Journal of Computational and Applied Mathematics 136 (2001) 99–107

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

On a structure formula for classical q-orthogonal polynomials

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Received 17 September 1999; received in revised form 15 June 2000

Abstract

The classical orthogonal polynomials are given as the polynomial solutions $P_n(x)$ of the differential equation

$$\sigma(x)y''(x)+\tau(x)y'(x)+\lambda_n y(x)=0,$$

where $\sigma(x)$ turns out to be a polynomial of at most second degree and $\tau(x)$ is a polynomial of first degree. In a similar way, the classical discrete orthogonal polynomials are the polynomial solutions of the difference equation

$$\sigma(x)\Delta\nabla y(x)+\tau(x)\Delta y(x)+\lambda_n y(x)=0,$$

where $\Delta y(x)=y(x+1)-y(x)$ and $\nabla y(x)=y(x)-y(x-1)$ denote the forward and backward difference operators, respectively. Finally, the classical q-orthogonal polynomials of the Hahn tableau are the polynomial solutions of the q-difference equation

$$\sigma(x)D_q D_{1/q} y(x)+\tau(x)D_q y(x)+\lambda_{q,n} y(x)=0,$$

where

$$D_q f(x)=\frac{f(qx)-f(x)}{(q-1)x},\quad q\neq 1,$$

denotes the q-difference operator. We show by a purely algebraic deduction — without using the orthogonality of the families considered — that a structure formula of the type

$$\sigma(x)D_{1/q}P_n(x)=\alpha_n P_{n+1}(x)+\beta_n P_n(x)+\gamma_n P_{n-1}(x)\quad (n\in\mathbb{N}:=\{1,2,3,\ldots\})$$

is valid. Moreover, our approach does not only prove this assertion, but *generates* the form of this structure formula. A similar argument applies to the discrete and continuous cases and yields

$$\sigma(x)\nabla P_n(x)=\alpha_n P_{n+1}(x)+\beta_n P_n(x)+\gamma_n P_{n-1}(x)\quad (n\in\mathbb{N})$$

and

$$\sigma(x)P_n'(x)=\alpha_n P_{n+1}(x)+\beta_n P_n(x)+\gamma_n P_{n-1}(x)\quad (n\in\mathbb{N}).$$

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PII: S0377-0427(00)00577-X

Whereas the latter formulas are well-known, their previous deduction used the orthogonality property. Hence our approach is also of interest in these cases.

MSC: 33D45; 33C45

Keywords: Structure formula; Classical q-orthogonal polynomials; q-Hahn tableau; q-Jacobi polynomials

1. Introduction

The classical q-orthogonal polynomials of the Hahn tableau [5,8] are the polynomial solutions

$$P_n(x) = k_n x^n + k'_n x^{n-1} + k''_n x^{n-2} + \cdots$$

of the q-difference equation

$$L_{q,n}\, y(x) := \sigma(x) D_q D_{1/q}\, y(x) + \tau(x) D_q\, y(x) + \lambda_{q,n}\, y(x) = 0, \tag{1}$$

where $\sigma(x) := ax^2 + bx + c$ and $\tau(x) := dx + e$ and

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1,$$

denotes the q-difference operator.

By equating the coefficients of x^n in (1) one gets

$$\lambda_{q,n} = -a[n]_{1/q}[n-1]_q - d[n]_q, \tag{2}$$

where the abbreviation

$$[n]_q = \frac{1-q^n}{1-q}$$

denotes the so-called q-brackets. Note that $\lim_{q\to 1} [n]_q = n$.

