frac{133}{3600}\,x^6\,y^2 + \frac{133}{1800}\,x^5\,y^3 + \frac{133}{1440}\,x^4\,y^4 \\
& + \frac{133}{1800}\,x^3\,y^5 + \frac{133}{3600}\,x^2\,y^6 + \frac{19}{1800}\,x\,y^7 + \frac{401}{1600}\,x^8\,y + \frac{401}{400}\,x^7\,y^2 + \frac{2807}{1200}\,x^6\,y^3 \\
& + \frac{2807}{800}\,x^5\,y^4 - \frac{7}{80}\,x^6\,y - \frac{21}{80}\,x^5\,y^2 - \frac{7}{16}\,x^4\,y^3 - \frac{7}{16}\,x^3\,y^4 - \frac{21}{80}\,x^2\,y^5 - \frac{7}{80}\,x\,y^6 \\
& + \frac{181}{360}\,x^9\,y + \frac{181}{80}\,x^8\,y^2 + \frac{181}{30}\,x^7\,y^3 + \frac{1267}{120}\,x^6\,y^4 + \frac{1267}{100}\,x^5\,y^5 - \frac{3}{1600}\,y^5 \\
& + \frac{401}{14400}\,y^9 - \frac{1}{80}\,y^7 + \frac{181}{3600}\,y^{10} + \frac{19}{14400}\,y^8 - \frac{53}{2400}\,x\,y^5 + \frac{7}{3600}\,x^3\,y \\
& + \frac{7}{2400}\,x^2\,y^2 + \frac{7}{3600}\,x\,y^3 - \frac{3}{320}\,x^4\,y - \frac{3}{160}\,x^3\,y^2 - \frac{3}{160}\,x^2\,y^3 - \frac{3}{320}\,x\,y^4 \\
& + \frac{1}{2880}\,y^3 - \frac{53}{14400}\,y^6 + \frac{7}{14400}\,y^4 + \frac{181}{3600}\,x^{10} + \frac{7}{14400}\,x^4 + \frac{1}{3600}\,x^2 \\
& - \frac{3}{1600}\,x^5 + \frac{1}{2880}\,x^3 + \frac{19}{14400}\,x^8 - \frac{53}{14400}\,x^6 - \frac{1}{80}\,x^7 + \frac{401}{14400}\,x^9 + x^{15} \\
& + \frac{7}{144}\,x^{11} + \frac{5}{36}\,x^{12} - \frac{1}{4}\,x^{13} - x^{14} - \frac{1}{14400} + \frac{1267}{120}\,x^4\,y^6 - \frac{715}{4}\,x^9\,y^4 \\
& - \frac{1287}{4}\,x^8\,y^5 - 429\,x^7\,y^6 - 429\,x^6\,y^7 - \frac{1287}{4}\,x^5\,y^8 - \frac{715}{4}\,x^4\,y^9 - \frac{143}{2}\,x^3\,y^{10} \\
& - \frac{39}{2}\,x^2\,y^{11} - \frac{13}{4}\,x\,y^{12} - 14\,x^{13}\,y - 91\,x^{12}\,y^2 - 364\,x^{11}\,y^3 - 1001\,x^{10}\,y^4 \\
& + \frac{5}{3}\,x^{11}\,y + \frac{55}{6}\,x^{10}\,y^2 + \frac{275}{9}\,x^9\,y^3 + \frac{275}{4}\,x^8\,y^4 + 110\,x^7\,y^5 + \frac{385}{3}\,x^6\,y^6 \\
& + 110\,x^5\,y^7 + \frac{275}{4}\,x^4\,y^8 + \frac{275}{9}\,x^3\,y^9 + \frac{55}{6}\,x^2\,y^{10} + \frac{5}{3}\,x\,y^{11} + 15\,x^{14}\,y \\
& + 105\,x^{13}\,y^2 + 455\,x^{12}\,y^3 + 1365\,x^{11}\,y^4 + 3003\,x^{10}\,y^5 + 5005\,x^9\,y^6 \\
& + \frac{77}{144}\,x\,y^{10} + \frac{7}{144}\,y^{11}
\end{aligned}
$$

This looks monstrous! Like magic Maple is capable to find the factors of this polynomial from its expanded form:

```
> factor(p);
```

$$\frac{1}{14400}(x+y-1)\,(4\,x^2+1+8\,x\,y+4\,y^2)$$
$$(25\,x^5+125\,x^4\,y+250\,x^3\,y^2+250\,x^2\,y^3+125\,x\,y^4+25\,y^5-1)$$
$$(9\,x^3-1+27\,x^2\,y+27\,x\,y^2+9\,y^3)\,(2\,x+2\,y+1)^2\,(2\,x+2\,y-1)^2$$

The `factor` command does factorization over $\mathbb{Q}$. Hence the following rational function is not factorized properly:

```
> factor(1/(1+x^4));
```

$$\frac{1}{1+x^4}$$

despite the fact that the integration command is successful:

```
> int(1/(1+x^4),x);
```

$$\frac{1}{8}\sqrt{2}\ln(\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1})+\frac{1}{4}\sqrt{2}\arctan(x\sqrt{2}+1)+\frac{1}{4}\sqrt{2}\arctan(x\sqrt{2}-1)$$

This result suggests that rational factorization can be carried out over $\mathbb{Q}(\sqrt{2})$ which can be invoked by

```
> factor(1/(1+x^4),sqrt(2));
```

$$\frac{1}{(x^2-x\sqrt{2}+1)\,(x^2+x\sqrt{2}+1)}$$

Another important and still very active field are Gröbner bases. The computation of Gröbner bases for polynomial ideals has many interesting applications. One application gives an algorithm to find the complete set of solutions of a given (nonlinear) polynomial equation system (see e.g. [24]–[25]). This algorithm is available through Maple's `solve` command.

```
> {solve({-5*x^2+y^2-2*x=1,x^2-1/15*y^2-2/15*x*y=1},{x,y})};
```

$$\{\{y=2,\,x=-1\},\,\{y=5,\,x=2\},\,\{y=-3,\,x=2\,\mathrm{RootOf}(5\,\_Z^2-2+\_Z)\}\}$$

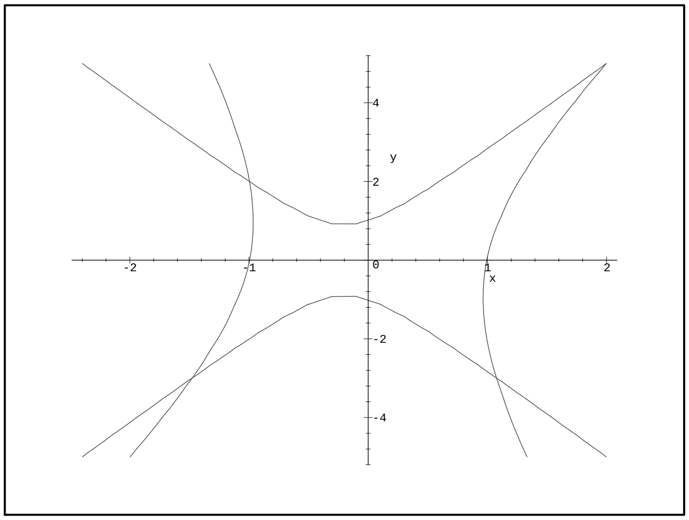
```
> convert(%,radical);
```

$$\{\{y=2,\,x=-1\},\,\{y=5,\,x=2\},\,\{y=-3,\,x=-\frac{1}{5}+\frac{1}{5}\sqrt{41}\}\}$$

Note that by the conversion to radicals one of the solutions is lost. We will see more advanced applications of the use of the `solve` command later.

