Discontinuous Galerkin time stepping with local projection stabilization for transient convection-diffusion-reaction problems

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Abstract

A time-dependent convection-diffusion-reactions problems is discretized in space by a continuous finite element method with local projection stabilization and in time by a discontinuous Galerkin method. We present error estimates for the semidiscrete problem after discretizing in space only and for the fully discrete problem. Numerical tests confirm the theoretical results.

Keywords: discontinuous Galerkin, stabilized finite elements, convection-diffusion-reaction equation

1. Introduction

The modeling of many technical and physical processes leads to descriptions which contain time-dependent convection-diffusion-reaction equations as sub-problems. Their accurate and efficient solution is often critical for accuracy and efficiency of the whole process.

There are several approach for discretizing time-dependent convection-diffusion-reaction problems by finite element methods. Firstly, space-time elements could be used. This results in very large systems of linear equations since all unknowns in the space-time cylinder are coupled. Secondly, semidiscretization as intermediate steps can be used. Here, we distinguish between the horizontal and vertical methods of lines. The vertical method of lines discretizes first in space and then in time while the horizontal method of lines (or Rothe’s method) applies first a time discretisation which is followed by a discretisation in space.
We will apply the vertical methods of lines. Since we are interested also in convection-dominated convection-diffusion-reaction problems, standard finite element methods will lead to solutions which contain global unphysical oscillations. In order to prevent this, stabilization techniques are applied. One of the most popular methods is the Streamline-Upwind Petrov–Galerkin method (SUPG) which was introduced by Hughes and Brooks [1] for steady problems. The main drawback of SUPG for time-dependent problems is the fact that for ensuring the consistency of the method the time derivative, the source term, and second order derivatives have to be included into the stabilization term. This leads to a wide (and generally unphysical) coupling of the unknowns.

An alternative to SUPG are symmetric stabilization methods such as the local projection stabilization (LPS) [2–4], the continuous interior penalty method (CIP) [5], the subgrid scale modeling (SGS) [6, 7] and the orthogonal subscales method (OSS) [8, 9], which have been investigated during the last decade. The stencil resulting from a discretisation of the stabilizing term is for the CIP and the OSS method larger than for the SUPG. This is not the case for the one-level variant of the LPS. Originally proposed for the Stokes problem [10], the LPS was extended successfully to transport problems [2]. The application of local projection methods to Oseen problems are studied in [3, 4, 11]. The local projection methods provides additional control on the fluctuations of the gradient or parts of its. Although, the methods is weakly consistent only, the consistency error can be bounded such that the optimal order of convergence is maintained.

The discontinuous Galerkin (dG) method was first introduced by Reed and Hill [12] for neutron transport equations. The analysis of dG methods starts with the works of Lesaint and Raviart [13] and Johnson and Pitkäranta [14]. Since then, many different aspects of dG methods have been investigated. We will just mention a few of them: Delfour, Hager and Trochu [15], Larsson, Thomée and Wahlbin [16], Schötzau and Schwab [17, 18], the survey article [19], and the books [20, 21].

Stabilized finite element methods for time-dependent convection-diffusion-reaction problems have been investigated by several authors. We refer to [22, 23] which consider different stabilization techniques including SUPG and to [8] using OSS. The stability property of consistent stabilization methods in the small time step limit have been discussed in [24, 25]. The CIP in space combined with the \( \theta \)-method in time have been investigated in [26]. The coupling of other stabilization techniques in the one dimensional case with the finite difference time integration in particular the vertical and horizontal method of lines has been discussed in [27]. The standard Galerkin method in space but on a layer adapted Shishkin mesh and different time discretisations have been studied in [28]. The dG method has been analyzed in space [29, 30] and in space and time [31]. A numerical study of SUPG applied to time-dependent convection diffusion problems with small diffusion parameter can be found in [32].

The aim of our paper is to combine the local projection stabilization in space with the discontinuous Galerkin method in time. We will give error estimates for the semidiscrete problem after discretizing in space by a finite element method with local projection stabilization and for the fully discrete problems.
The plan of the paper is as follows. Section 2 introduces the problem under consideration and defines the basic notations. The semidiscretization in space and the local projection stabilization are introduced in Section 3. Furthermore, an optimal error estimate for the semidiscretized problems will be given. Section 4 presents the error analysis for the fully discrete problem after a time discretisation by a discontinuous Galerkin method. Numerical results which confirm the theoretical predictions will be showed in Section 5. Finally, Section 6 will provide some concluding remarks.