The q-difference operator obeys the identity

$$D_q D_{1/q} f(x) = \frac{D_q f(x) - D_{1/q} f(x)}{(q-1)x}, \tag{3}$$

so that the q-difference equation can be rewritten in the form

$$(\sigma(x) + (q-1)x\tau(x)) D_q\, y(x) - \sigma(x) D_{1/q}\, y(x) + (q-1)x\lambda_{q,n}\, y(x)$$
$$= (q-1)x L_{q,n}\, y(x) = 0.$$

Furthermore, the q-difference operator obeys the product rule

$$D_q(f(x)g(x)) = f(qx) D_q g(x) + g(x) D_q f(x). \tag{4}$$

Since $D_q\, x = 1$, we get moreover

$$D_q(xg(x)) = qx D_q g(x) + g(x) \tag{5}$$

and similarly

$$D_{1/q}(xg(x)) = \frac{1}{q} x D_{1/q} g(x) + g(x). \tag{6}$$

Furthermore, it is well-known for orthogonal polynomial systems and can be shown algebraically (compare [9,5]) that the solution families $y(x)=P_n(x)$ of (1) satisfy a three-term recurrence equation of the form

$$P_{n+1}(x)=(A_nx+B_n)P_n(x)-C_nP_{n-1}(x) \quad (A_n,B_n,C_n\in\mathbb{R} \text{ for } n\in\mathbb{N}). \tag{7}$$

For orthogonal polynomial systems one knows moreover that $C_nA_nA_{n-1}>0$ [3, p. 20].

2. Main identity

Let

$$L_n:=L_{q,n}=\sigma(x)D_qD_{1/q}+\tau(x)D_q+\lambda_{q,n}$$

denote the q-difference operator connected with (1).

We would like to find a representation for $L_{n+1}P_{n+1}(x)$ in terms of $P_n(x)$. Since, by (7), $P_{n+1}(x)$ is a linear combination of $P_n(x)$, $P_{n-1}(x)$ and $xP_n(x)$, it is essential to compute $L_{n+1}[xP_n(x)]$.

Therefore we compute, using (5)–(6),

$$\begin{aligned}D_qD_{1/q}(xP_n(x))&=D_q\left(\frac{1}{q}xD_{1/q}P_n(x)+P_n(x)\right)\\&=\frac{1}{q}D_q(xD_{1/q}P_n(x))+D_qP_n(x)\\&=xD_qD_{1/q}P_n(x)+\frac{1}{q}D_{1/q}P_n(x)+D_qP_n(x);\end{aligned}$$

hence,

$$\sigma(x)D_qD_{1/q}(xP_n(x))=x\sigma(x)D_qD_{1/q}P_n(x)+\sigma(x)\left(\frac{1}{q}D_{1/q}P_n(x)+D_qP_n(x)\right). \tag{8}$$

Similarly, one gets using (5)

$$\tau(x)D_q(xP_n(x))=qx\tau(x)D_qP_n(x)+\tau(x)P_n(x). \tag{9}$$

Next, we use the identity

$$\lambda_{q,n+1}xP_n(x)=x\lambda_{q,n}P_n(x)+(\lambda_{q,n+1}-\lambda_{q,n})xP_n(x). \tag{10}$$

Hence, adding (8)–(10), we get

$$\begin{aligned}L_{n+1}[xP_n(x)]&=xL_n(P_n(x))+\sigma(x)\left(\frac{1}{q}D_{1/q}P_n(x)+D_qP_n(x)\right)\\&\quad+(q-1)x\tau(x)D_qP_n(x)+\tau(x)P_n(x)+(\lambda_{q,n+1}-\lambda_{q,n})xP_n(x)\\&=qxL_n(P_n(x))+\left(1+\frac{1}{q}\right)\sigma(x)D_{1/q}P_n(x)+\tau(x)P_n(x)\\&\quad-(q-1)x\lambda_{q,n}P_n(x)+(\lambda_{q,n+1}-\lambda_{q,n})xP_n(x),\end{aligned}$$

where we used (4).

Finally, we apply L_{n+1} to recurrence equation (7) to get

Theorem 1. *Applying* L_{n+1} *to recurrence equation* (7) *yields the identity*

$$\begin{aligned} L_{n+1}[P_{n+1}(x)] = {} & (A_n qx + B_n)L_n[P_n(x)] - C_n L_{n-1}[P_{n-1}(x)] \\ & + A_n\left(1+\frac{1}{q}\right)\sigma(x)D_{1/q}P_n(x) + A_n\tau(x)P_n(x) \\ & - A_n(q-1)x\lambda_{q,n}P_n(x) + A_n(\lambda_{q,n+1}-\lambda_{q,n})xP_n(x) \\ & + B_n(\lambda_{q,n+1}-\lambda_{q,n})P_n(x) - C_n(\lambda_{q,n+1}-\lambda_{q,n-1})P_{n-1}(x). \end{aligned}$$

3. Structure formula

Now we are able to present the structure formula sought for, compare [10, Proposición 23], and [4, Theorem 2.9].