Computer algebra systems can give graphical representations. Here is the intersection of the two hyperbolas above that are given implicitly:

```
> with(plots):
> implicitplot({-5*x^2+y^2-2*x=1,x^2-1/15*y^2-2/15*x*y=1},
> x=-5..5,y=-5..5,grid=[50,50]);
```



General purpose computer algebra systems can also deal with transcendental objects. They can compute derivatives, e.g.

```
> derivative:=diff(exp(x-x^2)*sin(x^6-1),x);
```

$$derivative := (1 - 2\,x)\,e^{(x-x^2)}\sin(x^6-1) + 6\,e^{(x-x^2)}\cos(x^6-1)\,x^5$$

and they can do integrations

```
> integral:=int(derivative,x);
```

$$integral := -\frac{1}{2} I(-e^{(-(-1+x)\,(I\,x^5+I\,x^4+I\,x^3+I\,x^2+x+I\,x+I))} + e^{((-1+x)\,(I\,x^5+I\,x^4+I\,x^3+I\,x^2-x+I\,x+I))})$$

Note that by the differentiation rules differentiation is a purely algebraic operation. On the other hand, it is not so clear that integration can also be carried out completely algebraically. Risch's algorithm [28]–[29] is an algebraic algorithm which decides after finitely many steps whether or not a given elementary function (rational composition of exp-log functions) has an elementary antiderivative, and finds it in the affirmative case. Risch's algorithm converts trigonometric functions in exponentials, hence in our case the resulting integral looks rather different from the input expression. This example shows that functions can come in quite different disguises.

To bring the resulting integral in a form to be comparable with the input expression, we can use the command

```
> factor(convert(integral,trig));
```

$$\sin((-1+x)\,(x+1)\,(x+x^2+1)\,(x^2-x+1))$$
$$(\cosh((-1+x)\,x)-\sinh((-1+x)\,x))$$

The original expression is converted towards the same expression

```
> factor(convert(exp(x-x^2)*sin(x^6-1),trig));
```

$$\sin((-1+x)\,(x+1)\,(x+x^2+1)\,(x^2-x+1))$$
$$(\cosh((-1+x)\,x)-\sinh((-1+x)\,x))$$

Computer algebra systems can solve ordinary differential equations as well. As an example we solve the Euler differential equation

$$x^2\,y''(x)-3x\,y'(x)+2y(x)=0$$

by

```
> dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+2*y(x)=0,y(x));
```

$$\mathrm{y}(x)=\_C1\;x^{(2+\sqrt{2})}+\_C2\;x^{(2-\sqrt{2})}$$

We see that Maple gives a basis of the solution space and introduces two integration constants.

To generate more differential equations which we would like to solve we load the share library package `FormalPowerSeries`. This package was written by Dominik Gruntz [12] and includes algorithms described in [16].

```
> with(share): with(FPS):
```

Now we are prepared to generate the differential equation for $\arcsin x\,e^x$, for example,

```
> DE:=SimpleDE(arcsin(x)*exp(x),x,F);
```

$$DE:=(-x-1+x^2)\,\mathrm{F}(x)$$
$$+(x+2-2\,x^2)\,(\frac{\partial}{\partial x}\,\mathrm{F}(x))+(-1+x)\,(x+1)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{F}(x))=0$$

and apply `dsolve` to compute its solution space

```
> dsolve(DE,F(x));
```

$$\mathrm{F}(x)=\frac{\_C1\;(x-1)^{(1/4)}\,(x+1)^{(1/4)}\,e^x}{((x-1)\,(x+1))^{(1/4)}}$$
$$+\frac{\_C2\;(x-1)^{(1/4)}\,(x+1)^{(1/4)}\,e^x\ln(x+\sqrt{(x-1)\,(x+1)})}{((x-1)\,(x+1))^{(1/4)}}$$

As another example, let's deal with the Bessel functions:

```
> DE:=SimpleDE(BesselJ(n,x),x,F);
```

$$DE:=(\frac{\partial^2}{\partial x^2}\,\mathrm{F}(x))\,x^2+(-n+x)\,(x+n)\,\mathrm{F}(x)+(\frac{\partial}{\partial x}\,\mathrm{F}(x))\,x=0$$

```
> dsolve(DE,F(x));
```

$$F(x) = \_C1 \,\mathrm{BesselJ}(n,\, x) + \_C2 \,\mathrm{BesselY}(n,\, x)$$

The result is returned immediately showing that Maple has the Bessel differential equation in a lookup-table. Whereas algorithmic techniques are available for elementary function solutions of differential equations, this is not so if special functions are involved. Maple fails to find a solution for the following simple example.

```
> DE:=SimpleDE(BesselJ(0,x)+exp(x),x,F);
```

$$DE := x\,(2\,x+1)\,(\frac{\partial^3}{\partial x^3}\,F(x)) - (x+1)\,(-1+2\,x)\,F(x) + (x+1)\,(2\,x-3)\,(\frac{\partial}{\partial x}\,F(x)) + (2+x-2\,x^2)\,(\frac{\partial^2}{\partial x^2}\,F(x)) = 0$$

```
> dsolve(DE,F(x));
```

$$F(x) = \_C1\, e^x + e^x\,\mathrm{DESol}\left(\left\{\_Y(x) + (\frac{\partial}{\partial x}\,\_Y(x)) + \frac{(2\,x^2+x)\,(\frac{\partial^2}{\partial x^2}\,\_Y(x))}{(2\,x+1)^2}\right\},\, \{\_Y(x)\}\right)$$

HYPERGEOMETRIC FUNCTIONS

The Laguerre polynomials satisfy the differential equation

```
> DE:=SimpleDE(LaguerreL(n,alpha,x),x,F);
```

$$DE := (\frac{\partial^2}{\partial x^2}\,F(x))\,x + n\,F(x) + (-x+\alpha+1)\,(\frac{\partial}{\partial x}\,F(x)) = 0$$

and Maple finds its general solution[3]

```
> dsolve(DE,F(x));
```

$$F(x) = \_C1\,\mathrm{hypergeom}([-n],\,[\alpha+1],\,x) + \_C2\,x^n\,\mathrm{hypergeom}([-n-\alpha,\,-n],\,[],\,-\frac{1}{x})$$

in terms of hypergeometric functions. Indeed, the first of these hypergeometric functions is a multiple of the Laguerre polynomial $L_n^{(\alpha)}(x)$.

The *generalized hypergeometric series* is defined by

$${}_pF_q\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{array} \middle|\, x\right) := \sum_{k=0}^{\infty} A_k\,x^k = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k\,k!} x^k$$

where $(a)_k := \prod_{j=1}^{k} (a+j-1) = \Gamma(a+k)/\Gamma(a)$, denotes the *Pochhammer symbol* (shifted factorial).

A_k is a *hypergeometric term* and fulfills the recurrence equation ($k \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$)

$$A_{k+1} := \frac{(k + a_1) \cdots (k + a_p)}{(k + b_1) \cdots (k + b_q)(k + 1)} \cdot A_k$$

with the initial value

$$A_0 := 1 \; .$$

In Maple one has the syntax `hypergeom(plist,qlist,x)`, where

$$\text{plist} = [a_1, a_2, \ldots, a_p] \qquad \text{and} \qquad \text{qlist} = [b_1, b_2, \ldots, b_q] \; .$$

Examples of hypergeometric functions are given by

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = {}_0F_0\left(\left.\begin{array}{c} - \\ - \end{array}\right| x\right) ,$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = {}_1F_0\left(\left.\begin{array}{c} -\alpha \\ - \end{array}\right| -x\right) ,$$

$$\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{1}{k} x^k = x \; {}_2F_1\left(\left.\begin{array}{c} 1, 1 \\ 2 \end{array}\right| x\right) ,$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x \; {}_0F_1\left(\left.\begin{array}{c} - \\ 3/2 \end{array}\right| -\frac{x^2}{4}\right) ,$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = {}_0F_1\left(\left.\begin{array}{c} - \\ 1/2 \end{array}\right| -\frac{x^2}{4}\right) ,$$