2. Notations and Preliminaries

Let \( \Omega \subset \mathbb{R}^d \) be a bounded polygonal for \( d = 2 \) or polyhedral for \( d = 3 \) domain with Lipschitz continuous boundary \( \Gamma = \partial \Omega \) and \( T > 0 \). We set \( Q_T := \Omega \times (0, T) \) and consider the following time-dependent convection-diffusion-reaction problem:

Find \( u : Q_T \to \mathbb{R} \) such that

\[
\begin{aligned}
    u_t - \varepsilon \Delta u + b \cdot \nabla u + \sigma u &= f & \text{in } Q_T, \\
    u &= 0 & \text{on } \partial \Omega \times (0, T), \\
    u(\cdot, 0) &= u_0 & \text{in } \Omega.
\end{aligned}
\]  

We assume that \( b, \sigma \) are independent on time \( t \), whereas \( f \) may depend on \( t \). Furthermore, let the data \( b, \sigma, u_0 \) and \( f \) are sufficiently smooth on \( \Omega \) and \( \Omega \times (0, T) \), respectively. The parameter \( \varepsilon \) is supposed to be positive. By changing the dependent variable we may also assume that

\[
\sigma - \frac{1}{2} \text{div } b \geq \sigma_0 > 0 \text{ in } \Omega.
\]  

Throughout this paper, standard notations and conventions will be used. Let \( H^m(\Omega) \) denote the Sobolev space of functions with derivatives up to order \( m \) in \( L^2(\Omega) \). We denote by \( (\cdot, \cdot) \) the inner product in \( L^2(\Omega) \) and by \( \| \cdot \| \) the associated \( L^2 \)-norm. The norm in \( H^m(\Omega) \) is defined as

\[
\| v \|_m = \left( \sum_{|\alpha| \leq m} \| D^\alpha v \|^2 \right)^{1/2}.
\]

We consider also certain Bochner spaces. For this let \( X \) be a Banach space equipped with the norm \( \| \cdot \|_X \) and seminorm \( | \cdot |_X \). Then, we define

\[
\begin{align*}
    C(0, T; X) &= \{ v : [0, T] \to X, \quad v \text{ continuous} \}, \\
    L^2(0, T; X) &= \{ v : (0, T) \to X, \quad \int_0^T \| v(t) \|_X^2 \, dt < \infty \}, \\
    H^m(0, T; X) &= \{ v \in L^2(0, T; X) : \frac{\partial^j v}{\partial \partial^j} \in L^2(0, T; X), \, 1 \leq j \leq m \},
\end{align*}
\]
where the derivatives $\partial^j v/\partial t^j$ are understood in the sense of distributions on $(0, T)$. In the following we use the short notation $Y(X) := Y(0, T; X)$. The norms and seminorms in the above defined spaces are given by

$$
\|v\|_{C^m(X)} = \sup_{t \in [0, T]} \|v(t)\|_X,
\|v\|_{L^2(X)}^2 = \int_0^T \|v(t)\|_X^2 dt,
|v|_{\dot{H}^m(X)}^2 = \int_0^T \|\partial^m v\|_X^2 dt,
\|v\|_{H^m(X)}^2 = \int_0^T \sum_{j=0}^m \|\partial^j v\|_X^2 dt.
$$

Let us introduce the space $V = H^1_0(\Omega)$, its dual space $H^{-1}(\Omega)$, and $\langle \cdot, \cdot \rangle$ for the duality product between these two spaces. Then, a function $u$ is a weak solution of problem (1), if

$$
u \in L^2(H^1_0), \quad u' \in L^2(H^{-1}),
$$

and for almost all $t \in (0, T)$,

$$
\begin{cases}
\langle u'(t), v \rangle + a(u(t), v) = \langle f(t), v \rangle \quad \forall v \in V,
\end{cases}
$$

where the bilinear form $a$ is given by

$$
a(u, v) := \varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u, v) + (\sigma u, v).
$$

Note that (3) implies the continuity of $u$ as a mapping of $[0, T] \rightarrow L^2(\Omega)$ such that the initial condition $u(0) = u_0$ is well-defined. In what follows, we shall denote by $f'$, $f''$, and $f^{(q)}$ the first, second and $q$-th order time derivative of $f$, respectively.