Theorem 2. *Assume* $P_n(x)$ *is a solution family of* (1). *Then a structure formula of the type*

$$\sigma(x)D_{1/q}P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \quad (\alpha_n, \beta_n, \gamma_n \in \mathbb{R} \text{ for } n \in \mathbb{N}) \tag{11}$$

is valid for $P_n(x)$.

Proof. Since any solution family of (1) satisfies recurrence equation (7), the identity of Theorem 1 is valid.

All terms of the first line vanish, and the first term of the second line is a multiple of the left-hand side of (11). If we expand the remaining terms, we get constant multiples of $P_{n-1}(x)$, $P_n(x)$ and $xP_n(x)$. Using (7) once more, $xP_n(x)$ can be rewritten as a linear combination of $P_{n-1}(x)$, $P_n(x)$ and $P_{n+1}(x)$, which completes the result. □

Once it is known that such a representation is valid, the coefficients α_n, β_n and γ_n can be easily determined in terms of the coefficients a, b, c, d, e of $\sigma(x) := ax^2 + bx + c$ and $\tau(x) := dx + e$.

Theorem 3. *Assume* $P_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \cdots$ *is a solution family of* (1). *Then structure formula* (11) *is valid with*

$$\alpha_n = a[n]_{1/q}\frac{k_n}{k_{n+1}},$$

$$\begin{aligned} \beta_n = {} & ((-1+N)(aN + dNq - dN - aq)(eNaq^2 - bdN^2q + Nqba \\ & - bqa + bdN^2 - baN^2 + bNa - eNa)q)/ \\ & ((-dN^2q + dN^2 + aq^2 - aN^2)(q-1)(dN^2q - a + aN^2 - dN^2)), \end{aligned}$$

$$\begin{aligned} \gamma_n = {} & ((-1+N)(aN + dNq - dN - aq)(aq^2 - dNq + dN - aN) \\ & (2b^2N^2aq^2 + q^4ca^2 + b^2q^3dN^2 - b^2q^2dN^2 + e^2N^2aq^2 \end{aligned}$$

$$+ cd^2N^4 + cN^4a^2 - 2cq^3adN^2 + eNbaq^3 - 2eN^2baq^2$$
$$- eq^4Nba - eq^2N^3ba + 2eq^2N^3bd + 2bq^3N^2ea$$
$$- eq^3N^3bd + eN^3baq + 2cqaN^4d + e^2q^4N^2a - 2e^2q^3N^2a$$
$$+ qb^2N^3d - q^3b^2Na - qb^2N^3a - b^2q^2N^3d - 2cqd^2N^4$$
$$- 2q^2N^2ca^2 - 2dN^4ca + cq^2d^2N^4 + 2q^2dcaN^2 - eN^3bdq)Nq)/$$
$$((-aq + dN^2q + aN^2 - dN^2)(-dN^2q + dN^2 + aq^3 - aN^2)$$
$$\times(-dN^2q + dN^2 + aq^2 - aN^2)^2(q-1))\frac{k_n}{k_{n-1}},$$

where $N = q^n$.

Proof ([7]). Substitute $P_n(x) = k_nx^n + k_n'x^{n-1} + k_n''x^{n-2} + \cdots$ in the q-difference equation (1).

Equating the coefficients of x^n yields $\lambda_{q,n}$, given by (2). Equating the coefficients of x^{n-1} and x^{n-2} gives k_n', and k_n'', respectively, as multiples – rational w.r.t. $N:=q^n$ – of k_n.

Substitute $P_n(x)$ in the proposed equation (11), and equate again the three highest coefficients. This yields α_n, β_n, and γ_n in terms of $a, b, c, q, q^n, k_{n-1}, k_n, k_{n+1}, k_n', k_{n+1}', k_n'', k_{n+1}''$ by linear algebra.