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x \; {}_2F_1\left(\left.\begin{array}{c} 1/2, 1 \\ 3/2 \end{array}\right| -x^2\right) ,$$

$$\arcsin x = x \; {}_2F_1\left(\left.\begin{array}{c} 1/2, 1/2 \\ 3/2 \end{array}\right| -x^2\right) .$$

As examples from the world of orthogonal polynomials we consider the Legendre polynomials which can be represented as any of

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \binom{-n-1}{k} \left(\frac{1-x}{2}\right)^k = {}_2F_1\left(\left.\begin{array}{c} -n, n+1 \\ 1 \end{array}\right| \frac{1-x}{2}\right) \\ &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k} (x+1)^k = \left(\frac{1-x}{2}\right)^n {}_2F_1\left(\left.\begin{array}{c} -n, -n \\ 1 \end{array}\right| \frac{1+x}{1-x}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} \\
&= \binom{2n}{n} \left(\frac{x}{2}\right)^n {}_2F_1\left(\left. \begin{array}{c} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2} \\ -n+1/2 \end{array} \right| \frac{1}{x^2} \right) \\
&= x^n \, {}_2F_1\left(\left. \begin{array}{c} -n/2, -n/2+1/2 \\ 1 \end{array} \right| 1-\frac{1}{x^2} \right) .
\end{aligned}
$$

Note that whenever one of the upper parameters of a hypergeometric function is a negative integer like $-n$ in the above cases, the hypergeometric series is finite. Again, we see that functions can come in quite different disguises.

Since orthogonal polynomials can be represented by hypergeometric functions we see furthermore that summation is an important issue.

Why does Maple give a simple *antidifference* for $a_k = (-1)^k \binom{n}{k}$

```
> sum((-1)^k*binomial(n,k),k);
```

$$-\frac{k\,(-1)^k \operatorname{binomial}(n,\,k)}{n}$$

and fails to give one for $a_k = \binom{n}{k}$?

```
> sum(binomial(n,k),k);
```

$$\sum_k \operatorname{binomial}(n,\,k)$$

The reason is that Gosper's algorithm for indefinite summation is used which is a discrete analogue of Risch's integration algorithm deciding whether or not a hypergeometric term a_k has a hypergeometric term antidifference s_k, i.e. $s_{k+1} - s_k = a_k$. Gosper's algorithm tells that the first expression has a hypergeometric term antidifference and the second has not. That's the whole story. Maple's `sumtools` package contains an implementation of Gosper's algorithm via the `gosper` procedure which is also directly accessible through the `sum` command [17].

If we consider the series

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k \in \mathbb{Z}} \binom{n}{k}$$

with *natural bounds* then Maple is successful, again:

```
> sum(binomial(n,k),k=0..n);
```

$$2^n$$

In such a case, Zeilberger's algorithm for definite summation can be used. If $F(n,k)$ is a hypergeometric term with respect to both n and k, i. e.

$$\frac{F(n+1,k)}{F(n,k)} \quad \text{and} \quad \frac{F(n,k+1)}{F(n,k)} \in \mathbb{Q}(n,k) \ ,$$

then Zeilberger's algorithm generates a *holonomic recurrence equation*, i.e. a homogeneous linear recurrence equation with polynomial coefficients, for

$$s_n := \sum_{k\in\mathbb{Z}} F(n,k) \ .$$

In the particular case that the recurrence equation is of first order, then given one initial value it defines a unique hypergeometric term.

Maple's `sumtools` package contains an implementation of Zeilberger's algorithm via the `sumrecursion` procedure which is also directly accessible through the `sum` command [17]. A newer implementation is available through the `hsum` package, developed in [18].

As an example, `sumrecursion` generates the same recurrence equation for two different hypergeometric representations of the Legendre polynomials.[4]

```
> read `hsum.mpl`;
```

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```
> sumrecursion(
> binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k,k,P(n));
```

$$(n+2)\,\mathrm{P}(n+2) - x\,(2\,n+3)\,\mathrm{P}(n+1) + (n+1)\,\mathrm{P}(n) = 0$$

```
> Sumtohyper(binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k,k);
```

$$\mathrm{Hypergeom}([n+1,\, -n],\, [1],\, \frac{1}{2} - \frac{1}{2}\,x)$$

```
> sumrecursion(1/2^n*(-1)^k*
> binomial(n,k)*binomial(2*n-2*k,n)*x^(n-2*k),k,P(n));
```

$$(n+2)\,\mathrm{P}(n+2) - x\,(2\,n+3)\,\mathrm{P}(n+1) + (n+1)\,\mathrm{P}(n) = 0$$

```
> Sumtohyper(1/2^n*(-1)^k*
> binomial(n,k)*binomial(2*n-2*k,n)*x^(n-2*k),k);
```

$$2^{(-n)}\,\mathrm{binomial}(2\,n,\,n)\,x^n\,\mathrm{Hypergeom}([-\frac{1}{2}\,n,\, -\frac{1}{2}\,n + \frac{1}{2}],\, [-n + \frac{1}{2}],\, \frac{1}{x^2})$$

The procedure `Sumtohyper` converts the series in hypergeometric notation.

Modulo two initial values these computations show that the different series represent the same functions.

We give some other examples: The computation

```
> sumrecursion(hyperterm([a,1+a/2,b,c,d,1+2*a-b-c-d+n,-n],
> [a/2,1+a-b,1+a-c,1+a-d,b+c+d-a-n,1+a+n],1,k),k,S(n));
```

$$-(a-d+1+n)\,(n+1+a-c)\,(n+1+a-b)\,(-b-c-d+a+n+1)$$
$$\mathrm{S}(n+1)+(1+a+n)\,(n+1-d+a-c)\,(n+1+a-b-d)$$
$$(n+1-b-c+a)\,\mathrm{S}(n)=0$$

generates *Dougall's identity*

$$ {}_7F_6\left(\begin{array}{c} a, 1+\frac{a}{2}, b, c, d, 1+2a-b-c-d+n, -n \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, b+c+d-a-n, 1+a+n \end{array}\middle| 1\right)$$
$$= \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}$$

(from left to right); and the computation

```
> sumrecursion(hyperterm([a,b],[a+b+1/2],1,j)*
> hyperterm([a,b],[a+b+1/2],1,k-j),j,C(k));
```

$$-(k+1)\,(1+2\,a+2\,b+2\,k)\,(2\,a+2\,b+k)\,\mathrm{C}(k+1)$$
$$+\,2\,(2\,b+k)\,(k+2\,a)\,(a+b+k)\,\mathrm{C}(k)=0$$

generates *Clausen's formula*

$$ {}_2F_1\left(\begin{array}{c} a, b \\ a+b+1/2 \end{array}\middle| x\right)^2 = {}_3F_2\left(\begin{array}{c} 2a, 2b, a+b \\ a+b+1/2, 2a+2b \end{array}\middle| x\right)$$

(from left to right) by computing the coefficient of the left hand series written as Cauchy product.

In connection with the book project [18] we implemented Zeilberger type algorithms for other purposes: The procedure `Sumrecursion` gives three-term recurrence equations for orthogonal polynomials in a special form. When applied to the *Wilson polynomials*

$$W_n(x) = {}_4F_3\left(\begin{array}{c} -n, a+b+c+d+n-1, a-x, a+x \\ a+b, a+c, a+d \end{array}\middle| 1\right)$$

one gets