3. Semidiscretization and local projection stabilization

For the finite element discretization of (4), let $\{T_h\}$ denote a family of shape regular triangulations of $\Omega$ into $d$-simplices, quadrilaterals or hexahedra such that $\Omega = \bigcup_{K \in T_h} K$. The diameter of $K \in T_h$ will be denoted by $h_K$ and the mesh size $h$ is defined by $h := \max_{K \in T_h} h_K$. We will consider the one-level LPS in which approximation and projection space live on the same mesh. For other variants of LPS we refer to [4, 33–37].

Let $V_h \subset V$ denote the approximation space of continuous, piecewise polynomials and $D_h$ be the projection space of discontinuous, piecewise polynomials. Let $D_h(K) = \{q_K : q_h \in D_h\}$ and $\pi_K : L^2(\Omega) \rightarrow D_h(K)$ the local $L^2$-projection.
into $D_h(K)$. Define the global projection $\pi_h : L^2(\Omega) \rightarrow D_h$ by $(\pi_h v) |_K := \pi_K(v|_K)$. The fluctuation operator $\kappa_h : L^2(\Omega) \rightarrow L^2(\Omega)$ is given by $\kappa_h := id - \pi_h$, where $id : L^2(\Omega) \rightarrow L^2(\Omega)$ is the identity. The stabilizing term $S_h(u_h, v_h)$ is given by

$$S_h(u_h, v_h) = \sum_{K \in T_h} \tau_K (\kappa_h \nabla u_h, \kappa_h \nabla v_h)_K$$

with user chosen non-negative constants $\tau_K$, $K \in T_h$. It gives additional control over the fluctuations of gradients. Note that one can replace the gradients $\nabla w_h$ by the derivative in streamline direction $b \cdot \nabla w_h$ or (even better [34, 38]) by $b_K \cdot \nabla w_h$ where $b_K$ is a piecewise constant approximation of $b$, which leads to similar results. Now, the stabilized semidiscrete problem reads:

For all $t \in (0, T)$, find $u_h(t) \in V_h$ such that

$$\begin{cases}
(u'_h(t), v_h) + a_h(u_h(t), v_h) = (f(t), v_h) & \forall v_h \in V_h, \\
u_h(0) = u_{h,0}
\end{cases}$$

(5)

where

$$a_h(u_h, v_h) := a(u_h, v_h) + S_h(u_h, v_h)$$

and $u_{h,0} \in V_h$ is a suitable approximation of $u_0 \in L^2(\Omega)$.

Thanks to (2) the bilinear form $a_h$ is coercive with respect to the mesh dependent norm

$$|||v||| = \left\{ \varepsilon |v|^2 + \sigma_0 |v|^2 + S_h(v, v) \right\}^{1/2},$$

(6)

i.e.

$$a_h(v_h, v_h) \geq |||v_h|||^2 \forall v_h \in V_h,$$

(7)

The stability and convergence properties of the LPS method (5) are based on the following assumptions with respect to the pair $(V_h, D_h)$, see [4, 33].

**Assumption A1:** There is an interpolation operator $j_h : H^2(\Omega) \rightarrow V_h$ such that for all $K \in T_h$, $v \in H^l(K)$ and $2 \leq l \leq r + 1$,

$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \leq Ch_k^l \|v\|_{l,K}$$

(8)

and the orthogonality

$$(v - j_h v, q_h) = 0 \forall q_h \in D_h, \forall v \in H^2(\Omega)$$

(9)

hold true.
**Assumption A2:** The fluctuation operator $\kappa_h$ satisfy the following approximation property

$$
\|\kappa_h q\|_{0,K} \leq C h^l K \|q\|_{l,K} \quad \forall K \in T_h, \forall q \in H^l(K), \ 0 \leq l \leq r. \quad (10)
$$