Substituting the values of k_n', k_{n+1}', k_n', and k_{n+1}'' that were given before yields the above formulas. □

4. Further results

In this section, we list some further consequences of Theorems 1–3.

(1) The classical orthogonal polynomials are given as the polynomial solutions $P_n(x)$ of the differential equation

$$L_n y(x) := \sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where $\sigma(x)$ is a polynomial of at most second degree and $\tau(x)$ is a polynomial of first degree. They satisfy a recurrence equation of type (7).

Essentially, the same deduction as in Section 2 can be applied to this case leading to

Theorem 4. *Applying* L_{n+1} *to recurrence equation* (7) *yields the identity*

$$L_{n+1}[P_{n+1}(x)] = (A_nx + B_n)L_n[P_n(x)] - C_nL_{n-1}[P_{n-1}(x)] + 2A_n\sigma(x)P_n'(x)$$
$$+ A_n\tau(x)P_n(x) + A_n(\lambda_{n+1} - \lambda_n)xP_n(x)$$
$$+ B_n(\lambda_{n+1} - \lambda_n)P_n(x) - C_n(\lambda_{n+1} - \lambda_{n-1})P_{n-1}(x).$$

Note that this identity also follows by taking the limit $q \to 1$ in Theorem 1. An equivalent identity can be found in [6, Eq. (4)], which was the starting point for our development.

Theorem 4 obviously implies the well-known derivative rule

$$\sigma(x)P'_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \quad (n \in \mathbb{N})$$

for the classical orthogonal polynomials. For a kind of inverse statement see [1].

(2) The classical discrete orthogonal polynomials are given as the polynomial solutions $P_n(x)$ of the difference equation

$$L_n y(x) := \sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0,$$

where $\Delta y(x) = y(x+1) - y(x)$ and $\nabla y(x) = y(x) - y(x-1)$ denote the forward and backward difference operators, respectively, $\sigma(x)$ is a polynomial of at most second degree and $\tau(x)$ is a polynomial of first degree. They satisfy a recurrence equation of type (7).
Again, essentially the same deduction as in Section 2 can be applied leading to

Theorem 5. *Applying L_{n+1} to recurrence equation* (7) *yields the identity*

$$\begin{aligned} L_{n+1}[P_{n+1}(x)] = {} & (A_n x + B_n) L_n[P_n(x)] - C_n L_{n-1}[P_{n-1}(x)] + 2A_n \sigma(x) \nabla P_n(x) \\ & + A_n \tau(x) P_n(x) - A_n \lambda_n P_n(x) + A_n(\lambda_{n+1} - \lambda_n) x P_n(x) \\ & + B_n(\lambda_{n+1} - \lambda_n) P_n(x) - C_n(\lambda_{n+1} - \lambda_{n-1}) P_{n-1}(x). \end{aligned}$$

This result obviously implies the well-known structural formula

$$\sigma(x)\nabla P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \quad (n \in \mathbb{N}).$$

for the classical discrete orthogonal polynomials.

(3) The most important orthogonal polynomials of the q-Hahn tableaux can be found in Table 1. Tables 2–4 give the coefficients of the structure formula for the monic polynomial systems $\tilde{P}_n(x) = P_n(x)/k_n$ that are standardized with $\tilde{k}_n = 1$. Note that in these tables the notation $N := q^n$ is used. We omitted the results for the q-Hahn polynomials since they are rather complicated. Anyway, they are special cases of the Big q-Jacobi polynomials.

(4) We show that structure formula (11) implies an important connection relation between the families $(P_n(x))$ and $(D_q P_n(x))$.

Theorem 6. *Assume $P_n(x)$ is a solution family of* (1). *Then a structure formula of the type*

$$P_n(x) = \hat{a}_n D_q P_{n+1}(x) + \hat{b}_n D_q P_n(x) + \hat{c}_n D_q P_{n-1}(x) \tag{12}$$

is valid for $P_n(x)$.