```
> Sumrecursion(hyperterm([-n,n+a+b+c+d-1,a+x,a-x],
> [a+b,a+c,a+d],1,k),k,W(n,x));
```

$$(a+x)\,(-a+x)\,\mathrm{W}(n,\,x)=(n+d+a)\,(a+n+c)\,(a+b+n)$$
$$(n+a+b+c+d-1)\,\mathrm{W}(n+1,\,x)/((a+b+c+d+2\,n)$$
$$(a+2\,n+c+d+b-1))-($$
$$\frac{(-1+d+c+n)\,(n-1+b+d)\,(b+n-1+c)\,n}{(a+2\,n+c+d+b-1)\,(-2+b+2\,n+c+a+d)}$$

$$+\frac{(n+d+a)\,(a+n+c)\,(a+b+n)\,(n+a+b+c+d-1)}{(a+b+c+d+2\,n)\,(a+2\,n+c+d+b-1)})\mathrm{W}(n,\,x)$$
$$+\frac{(-1+d+c+n)\,(n-1+b+d)\,(b+n-1+c)\,n\,\mathrm{W}(n-1,\,x)}{(a+2\,n+c+d+b-1)\,(-2+b+2\,n+c+a+d)}$$

The Wilson polynomials include all classical systems like the Jacobi and Hahn polynomials as limiting cases.

A version of Zeilberger's algorithm finds differential equations for sums: Given the representation

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{k}\left(\frac{1-x}{2}\right)^k ,$$

the computation

```
> sumdiffeq(binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k,k,P(x));
```

$$-(-1+x)\,(x+1)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{P}(x))+\mathrm{P}(x)\,n\,(n+1)-2\,x\,(\frac{\partial}{\partial x}\,\mathrm{P}(x))=0$$

generates the differential equation of the Legendre polynomials.

A version of Zeilberger's algorithm [1] finds recurrence or differential equations for definite integrals. The *Bateman Integral Representation*

$$\int_0^1 t^{c-1}\,(1-t)^{d-1}\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle|\, tx\right)dt=\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\,{}_2F_1\left(\begin{array}{c} a,b \\ c+d \end{array}\middle|\, x\right)$$

is deduced by

```
> intrecursion(t^(c-1)*(1-t)^(d-1)*
> hyperterm([a,b],[c],t*x,k),t,B(k));
```

$$-(k+1)\,(k+d+c)\,\mathrm{B}(k+1)+\mathrm{B}(k)\,x\,(b+k)\,(k+a)=0$$

The initial value is given by a Beta type integral.

The previous examples were mainly one-liners, but obviously not all questions are of this type. Hence let's consider a more difficult one: In Ramanujan's second notebook [26] on p. 258 one finds the identity

$${}_2F_1\left(\begin{array}{c} \frac{1}{3},\frac{2}{3} \\ 1 \end{array}\middle|\, 1-\left(\frac{1-x}{1+2x}\right)^3\right)=(1+2x)\,{}_2F_1\left(\begin{array}{c} \frac{1}{3},\frac{2}{3} \\ 1 \end{array}\middle|\, x^3\right).$$

With Garvan [9] we might ask the question whether there is an extension of this formula which has the form

$${}_2F_1\left(\begin{array}{c} A,B \\ C \end{array}\middle|\, 1-\left(\frac{1-x}{1+2x}\right)^3\right)=(1+2x)^d\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle|\, x^3\right) \qquad (1.1)$$

for some A, B, C, a, b, c, d. The following computation with Maple gives

$$ {}_2F_1\left(\begin{array}{c} \frac{d}{3}, \frac{1+d}{3} \\ (1+d)/2 \end{array} \middle| 1-\left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)^d \, {}_2F_1\left(\begin{array}{c} \frac{d}{3}, \frac{1+d}{3} \\ (5+d)/6 \end{array} \middle| x^3\right), \quad (1.2) $$

and moreover shows that this is the only possible extension of the given form (compare [9]).

We define the left and right hand *summands*

```
> first:=hyperterm([A,B],[C],1-((1-x)/(1+2*x))^3,k):
> second:=(2*x+1)^d*hyperterm([a,b],[c],x^3,k):
```

and compute the differential equations for the left and right hand *sums*

```
> DE1:=sumdiffeq(first,k,S(x));
```

$$ DE1 := x(-1+x)(x+x^2+1)(1+2x)^2 \left(\frac{\partial^2}{\partial x^2}\mathrm{S}(x)\right) + (1+2x)(4x^4+9x^3A - 8x^3C + 3x^3 + 9x^3B + 9x^2A + 3x^2 + 9x^2B - 12x^2C + 9xA + 9xB - x - 6xC - C)\left(\frac{\partial}{\partial x}\mathrm{S}(x)\right) + 9(-1+x)^2 AB\,\mathrm{S}(x) = 0 $$

```
> DE2:=sumdiffeq(second,k,S(x));
```

$$ DE2 := x(-1+x)(x+x^2+1)(1+2x)^2 \left(\frac{\partial^2}{\partial x^2}\mathrm{S}(x)\right) + (1+2x)(2x^4+6x^4b - 4dx^4 + 6x^4a + 3x^3b + 3x^3a + x^3 + 4dx - 6cx + 4x - 3c + 2) \left(\frac{\partial}{\partial x}\mathrm{S}(x)\right) + (4d^2x^4 - 12x^4bd - 12x^4ad + 36x^4ba - 6x^3ad - 2x^3d - 6x^3bd + 36x^3ba + 9x^2ba + 12cxd - 12dx - 4d^2x - 4d + 6cd)\,\mathrm{S}(x) = 0 $$

Both differential equation must be equivalent. After elimination of the highest derivative order

```
> DE:=collect(collect(op(1,DE1)-op(1,DE2),S(x)),diff(S(x),x));
```

$$ DE := ((1+2x)(4x^4+9x^3A - 8x^3C + 3x^3 + 9x^3B + 9x^2A + 3x^2 + 9x^2B - 12x^2C + 9xA + 9xB - x - 6xC - C) - (1+2x)(2x^4 + 6x^4b - 4dx^4 + 6x^4a + 3x^3b + 3x^3a + x^3 + 4dx - 6cx + 4x - 3c + 2))\left(\frac{\partial}{\partial x}\mathrm{S}(x)\right) + (9(-1+x)^2AB - 4d^2x^4 + 12x^4bd + 12x^4ad - 36x^4ba + 6x^3ad + 2x^3d + 6x^3bd - 36x^3ba - 9x^2ba - 12cxd + 12dx + 4d^2x + 4d - 6cd)\mathrm{S}(x) $$

the coefficient polynomials of $S(x)$ and $S'(x)$ must be identical to zero which leads to the system of equations

```
> firstcoeff:=collect(frontend(coeff,[DE,S(x)]),x):
> secondcoeff:=collect(frontend(coeff,[DE,diff(S(x),x)]),x):
> LIST:={coeffs(firstcoeff,x)} union {coeffs(secondcoeff,x)};
```

$$
\begin{aligned}
LIST := \{&12\,b\,d + 12\,a\,d - 36\,b\,a - 4\,d^2,\ 2\,d + 6\,b\,d - 36\,b\,a + 6\,a\,d,\\
&-18\,A\,B - 12\,c\,d + 12\,d + 4\,d^2,\ -9\,b\,a + 9\,A\,B,\ 9\,A\,B + 4\,d - 6\,c\,d,\\
&4 - 12\,b + 8\,d - 12\,a,\ -3\,b - 3\,a + 8 + 27\,A - 32\,C + 27\,B,\\
&-8\,d + 12\,c - 7 + 27\,A + 27\,B - 24\,C,\ 9\,A + 9\,B - 9 - 8\,C - 4\,d + 12\,c,\\
&6 + 18\,A - 16\,C + 18\,B + 4\,d - 12\,b - 12\,a,\ -C + 3\,c - 2\}
\end{aligned}
$$

All the members of `LIST` must equal zero which constitutes a nonlinear polynomial system for the unknowns A, B, C, a, b, c, d.

Remember that Maple is capable to find the complete set of solutions of such a system leading to