Examples of spaces $(V_h, D_h)$ satisfying A1 and A2 are given, for example, in [4, 33]. In order to analyze the semi-discrete error, we define the Ritz-projection $R_h : V \rightarrow V_h$ associated with the stabilized bilinear form $a_h$ as

$$
a_h(R_h w, v_h) = a(w, v_h) \quad \forall v_h \in V_h. \quad (11)
$$

For the stationary problem associated with (1) we have

**Theorem 1.** Suppose A1 and A2, $\tau_K \sim h_K$ for all $K \in T_h$, and let the data of the problem be sufficiently smooth. Then, there exists a positive constant $C$, independent of $\varepsilon$ and $h$, such that

$$
|||R_h w||| \leq C \|w\|_1 \quad \forall w \in H^1(\Omega) \quad (12)
$$

and

$$
|||w - R_h w||| \leq C(\varepsilon^{1/2} + h^{1/2}) h^r \|w\|_{r+1} \quad (13)
$$

for all $w \in H^1_0(\Omega) \cap H^{r+1}(\Omega)$.

**Proof.** From (7) and (11) we have

$$
|||R_h w|||^2 \leq a_h(R_h w, R_h w) = a(w, R_h w)
$$

$$
\leq C\|w\|_1 |||R_h w|||,
$$

from which (12) follows. For (13), see [33, Theorem 3.74].

Now we can state the main result of this section.

**Theorem 2.** Let $u(t)$ and $u_h(t)$ be the solutions of the continuous problem (4) and the semi-discrete problem (5), respectively. If $\tau_K \sim h_K$ for all $K \in T_h$, then there exists a positive constant $C$ independent of $t$, $\varepsilon$, and $h$, such that for all $t \in [0, T]$

$$
\|u_h(t) - u(t)\| \leq \|u_{h,0} - u_0\| + C(\varepsilon^{1/2} + h^{1/2}) h^r \|u_0\|_{r+1}
$$

$$
+ C(\varepsilon^{1/2} + h^{1/2}) h^r \int_0^t \|u'\|_{r+1} dt \quad (14)
$$

and

$$
\int_0^t |||u_h(s) - u(s)|||^2 ds
$$

$$
\leq C \left[ ||u_{h,0} - u_0||^2
$$

$$
+ (\varepsilon + h^2) \int_0^t \left( \|u(s)\|^2_{r+1} + \|u'(s)\|^2_{r+1} \right) ds \right]. \quad (15)
$$
Proof. We split the error into two parts

\[ u_h(t) - u(t) = u_h(t) - R_h u(t) + R_h u(t) - u(t) = \xi(t) + \eta(t) \]

where

\[ \xi := u_h - R_h u, \quad \eta := R_h u - u. \]

The estimate for the projection error \( \eta(t) \) follows from (13)

\[
\|\eta(t)\| = \|R_h u(t) - u(t)\| \leq Ch^r (\epsilon^{1/2} + h^{1/2}) \|u(t)\|_{r+1} \\
\leq Ch^r (\epsilon^{1/2} + h^{1/2}) \left\{ \|u_0\|_{r+1} + \int_0^t \|u'(s)\|_{r+1} ds \right\} \tag{16}
\]

where we used in the second step

\[
\|u(t)\|_{r+1} \leq \|u_0\|_{r+1} + \int_0^t \|u'(s)\|_{r+1} ds.
\]

In order to bound \( \xi(t) \), we use (4), (5), the definition (11) of the Ritz-projection \( R_h \), and the fact that \( R_h \) commutes with time derivative to get

\[
(\xi'(t), v_h) + a_h(\xi(t), v_h) = (u'(t) - (R_h u)'(t), v_h) \\
= -(\eta'(t), v_h) \quad \forall v_h \in V_h.
\]

Setting \( v_h = \xi(t) \) and taking into consideration the nonnegativity of the bilinear form \( a_h \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\xi(t)\|^2 \leq (\xi'(t), \xi(t)) + a_h(\xi(t), \xi(t)) \\
= -(\eta'(t), \xi(t)) \leq \|\eta'(t)\| \|\xi(t)\|.
\]

A usual regularization trick to avoid problems with the differentiability of \( t \mapsto \|\xi(t)\| \) when \( \xi = 0 \) and integration over time from 0 to \( t \) yields

\[
\|\xi(t)\| \leq \|\xi(0)\| + \int_0^t \|\eta'(s)\| \, ds.
\]