Proof. Applying D_q to the q-difference equation (1) for $P_n(x)$ and using relation (3), the product rule (4) and

$$D_{1/q} D_q = q D_q D_{1/q},$$

it follows for $y(x) := D_q P_n(x)$ that

$$\frac{1}{q}\sigma(x) D_q D_{1/q} y(x) + \left(\frac{1}{q} D_q \sigma(xz) + \tau(x)\right) D_q y(x) + (D_q \tau(x) + \lambda_{q,n}) y(x) = 0,$$

Table 1
The q-orthogonal polynomials

$\sigma(x)$	$\tau(x)$	$p_n(x)$	family
1	$1-x$	$\tilde{h}_n(x;q)$	discrete q-Hermite II pols.
1	$\frac{a+1-x}{a(q-1)}$	$V_n^{(a)}(x;q)$	Al-Salam-Carlitz II pols.
x	$\frac{xq+a+q}{a(q-1)}$	$C_n(x;a;q)$	q-Charlier polynomials
x	$-q^{\alpha+1}x+\frac{q^{\alpha+1}-1}{q-1}$	$L_n^{(\alpha)}(x;q)$	q-Laguerre polynomials
x	$\frac{xq-1}{q-1}$	$S_n(x;q)$	Stieltjes-Wigert pols.
$x-bq$	$\frac{xq-q-c+qbc}{c(q-1)}$	$M_n(x;b,c;q)$	q-Meixner polynomials
$x(x-1)$	$-\frac{x-1+aq}{q-1}$	$p_n(x;a\vert q)$	Little q-Laguerre pols.
$x(x-1)$	$\frac{x+aqx-1}{q-1}$	$K_n(x;a;q)$	alternative q-Charlier pols.
$x(x-1)$	$\frac{1-aq-x+xabq^2}{q-1}$	$p_n(x;a,b\vert q)$	Little q-Jacobi pols.
$(x-1)(x+1)$	$-\frac{x}{q-1}$	$h_n(x;q)$	discrete q-Hermite I pols.
$(x-1)(x-a)$	$\frac{a+1-x}{q-1}$	$U_n^{(a)}(x;q)$	Al-Salam-Carlitz I pols.
$(x-aq)(x-bq)$	$\frac{aq+bq-abq^2-x}{q-1}$	$P_n(x;a,b;q)$	Big q-Laguerre polynomials
$(q^Mx-1)(x-\alpha q)$	$\frac{q^{M+2}\alpha\beta(x-1)+q^{M+1}\alpha-\alpha q+1-xq^M}{q-1}$	$Q_n(x;\alpha,\beta,M\vert q)$	q-Hahn polynomials
$(x-aq)(x-bq)$	$\frac{q(a+c-abq-acq)-x+abq^2x}{q-1}$	$P_n(x;a,b,c;q)$	Big q-Jacobi polynomials

Table 2
Structure formula for the Monic q-polynomials I

α_n	β_n	γ_n	$p_n(x)$
0	0	$\frac{q(N-1)}{N(q-1)}$	$\tilde{\tilde{h}}_n(x;q)$
0	0	$\frac{q(N-1)}{N(q-1)}$	$\tilde{V}_n^{(a)}(x;q)$
0	$\frac{q(N-1)}{N(q-1)}$	$-\frac{(Nq-aq+2aN)q(N-1)}{N^3(q-1)}$	$\tilde{C}_n(x;a;q)$
0	$\frac{q(N-1)}{N(q-1)}$	$\frac{(-1+q^\alpha N)q^2(N-1)}{N^3q^\alpha(q-1)^2}$	$\tilde{L}_n^{(\alpha)}(x;q)$
0	$\frac{q(N-1)}{N(q-1)}$	$\frac{q^2(N-1)}{N^3(q-1)}$	$\tilde{S}_n(x;q)$
0	$\frac{q(N-1)}{N(q-1)}$	$-\frac{(c+N)(Nb-1)q^2(N-1)}{N^3(q-1)}$	$\tilde{M}_n(x;b,c;q)$
$\frac{q(N-1)}{N(q-1)}$	$-\frac{(Naq-1+aN)q(N-1)}{q-1}$	$\frac{(-1+aN)aN^2(N-1)}{q-1}$	$\tilde{p}_n(x;a\vert q)$
$\frac{q(N-1)}{N(q-1)}$	0	$-\frac{N(N-1)}{q-1}$	$\tilde{h}_n(x;q)$
$\frac{q(N-1)}{N(q-1)}$	$\frac{(a+1)q(N-1)}{q-1}$	$\frac{Na(N-1)}{q-1}$	$\tilde{U}_n^{(a)}(x;q)$
$\frac{q(N-1)}{N(q-1)}$	$-\frac{(Nbqa-a-b+bNa)q^2(N-1)}{q-1}$	$\frac{N(-1+aN)(Nb-1)baq^2(N-1)}{q-1}$	$\tilde{P}_n(x;a,b;q)$