```
> solve(LIST,{A,B,C,a,b,c,d});
```

$$\{d = d,\ c = \frac{1}{6}\,d + \frac{5}{6},\ B = \frac{1}{3}\,d,\ b = \frac{1}{3}\,d,\ a = \frac{1}{3} + \frac{1}{3}\,d,\ C = \frac{1}{2} + \frac{1}{2}\,d,\ A = \frac{1}{3} + \frac{1}{3}\,d\},$$

$$\{b = \frac{1}{3} + \frac{1}{3}\,d,\ d = d,\ c = \frac{1}{6}\,d + \frac{5}{6},\ B = \frac{1}{3}\,d,\ a = \frac{1}{3}\,d,\ C = \frac{1}{2} + \frac{1}{2}\,d,\ A = \frac{1}{3} + \frac{1}{3}\,d\},$$

$$\{d = d,\ c = \frac{1}{6}\,d + \frac{5}{6},\ B = \frac{1}{3} + \frac{1}{3}\,d,\ b = \frac{1}{3}\,d,\ a = \frac{1}{3} + \frac{1}{3}\,d,\ C = \frac{1}{2} + \frac{1}{2}\,d,\ A = \frac{1}{3}\,d\},$$

$$\{b = \frac{1}{3} + \frac{1}{3}\,d,\ d = d,\ c = \frac{1}{6}\,d + \frac{5}{6},\ B = \frac{1}{3} + \frac{1}{3}\,d,\ a = \frac{1}{3}\,d,\ C = \frac{1}{2} + \frac{1}{2}\,d,\ A = \frac{1}{3}\,d\}.$$

Since in the hypergeometric sums in (1.1) A and B as well as a and b can be interchanged, all four different solutions correspond to (1.2).

Finally, we consider q-hypergeometric sums, also called *basic hypergeometric series*. There is a summation theory for q-hypergeometric terms a_k for which a_{k+1}/a_k is rational w.r.t. q^k, and for many of the results and algorithms corresponding q-versions exist, see e.g. [18].

By a q-analogue of Zeilberger's algorithm [20], we get for the *q-Laguerre polynomials*

$$L_n^{(\alpha)}(x;q) = \frac{\left(q^{\alpha+1};\,q\right)_n}{\left(q;\,q\right)_n}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array}\middle|\, q, -xq^{n+\alpha+1}\right)$$

a three-term recurrence equation which is computed by an implementation of Harald Böing [5], see [18].

```
> read `qsum.mpl`;
```

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```
> qsumrecursion(qpochhammer(q^(alpha+1),q,n)/
> qpochhammer(q,q,n)*qphihyperterm([q^(-n)],
> [q^(alpha+1)],q,-x*q^(n+alpha+1),k),q,k,L(n));
```

$$q\,(-1+q^n)\,\mathrm{L}(n) - (-q^2 + q^{(2\,n+\alpha)}\,x - q + q^{(n+1)} + q^{(n+\alpha+1)})\,\mathrm{L}(n-1) \\ - (q - q^{(\alpha+n)})\,q\,\mathrm{L}(n-2) = 0$$

ORTHOGONAL POLYNOMIALS

Assume that a scalar product

$$\langle f, g\rangle := \int_a^b f(x)\,g(x)\,d\mu(x)$$

is given with nonnegative measure μ supported in the interval (a, b). Particular cases are:

- absolutely continuous measure $d\mu(x) = \rho(x)\,dx$ with weight function $\rho(x)$,
- discrete measure $\rho(x_k)$ supported by $\mathbb{Z}$.

A family $p_n(x)$ of polynomials

$$p_n(x) = k_n\,x^n + k_n'\,x^{n-1} + \ldots\,, \qquad k_n \neq 0 \tag{1.3}$$

is called *orthogonal* w.r.t. the measure $\mu(x)$ if

$$\langle p_n, p_m\rangle \quad = \quad \begin{cases} 0 & \text{if } m \neq n \\ d_n^2 \neq 0 & \text{if } m = n \end{cases} \quad .$$

The *classical orthogonal polynomials* can be alternatively defined as the polynomial solutions (1.3) of the *differential equation*

$$\sigma(x)\,y''(x) + \tau(x)\,y'(x) + \lambda_n\,y(x) = 0\,. \tag{1.4}$$

Substituting (1.3) in (1.4) one gets the conclusions:

- $n = 1 \quad \Rightarrow \quad \tau(x) = d\,x + e$,
- $n = 2 \quad \Rightarrow \quad \sigma(x) = a\,x^2 + b\,x + c$,
- equating coefficient of $x^n \quad \Rightarrow \quad \lambda_n = -n\,(a\,(n-1) + d)$.

From this one deduces [4] that the classical orthogonal polynomials can be classified modulo linear transformations according to the scheme

1. $a = b = c = e = 0,\ d = 1 \implies p_n(x) = x^n$,
2. $a = b = e = 0,\ c = 1,\ d = -2 \implies p_n(x) = H_n(x)$, the *Hermite polynomials*,
3. $a = c = 0,\ b = 1,\ d = -1,\ e = \alpha + 1 \implies p_n(x) = L_n^{(\alpha)}(x)$, the *Laguerre polynomials*,

4a. $a = 1,\ b = c = d = e = 0 \implies p_n(x) = x^n$,

4b. $a = 1,\ b = c = 0,\ d = \alpha + 2,\ e = 2 \implies p_n(x) = B_n^{(\alpha)}(x)$, the *Bessel polynomials*,

5. $a = 1, b = 0, c = -1, d = \alpha + \beta + 2, e = \alpha - \beta \Rightarrow p_n(x) = P_n^{(\alpha,\beta)}(x)$, the *Jacobi polynomials*.

Table 1.1 Normal Forms of Classical Polynomials

The weight function $\rho(x)$ corresponding to the differential equation satisfies *Pearson's differential equation*

$$\frac{d}{dx}\Big(\sigma(x)\,\rho(x)\Big) = \tau(x)\,\rho(x)\ .$$

Hence it is given as[5]

$$\rho(x) := \frac{C}{\sigma(x)}\, e^{\int \frac{\tau(x)}{\sigma(x)}\,dx}\ .$$

The multiplication with $\rho(x)$ makes the differential equation *self-adjoint*

$$\frac{d}{dx}\Big(\sigma(x)\,\rho(x)\,y'(x)\Big) + \lambda_n\,\rho(x)\,y(x) = 0\ .$$

The *classical discrete orthogonal polynomials* can be analogously defined as the polynomial solutions (1.3) of the *difference equation*

$$\sigma(x)\,\Delta\nabla y(x) + \tau(x)\,\Delta y(x) + \lambda_n\,y(x) = 0 \qquad (1.5)$$

where

$$\Delta y(x) = y(x+1) - y(x)\ , \quad \nabla y(x) = y(x) - y(x-1).$$

Again it turns out that $\tau(x) = d\,x + e$, $\sigma(x) = a\,x^2 + b\,x + c$, and $\lambda_n = -n\,(a\,(n-1) + d)$, and the classical discrete orthogonal polynomials can be classified modulo linear transformations according to the scheme

1. $\sigma(x) = 1,\ \tau(x) = \alpha\, x + \beta \quad\Longrightarrow\quad p_n(x) = K_n^{(\alpha,\beta)}(x)$,

2a. $\sigma(x) = x,\ \sigma(x) + \tau(x) = 0 \quad\Longrightarrow\quad p_n(x) = x^{\underline{n}} := x(x-1)\cdots(x-n+1)$,

2b. $\sigma(x) = x,\ \sigma(x) + \tau(x) = \mu\ (\mu \neq 0) \quad\Longrightarrow\quad p_n(x) = c_n^{(\mu)}(x)$, the *Charlier polynomials*,

3 $\sigma(x) = x,\ \sigma(x) + \tau(x) = \mu\,(\gamma + x) \quad\Longrightarrow\quad p_n(x) = m_n^{(\gamma,\mu)}(x)$, the *Meixner polynomials*,

4 $\sigma(x) = x,\ \sigma(x)+\tau(x) = \frac{p}{1-p}\,(N-x) \ \Rightarrow\ p_n(x) = k_n^{(p)}(x,N)$, the *Krawtchouk polynomials*,

5 $\sigma(x) = x(N+\alpha-x),\ \sigma(x)+\tau(x) = (x+\beta+1)(N-1-x) \Rightarrow p_n(x) = h_n^{(\alpha,\beta)}(x,N)$, the *Hahn polynomials*,

6 $\sigma(x) = x(x+\mu),\ \sigma(x)+\tau(x) = (\nu+N-1-x)(N-1-x) \Rightarrow p_n(x) = \tilde{h}_n^{(\mu,\nu)}(x,N)$, the *Hahn-Eberlein polynomials)*.