The terms on the right hand side can be estimated as follows

\[
\|\xi(0)\| \leq \|u_h(0) - u(0)\| + \|u(0) - R_h u(0)\| \\
\leq \|u_h,0 - u_0\| + Ch^r (\epsilon^{1/2} + h^{1/2}) \|u_0\|_{r+1} \\
\|\eta'(s)\| = \|u'(s) - R_h u'(s)\| \\
\leq Ch^r (\epsilon^{1/2} + h^{1/2}) \|u'(s)\|_{r+1}.
\]

Thus, for the error to the Ritz-projection we have

\[
\|\xi(t)\| \leq \|u_h,0 - u_0\| \\
+ Ch^r (\epsilon^{1/2} + h^{1/2}) \left\{ \|u_0\|_{r+1} + \int_0^t \|u'(s)\|_{r+1} ds \right\}.
\]
Combining this with (16), we get (14).

Above we used only the nonnegativity of \( a_h(\xi(t), \xi(t)) \) instead of the stronger coercivity estimate

\[
a_h(\xi(t), \xi(t)) \geq |||\xi(t)|||^2.
\]

Now starting with

\[
\frac{1}{2} \frac{d}{dt} |||\xi(t)|||^2 + |||\xi(t)|||^2 \leq |||\eta'(t)||| \cdot |||\xi(t)|||,
\]

applying

\[
|||\eta'(t)||| \cdot |||\xi(t)||| \leq \sigma_0 \|\xi(t)\|^2 + \frac{1}{2\sigma_0} |||\eta'(t)|||^2
\]

and integrating over \( t \), we obtain

\[
|||\xi(t)|||^2 + \int_0^t |||\xi(s)|||^2 ds \leq |||\xi(0)|||^2 + \frac{1}{\sigma_0} \int_0^t |||\eta'(s)|||^2 ds.
\]

Using again the estimates for \( |||\xi(0)||| \) and \( |||\eta'(s)||| \) above, we have

\[
|||\xi(t)|||^2 + \int_0^t |||\xi(s)|||^2 ds \leq C \|u_{h,0} - u_0\|^2 + \\
+ C(\epsilon + h) h^{2r} \left\{ \|u_0\|_{r+1}^2 + \int_0^t \|u'(s)\|_{r+1}^2 ds \right\}.
\]

Finally, we use (13) and the triangle inequality to get (15).

4. Discontinuous Galerkin in time

For getting a fully discrete version of (1) we apply the discontinuous Galerkin method to problem (5). Let \( 0 = t_0 < t_1 < \cdots < t_N = T \) be a partition of the time interval \([0, T]\), \( J_n = (t_{n-1}, t_n), k_n = t_n - t_{n-1} \), and \( K = \max k_n \). For a given non-negative integer \( q \), we define the semi-discrete space

\[
S^q_k := \left\{ v : [0, T] \to V : v|_{J_n}(t) = \sum_{j=0}^q v_j t^j \text{ with } v_j \in V \right\}
\]

and the fully discrete space

\[
S^{r,q}_{h,k} := \left\{ v : [0, T] \to V_h : v|_{J_n}(t) = \sum_{j=0}^q v_j t^j \text{ with } v_j \in V_h \right\}. \tag{17}
\]

The functions in these spaces are allowed to be discontinuous at the nodes \( t_n \).

For discontinuous in time functions we use the notations

\[
\varphi^\pm_m = \varphi(t^\pm_m) = \lim_{t \to t^\pm_m} \varphi(t). \tag{18}
\]
The jumps across the nodes are given by \( \varphi_m := \varphi^+_m - \varphi^-_m \). Now, let us introduce the bilinear form \( B \) by

\[
B(u, v) := \sum_{n=1}^{N} \int_{J_n} \{(u', v) + a_h(u, v)\} \, dt + \sum_{n=1}^{N-1} ([u]_n, v^+_n) + (u^+_0, v^-_0).
\]  

(19)