hence, $y(x)=D_qP_n(x)$ satisfies a q-difference equation of the same type. As a consequence, a recurrence equation (7) is valid for $y(x)$ which can be brought into the form

$$xD_qP_n(x)=\alpha_n^\star D_qP_{n+1}(x)+\beta_n^\star D_qP_n(x)+\gamma_n^\star D_qP_{n-1}(x). \tag{13}$$

Table 3
Structure formula for the Monic q-polynomials II

α_n	$\frac{q(N-1)}{N(q-1)}$	
β_n	$-\frac{q(-2Nq+q+N^2aq+2N^2-2N)(-q+Naq+2N)(N-1)}{(q-1)(-q^2+N^2aq+2N^2)(N^2aq+2N^2-1)}$	$\tilde{K}_n(x;a;q)$
γ_n	$-\frac{(-q+2N)(Naq+2N-2q)q^2N^2(-q+Naq+2N)(-q^2+Naq+2N)(N-1)}{(q-1)(-q+N^2aq+2N^2)(-q^2+N^2aq+2N^2)^2(-q^3+N^2aq+2N^2)}$	
α_n	$\frac{q(N-1)}{N(q-1)}$	
β_n	$-\frac{(-Naq+N^2abq+1-aN)(Nbqa-1)q(N-1)}{(q-1)(bN^2a-1)(N^2abq^2-1)}$	$\tilde{p}_n(x;a,b\vert q)$
γ_n	$-\frac{(-1+aN)(Nb-1)aN^2(Nbqa-1)(bNa-1)q(N-1)}{(q-1)(N^2abq-1)(bN^2a-1)^2(-q+bN^2a)}$	

Table 4
Structure formula for the Monic Big q-Jacobi polynomials

α_n	$\frac{q(N-1)}{N(q-1)}$
β_n	$-\frac{(N^2q^2b^2a+N^2bq^2a^2-Nabq^2-q^2Nca+bq-Nbq+aq+qNc-Nbqa-Ncqa-Nb+Nc)(Nbqa-1)q(N-1)}{(q-1)(bN^2a-1)(q^2N^2ba-1)}$
γ_n	$\frac{N(Nc-Ncqa+aq-Nb-Nbqa+qN^2b^2a)(Nc-Ncqa+bq-Nb-Nbqa+bqa^2N^2)(Nbqa-1)(bNa-1)q(N-1)}{(q-1)(qN^2ba-1)(bN^2a-1)^2(-q+bN^2a)}$

Applying D_q to structure formula (11) yields an identity of the form

$$\sigma(x)D_qD_{1/q}P_n(x) = a'_nD_qP_{n+1}(x) + b'_nD_qP_n(x) + c'_nD_qP_{n-1}(x).$$

Substituting the left-hand side of this equation into the q-difference equation, and using (13) to get rid of the $xD_qP_n(x)$ terms, results in a connection relation of the type (12). □

The coefficients $\hat{a}_n, \hat{b}_n$ and $\hat{c}_n$ again can be written in terms of a, b, c, d and e, see e.g. [10,2]. These coefficients can be computed by a method similar to that given in the proof of Theorem 3.

Note added in proof

We would like to thank Prof. Lesky who informed us about the PhD dissertation "Polynomiale Eigenwertprobleme zweiter Ordnung mit Hahnschen q-Operatoren" by Steffen Häcker, University of Stuttgart, 1993. Häcker used orthogonality functionals to prove the existence of a structure formula (11).

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