Table 1.2 Normal Forms of Classical Discrete Polynomials

Here $K_n^{(\alpha,\beta)}(x)$ are connected with the Charlier polynomials by

$$K_n^{(\alpha,\beta)}(x) = (-1)^n\, c_n^{(-1/\alpha)}\left(-x - \frac{1+\beta}{\alpha}\right)\ .$$

The discrete measure $\rho(x)$ corresponding to the difference equation satisfies the Pearson type difference equation

$$\Delta\Big(\sigma(x)\,\rho(x)\Big) = \tau(x)\,\rho(x)\ .$$

Hence it is a hypergeometric term, given by the term ratio

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x) + \tau(x)}{\sigma(x+1)}\ .$$

The multiplication with $\rho(x)$ makes the difference equation self-adjoint

$$\Delta\Big(\sigma(x)\,\rho(x)\,\nabla y(x)\Big) + \lambda_n\, y(x) = 0\ .$$

Orthogonal polynomials have some structural properties. Most importantly, they satisfy a *three-term recurrence equation* of the special form

$$p_{n+1}(x) = (A_n\, x + B_n)\, p_n(x) - C_n\, p_{n-1}(x)\ . \tag{1.6}$$

Favard's Theorem (see e.g. [7], Theorem 4.4) states that if on the other hand $p_n(x)$ satisfies (1.6) and if $C_n/A_n > 0$ for all $n \geq 0$, then $p_n(x)$ forms a family of orthogonal polynomials.

CAOP: Computer Algebra and Orthogonal Polynomials

CAOP is a package for calculating formulas for orthogonal polynomials belonging to the Askey-scheme by Maple. With the present version users can compute recurrence relations, differential and difference equations or make a plot of every polynomial in the Askey scheme, without having Maple installed on their own computer. It is also possible to multiply the polynomial by a scaling function, to change the argument and to give values to the parameters by filling out a form, before doing the calculation. As an extra option the user can choose the layout of the output: prettyprint, lineprint or LaTeX. The latter two options make it possible to insert the output in another Maple worksheet respectively in a LaTeX document by a simple mouse-action.

Furthermore there are some help pages available for users who are not familiar with Maple. They can be viewed simultaniously while filling in the form.

The algorithms used to calculate the various formulas are developed by Wolfram Koepf. The code is not restricted to Maple Version V.3.

If you want to use CAOP choose one of the following options:

- Calculate a recurrence relation
- Calculate a differential/ce equation
- Make a plot

To Home Page of the Askey-Wilson-scheme project

Figure 1.1 The CAOP Homepage

At this point I would like to point you to the `CAOP` Web site at the URL `http://www.can.nl/~demo/CAOP/CAOP.html` which was developed by René Swarttouw [31] and with which the on-line computation of differential/difference and recurrence equations for the orthogonal families of the Askey-Wilson scheme ([3], see also [15]) can be carried out. Figure 1.1 shows the CAOP home page.

If we then click on `Calculate a recurrence relation`, Figure 1.2 appears and gives us the option to choose a family of the Askey-Wilson scheme.

Let's choose the Laguerre polynomials. This opens Figure 1.3. This page gives the definition of the Laguerre polynomials in terms of a hypergeometric function. Next, the user has the possibility to multiply by an arbitrary scale factor (depending on n), and then invisibly for the user our Maple package will compute the recurrence equation valid for this particularly standardized polynomial system. Observe that this is more than a mathematical dictionary can offer.

Recurrence relations with CAOP

One of the features of the CAOP package is the calculation of a three-term recurrence relation of certain orthogonal polynomials belonging to the Askey-scheme. The package uses Maple 5.3 and several algorithms written by W. Koepf. Please choose one of the listed orthogonal polynomials.

- Wilson polynomials
- Racah polynomials
- Continuous Dual Hahn polynomials
- Continuous Hahn polynomials
- Hahn polynomials
- Dual Hahn polynomials
- Meixner-Pollaczek polynomials
- Jacobi polynomials
- Meixner polynomials
- Krawtchouk polynomials
- Laguerre polynomials
- Charlier polynomials
- Hermite polynomials

Go to:
CAOP Home page

Figure 1.2 CAOP: Computation of Recurrence Equations

As an example, we choose the scale factor `1/binomial(a,n)`, and CAOP answers

$$(a-n)\,(-1+a-n)\,p(n+2)+(a-n)\,(-a+x-3-2\,n)\,p(n+1)$$
$$+\,p(n)\,(n+1)\,(a+1+n)=0$$

where

$$p(n)=\frac{L_n(x,a)}{\binom{a}{n}}\ .$$

Recurrence relation for Laguerre polynomials

You have chosen Laguerre polynomials which are defined by

$$\mathbf{L_n^{(a)}(x)} = \frac{\mathbf{(a+1)_n}}{\mathbf{n!}} {}_1\mathbf{F}_1\left(\left.\begin{matrix}\mathbf{-n}\\ \mathbf{a+1}\end{matrix}\right| \mathbf{x}\right)$$

Before you let Maple do the calculation, you can multiply by a scale factor, change the argument and give a value for the parameter a. Helppages on Maple-style and Maple functions are available.

Type a scale factor (in the variable n), using Maple-style input: 1

Type an argument (in the variable x), using Maple-style input: x

Type a value for a (a > -1): a

Select the output format Pretty Print
If you want to submit your input to the computer please press Submit

Figure 1.3 CAOP: Laguerre Polynomials

Similarly CAOP can compute differential and difference equations for orthogonal polynomials multiplied by a factor (depending on x).

The classical families have many more interesting properties: They satisfy a *derivative/difference rule*

$$\sigma(x)\, p_n'(x) = \alpha_n\, p_{n+1}(x) + \beta_n\, p_n(x) + \gamma_n\, p_{n-1}(x) \ ,$$

or

$$\sigma(x)\, \nabla p_n(x) = \alpha_n\, p_{n+1}(x) + \beta_n\, p_n(x) + \gamma_n\, p_{n-1}(x) \ ,$$

respectively, their derivatives $q_n := p_{n+1}'$ (and differences $q_n := \Delta p_{n+1}$) are of the same type, again,

$$\sigma'(x)\, q_n''(x)(x) + \tau'(x)\, q_n'(x) + \lambda_n'\, q_n(x) = 0 \ ,$$

$$\sigma'(x)\, \Delta\nabla q_n(x) + \tau'(x)\, \Delta q_n(x) + \lambda_n'\, q_n(x) = 0 \ ,$$

and therefore satisfy a three-term recurrence equation of the type

$$x\, p_n'(x) = \alpha_n^*\, p_{n+1}'(x) + \beta_n^*\, p_n'(x) + \gamma_n^*\, p_{n-1}'(x) \ ,$$

$$x \Delta p_n(x) = \alpha_n^*\, \Delta p_{n+1}(x) + \beta_n^*\, \Delta p_n(x) + \gamma_n^*\, \Delta p_{n-1}(x) \ .$$

A combination of these identities finally leads to the *connection identities*

$$p_n(x) = \widehat{a}_n\, p'_{n+1}(x) + \widehat{b}_n\, p'_n(x) + \widehat{c}_n\, p'_{n-1}(x)$$

or

$$p_n(x) = \widehat{a}_n\, \Delta p_{n+1}(x) + \widehat{b}_n\, \Delta p_n(x) + \widehat{c}_n\, \Delta p_{n-1}(x)\ ,$$

respectively.