Integration by parts

\[
\int_{J_n} (u', v) \, dt = (u^-_n, v^-_n) - (u^+_n-1, v^+_n-1) - \int_{J_n} (u, v') \, dt
\]

leads to the representation

\[
B(u, v) = \sum_{n=1}^{N} \int_{J_n} \{- (u, v') + a_h(u, v)\} \, dt - \sum_{n=1}^{N-1} (u^-_n, [v]_n) + (u^-_N, v^-_N).
\]

Now, the fully discrete problem reads:

Find \( U_h \in S_{h,k}^{r,q} \) such that

\[
B(U_h, X) = (u_h, 0, X^+_0) + \int_0^T (f, X) \, dt, \quad \forall X \in S_{h,k}^{r,q}.
\]  

(20)

We consider two mesh-dependent norms given by

\[
\|v\|_w = \left( \sum_{n=1}^{N} \int_{J_n} \|v\|^2 \, dt + \frac{1}{2} \|v^-_N\|^2 \right)^{1/2}
\]

(21)

\[
\|v\|_s = \left( \sum_{n=1}^{N} \int_{J_n} \|v\|^2 \, dt + \frac{1}{2} \|v^+_0\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} \|v\|^2 + \frac{1}{2} \|v^-_N\|^2 \right)^{1/2}.
\]

(22)

Here, and in the following we assume the regularity of \( v \) needed such that \( B \) and \( \| \cdot \|_s \) are well defined.

**Lemma 3.** The bilinear form \( B \) is coercive with respect to the strong norm \( \| \cdot \|_s \), i.e., there is a positive constant \( \alpha \) such that

\[
B(v, v) \geq \alpha \|v\|_s^2.
\]

(23)

**Proof.** Adding the two representations for \( B \), setting \( u = v \), and using the coercivity of \( a_h \) with respect to \( \| \cdot \| \), we get the statement of the lemma.

**Lemma 4.** The solution \( U_h \) of the fully discrete problem (20) is uniquely determined and satisfies the stability estimate

\[
\|U_h\|_s \leq C \left( \|u_h, 0\| + \|f\|_{L^2(L^2)} \right).
\]

(24)

**Proof.** Setting \( X = U_h \) in (20) and using the coercivity of \( B \), (24) follows by means of Cauchy-Schwarz type inequalities.
For analyzing the error of the discontinuous Galerkin method, we define an interpolant \( \tilde{u} \in S^q_k \) of the exact solution \( u(t) \) of (1) by, for \( n = 1, \ldots, N \),

\[
\tilde{u}(t_n) = u(t_n),
\]

(25)

\[
\int_{J_n} (\tilde{u}(t) - u(t)) \, t^l \, dt = 0, \quad \text{for } l = 0, 1, \ldots, q - 1,
\]

(26)

i.e., \( \tilde{u} \) interpolates at the nodal points, and the interpolation error is \( L^2 \) orthogonal to the space of polynomials of degree \( q - 1 \) on \( J_n \). Note that \( \tilde{u} \) is on each \( J_n \) a polynomial in \( t \) with values in \( V \). For this type of interpolation, we have the following error estimates [39], for \( i, j = 0, 1 \)

\[
\sup_{0 \leq t \leq T} |u(t) - \tilde{u}(t)|_j \leq C k^{q+1} \sup_{0 \leq t \leq T} |u^{(q+1)}(t)|_j,
\]

(27)

\[
\int_{J_n} |u^{(i)}(t) - \tilde{u}^{(i)}(t)|^2 \, dt \leq C k^{2(q+1-i)} \int_{J_n} |u^{(q+1)}(t)|^2 \, dt.
\]

(28)

**Lemma 5.** For the solution \( U_h \) of the fully discrete problem (20) the following estimate holds true

\[
\| U_h - R_h \tilde{u} \|_s \leq C \left[ \| u_{h,0} - R_h u_0^+ \| + (\varepsilon^{1/2} + h^{1/2}) h^r \| u \|_{H^1(H^{r+1})} + k^{q+1} \| u \|_{H^{q+1}(H^1)} \right].
\]

Proof. For \( \xi = U_h - R_h \tilde{u} \in S_{h,k}^r \) we have the error equation

\[
B(U_h - R_h \tilde{u}, \xi) = B(U_h - R_h u, \xi) + B(R_h u - R_h \tilde{u}, \xi).
\]