Note that in the Jacobi and Hahn cases, the explicit connection identities read as follows:

$$\int P_n^{(\alpha,\beta)}(x)\, dx = \frac{2\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)\,(2n\,!+\alpha+\beta+2)}\, P_{n+1}^{(\alpha,\beta)}(x)$$
$$+\frac{2\,(\alpha-\beta)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)}\, P_n^{(\alpha,\beta)}(x)$$
$$-\frac{2\,(n+\alpha)\,(n+\beta)}{(n+\alpha+\beta)\,(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)}\, P_{n-1}^{(\alpha,\beta)}(x)$$

and

$$\sum_x h_n^{(\alpha,\beta)}(x,N) = \frac{n+\alpha+\beta+1}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta+2)}\, h_{n+1}^{(\alpha,\beta)}(x,N)$$
$$-\frac{2n^2+2n+2n\alpha+2n\beta+\alpha-\alpha N+\beta N+\alpha\beta+\beta+\beta^2}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)}\, h_n^{(\alpha,\beta)}(x,N)$$
$$+\frac{(n+\alpha)\,(n+\beta)\,(n-N)\,(n+\alpha+\beta+N)}{(n+\alpha+\beta)\,(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)}\, h_{n-1}^{(\alpha,\beta)}(x,N)\ .$$

and can be interpreted as a definite integral or definite sum, respectively.

Let's assume a family of polynomials (1.3) satisfies the differential equation (1.4) or difference equation (1.5), and we would like to know the recurrence equation (1.6) in terms of the coefficients a, b, c, d, e of σ and τ.

Using computer algebra (or by hand computations) the following method gives the coefficients A_n, B_n and C_n of the desired relation in terms of a, b, c, d, e, n, k_{n-1}, k_n, and k_{n+1} by linear algebra:

1. Substitute
$$p_n(x) = k_n\, x^n + k'_n\, x^{n-1} + k''_n\, x^{n-2} + \ldots$$
in the differential/difference equation.

2. Equate the coefficients of x^n to determine λ_n. As already mentioned this gives $\lambda_n = -n\,(a\,(n-1)+d)$.

3. Equate the coefficients of x^{n-1} and x^{n-2}. This gives k_n', and k_n'', respectively, in terms of k_n.

4. Substitute $p_n(x)$ in the proposed equation, and equate again the three highest coefficients. This computes the three unknowns A_n, B_n and C_n successively.

As an example, we give the recurrence equation coefficients in the continuous case (see [22], [19]):

$$A_n = \frac{k_{n+1}}{k_n} \, ,$$

$$B_n = \frac{2\,b\,n\,(\,a\,n + d - a\,) - e\,(\,-d + 2\,a\,)}{(\,d + 2\,a\,n\,)\,(\,d - 2\,a + 2\,a\,n\,)} \cdot \frac{k_{n+1}}{k_n}$$

and

$$C_n = \frac{-(an+d-2a)n}{(d-2a+2an)^2\,(2an-3a+d)\,(2an-a+d)} \cdot \frac{k_{n+1}}{k_{n-1}} \cdot$$

$$\Big((an+d-2a)n(4ca-b^2)+4a^2c-ab^2+ae^2-4acd+db^2-bed+d^2c\Big).$$

Similar representations hold in the discrete case, as well as for the other coefficients $\alpha_n, \beta_n, \gamma_n, \alpha_n^*, \beta_n^*, \gamma_n^*, \widehat{a}_n, \widehat{b}_n$ and $\widehat{c}_n$ all of which can be easily determined by the use of computer algebra [19].

As soon as we have these explicit formulas, we can determine the classical orthogonal polynomial solutions of a given holonomic recurrence equation by the following algorithm:

1. Given A_n, B_n and C_n by

   $$p_{n+1}(x) = (A_n\,x + B_n)\,p_n(x) - C_n\,p_{n-1}(x) \qquad (A_n \neq 0, B_n, C_n \in \mathbb{Q}(n)) \, ,$$

   define

   $$\frac{k_{n+1}}{k_n} := A_n = \frac{v_n}{w_n} \qquad (v_n, w_n \in \mathbb{Q}[n])$$

2. Use the explicit formulas for B_n and C_n and this term ratio to receive two polynomial identities w.r.t. n, in terms of the unknowns a, b, c, d, e.

3. Equate the coefficients, and solve the corresponding polynomial system for a, b, c, d and e.

Note that, again, the crucial step 3. is to find the complete set of solutions of a nonlinear polynomial system for the unknowns a, b, c, d, e. Note, moreover, that this system might have several solutions. With Koornwinder and Swarttouw we consider the example recurrence equation

$$p_{n+2}(x) - (x-n-1)\,p_{n+1}(x) + \alpha\,(n+1)^2\,p_n(x) = 0\;. \tag{1.7}$$

Note that a parameter α is involved, and the solution might depend on its value.

The algorithm gives the unique solution

$$\left\{b = 2\,c, c = c, d = -4\,c, e = 0, a = 0, \alpha = \frac{1}{4}\right\}\;.$$

Hence $\alpha = 1/4$,

$$\left(x + \frac{1}{2}\right)\, p_n''(x) - 2\,x\,p_n'(x) - 2\,n\,p_n(x) = 0$$

and

$$\rho(x) = 2\,e^{-2x}$$

in the interval $[-1/2, \infty]$, corresponding to shifted Laguerre polynomials.

In Maple, this is given by

```
> read retode;
> RE:=p(n+2)-(x-n-1)*p(n+1)+alpha*(n+1)^2*p(n);
```

$$RE := \mathrm{p}(n+2) - (x-n-1)\,\mathrm{p}(n+1) + \alpha\,(n+1)^2\,\mathrm{p}(n)$$

```
> REtoDE(RE,p(n),x);
```

$$\mathit{Warning}:\ \mathit{parameters\ have\ the\ values},$$
$$\{b = 2\,c,\ \alpha = \frac{1}{4},\ d = -4\,c,\ c = c,\ a = 0,\ e = 0\}$$

$$[\frac{1}{2}\,(2\,x+1)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{p}(n,\,x)) - 2\,x\,(\frac{\partial}{\partial x}\,\mathrm{p}(n,\,x)) - 2\,n\,\mathrm{p}(n,\,x) = 0,$$
$$[[\frac{-1}{2},\ \infty],\ 2\,e^{(-2\,x)}],\ 1]$$

Obviously there is a corresponding algorithm for the discrete case. Let's check whether the given recurrence equation (1.7) has classical discrete orthogonal polynomial solutions.

Using the linear transformation $x \mapsto \frac{x-g}{f}$, the discrete version of the algorithm gives the rational solution

$$\left\{a = 0, b = b, c = -\frac{b\,(-e+d+b)}{d}, d = d, e = e,\right.$$

$$f=-\frac{d+2\,b}{d}, g=-\frac{e}{d}, \alpha=\frac{b\,(\,d+b\,)}{(\,d+2\,b\,)^2}\Bigg\}\,.$$

This yields

$$b=-\frac{d}{2}\left(1\pm\frac{1}{\sqrt{1-4\alpha}}\right)\,.$$

For $1-4\,\alpha>0$ this corresponds to Meixner or Krawtchouk polynomials.

Finally, we consider the computation of *connection coefficients*. Let $P_n(x)=k_n\,x^n+\ldots$ $(n\in\mathbb{N}_0)$ denote a family of polynomials of degree exactly n and $Q_m(x)=\overline{k}_m\,x^m+\ldots$ $(m\in\mathbb{N}_0)$ denote a family of polynomials of degree exactly m. Then

$$P_n(x)=\sum_{m=0}^{n} C_m(n)\,Q_m(x)\,.$$

The coefficients $C_m(n)$ $(n\in\mathbb{N}_0,\ m=0,\ldots,n)$ are called the connection coefficients between the systems $P_n(x)$ and $Q_m(x)$. Interesting subproblems are given if

- $Q_m(x)=x^m$ (power series representation)
- $P_n(x)=x^n$ (representation of powers)

Note that if both subproblems have *hypergeometric term* solutions, then an application of Zeilberger's algorithm yields $C_m(n)$: Combining

$$P_n(x)=\sum_{j\in\mathbb{Z}} A_j(n)\,x^j$$

and

$$x^j=\sum_{m\in\mathbb{Z}} B_m(j)\,Q_m(x)$$

yields

$$P_n(x)=\sum_{m=0}^{n} C_m(n)\,Q_m(x)$$

with

$$C_m(n)=\sum_{j\in\mathbb{Z}} A_j(n)\,B_m(j)\,.$$

The corresponding subproblems in the discrete case are

- $Q_m(x)=x^{\underline{m}}$ (series representation)
- $P_n(x)=x^{\underline{n}}$ (representation of falling factorials)

Again, if both have hypergeometric term solutions, then Zeilberger's algorithm yields $C_m(n)$.

How can we determine $C_m(n)$? By rewriting the recurrence equations

$$x\,P_n(x) = a_n\,P_{n+1}(x) + b_n\,P_n(x) + c_n\,P_{n-1}(x)$$

and

$$x\,Q_m(x) = \overline{a}_m\,Q_{m+1}(x) + \overline{b}_m\,Q_m(x) + \overline{c}_m\,Q_{m-1}(x)\ ,$$

and equating coefficients one gets the first "cross rule" [30]

$$a_n\,C_m(n+1) + b_n\,C_m(n) + c_n\,C_m(n-1) =$$

$$\overline{a}_{m-1}\,C_{m-1}(n) + \overline{b}_m\,C_m(n) + \overline{c}_{m+1}\,C_{m+1}(n)\ .$$

Using both recurrence equations for the derivatives

$$x\,P_n'(x) = \alpha_n^*\,P_{n+1}'(x) + \beta_n^*\,P_n'(x) + \gamma_n^*\,P_{n-1}'(x)$$

and

$$x\,Q_m'(x) = \overline{\alpha}_m^*\,Q_{m+1}'(x) + \overline{\beta}_m^*\,Q_m'(x) + \overline{\gamma}_m^*\,Q_{m-1}'(x)\ ,$$

(or the analogous ones in the discrete case), results in the second cross rule [19]

$$\alpha_n^*\,C_m(n+1) + \beta_n^*\,C_m(n) + \gamma_n^*\,C_m(n-1) =$$

$$\overline{\alpha}_{m-1}^*\,C_{m-1}(n) + \overline{\beta}_m^*\,C_m(n) + \overline{\gamma}_{m+1}^*\,C_{m+1}(n)\ .$$

A third cross rule derived from the connection identity turns out to be linearly dependent.

Now we assume $\overline{\sigma}(x) = \sigma(x)$. Then, using both derivative rules

$$\sigma(x)\,P_n'(x) = \alpha_n\,P_{n+1}(x) + \beta_n\,P_n(x) + \gamma_n\,P_{n-1}(x)$$

and

$$\overline{\sigma}(x)\,Q_m'(x) = \overline{\alpha}_m\,Q_{m+1}(x) + \overline{\beta}_m\,Q_m(x) + \overline{\gamma}_m\,Q_{m-1}(x)\ ,$$

(or the analogous ones in the discrete case), leads to the third cross rule [19]

$$\alpha_n\,C_m(n+1) + \beta_n\,C_m(n) + \gamma_n\,C_m(n-1) =$$

$$\overline{\alpha}_{m-1}\,C_{m-1}(n) + \overline{\beta}_m\,C_m(n) + \overline{\gamma}_{m+1}\,C_{m+1}(n)\ .$$

With the use of computer algebra we can eliminate two of the five variables $C_m(n+1)$, $C_m(n)$, $C_m(n-1)$, $C_{m-1}(n)$ and $C_{m+1}(n)$. This gives three-term recurrence equations for $C_m(n)$ w.r.t. m and w.r.t. n which define $C_m(n)$ uniquely.

$P_n(x)$	$\to$	$Q_m(x)$
$P_n^{(\alpha,\beta)}(x)$	$\to$	$P_m^{(\gamma,\beta)}(x)$
$P_n^{(\alpha,\beta)}(x)$	$\to$	$P_m^{(\alpha,\delta)}(x)$
$P_n^{(\alpha,\alpha)}(x)$	$\to$	$P_n^{(\beta,\beta)}(x)$
$L_n^{(\alpha)}(x)$	$\to$	$L_m^{(\beta)}(x)$
$B_n^{(\alpha)}(x)$	$\to$	$B_m^{(\beta)}(x)$

Table 1.3 Hypergeometric Term Connection Coefficients: Continuous Case

$P_n(x)$	$\to$	$Q_m(x)$
$h_n^{(\alpha,\beta)}(x,N)$	$\to$	$h_m^{(\alpha,\delta)}(x,N)$
$h_n^{(\alpha,\beta)}(x,N)$	$\to$	$h_m^{(\gamma,\beta)}(x,N)$
$h_n^{(\alpha,\alpha)}(x,N)$	$\to$	$h_m^{(\beta,\beta)}(x,N)$
$m_n^{(\gamma,\mu)}(x)$	$\to$	$m_m^{(\delta,\mu)}(x)$
$m_n^{(\gamma,\mu)}(x)$	$\to$	$m_m^{(\gamma,\nu)}(x)$
$k_n^{(p)}(x,N)$	$\to$	$k_m^{(q)}(x,N)$
$k_n^{(p)}(x,N)$	$\to$	$k_m^{(p)}(x,M)$
$c_n^{(\mu)}(x)$	$\to$	$c_m^{(\nu)}(x)$
$K_n^{(\alpha,\beta)}(x)$	$\to$	$K_m^{(\alpha,\delta)}(x)$

Table 1.4 Hypergeometric Term Connection Coefficients: Discrete Case

In many instances the recurrence equations reduce to two terms. Then their hypergeometric term solutions are easily identified. In Tables 1.3–1.4 it is shown between which polynomial systems such a hypergeometric connection relation is valid.

Note that Askey and Gasper [2] were the first who used recurrence equations to prove the positivity of the connection coefficients between certain instances of the Jacobi polynomials. In Gasper [10], almost all the results of Tables 1.3–1.4 were published, but the use of computer algebra unifies this development.

Using a similar approach, it turns out that *all* classical continuous and discrete polynomials have hypergeometric representations (where the Jacobi polynomials are developed at $x = \pm 1$) and hypergeometric type representations for the powers/falling factorials. Again, these are well-known, but the given approach unifies their treatment.

A similar technique can be used for polynomial solutions of higher order differential/difference equations [30]; *parameter derivatives* of the orthogonal polynomials can be computed using their connection formulas [19], e.g.

$$\frac{\partial}{\partial\alpha} L_n^{(\alpha)}(x) = \sum_{k=0}^{n-1} \frac{1}{n-k}\, L_k^{(\alpha)}(x)\ .$$

CONCLUSION

We gave some examples of the use of computer algebra in the work with orthogonal polynomials. I am convinced that the more people know about the algorithmic methods available in that field, and the more computer algebra systems are used in research and applications, the more these methods will be used. This lecture might help in this direction.

Note that the Maple packages `hsum`, `qsum` and `retode` can be obtained from the author.

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Notes

1. Therefore computer algebra systems are generally slower than numerically oriented systems.
2. `%` refers to the previous result. In older Maple versions you must use `"` instead.
3. This is the result of Maple V.4. Release V.5 gives the result in terms of the Whittaker functions.
4. The other hypergeometric representations can be handled similarly.
5. For the Bessel polynomials $\rho(x)$ is not a weight function since the integrals diverge. They can be interpreted as orthogonal polynomials on the unit circle though.

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