(29)

Having in mind that \( U_h, u \) are solutions of the fully discrete and continuous problem, respectively, that \( [R_h u]_n = 0 \), and that the stabilized Ritz projection commutes with the time derivative, we obtain for the first term in (29)

\[
B(U_h - R_h u, \xi) = \sum_{n=1}^N \int_{J_n} (u' - R_h u')(\xi) \, dt
\]

\[
+ (u_{h,0} - R_h u_0^+, \xi_0^+).
\]

Now, applying Cauchy-Schwarz’s inequality and (13), we conclude

\[
|B(U_h - R_h u, \xi)|
\]

\[
\leq C (\varepsilon^{1/2} + h^{1/2}) h^r \sqrt{\int_0^T \| u'(s) \|_{r+1}^2 \, ds} \sqrt{\sum_{n=1}^N \int_{J_n} \| \xi \|^2 \, ds}
\]

\[
+ \| u_{h,0} - R_h u_0^+ \| \| \xi_0^+ \|.
\]
Taking into consideration that \( u(t_n) = \tilde{u}(t_n^-) \), \( n = 1, \ldots, N \), and using the second representation of the bilinear form \( B \), we get for the second term in (29)

\[
B(R_h u - R_h \tilde{u}, \xi) = \sum_{n=1}^{N} \int_{J_n} (- (R_h u - R_h \tilde{u}, \xi') + a_h (R_h u - R_h \tilde{u}, \xi)) dt
\]

\[
= \sum_{n=1}^{N} \int_{J_n} a(u - \tilde{u}, \xi) dt
\]
due to the orthogonality (26). By means of the interpolation error estimate we conclude

\[
|B(R_h u - R_h \tilde{u}, \xi)| \leq C k^{q+1} \sqrt{\int_0^T \| u^{q+1}(s) \|^2_2 ds} \sqrt{\sum_{n=1}^{N} \int_{J_n} ||| \xi |||^2_2 ds}.
\]

Collecting the estimates and using the coercivity of \( B \), we end up with the statement of the lemma.

**Lemma 6.** We have following estimates for the interpolation and the projection error, respectively:

\[
\| R_h \tilde{u} - R_h u \|_s \leq C k^{q+1/2} |u|_{H^{s+1}(H^1)},
\]

\[
\| R_h \tilde{u} - R_h u \|_w \leq C k^{q+1} |u|_{H^{s+1}(H^1)},
\]

\[
\| R_h u - u \|_s \leq C (\varepsilon^{1/2} + h^{1/2}) h^r \left(\| u\|_{L^2(H^{s+1})} + \| u\|_{C(H^{s+1})}\right).
\]

**Proof.** The interpolation \( \tilde{u} \) satisfies \( \tilde{u}(t_n^-) = u(t_n) \), \( n = 1, \ldots, N \), thus from the second representation of the bilinear form \( B \) we get for the Ritz projection of the interpolation error \( \eta := R_h \tilde{u} - R_h u = R_h (\tilde{u} - u) \)

\[
\alpha \| \eta \|^2_s \leq B(\eta, \eta) = \sum_{n=1}^{N} \int_{J_n} [-(\eta, \eta') + a_h(\eta, \eta)] dt
\]

\[
\leq C \sum_{n=1}^{N} \int_{J_n} [\| \eta \| \| \eta' \| + ||| \eta |||^2] dt
\]

\[
\leq C \sum_{n=1}^{N} \int_{J_n} [||| \tilde{u}' - u' |||_1 + \| \tilde{u} - u \|_1 + ||| \tilde{u} - u |||^2_1] dt
\]

\[
\leq C k^{2q+1} |u|^2_{H^{s+1}(H^1)},
\]

Here, we used the stability of the Ritz projection (12) and the interpolation estimate (28).
For the improved error estimate with respect to the weak norm we use again the stability of the Ritz projection (12), the interpolation estimates (27), condition (28), and \( \tilde{u}_N = u(t_N) \) to get

\[
\|\eta\|_w = \left( \sum_{n=1}^{N} \int_{J_n} |||\eta|||^2 \, dt \right)^{1/2} 
\leq C \left( \sum_{n=1}^{N} \int_{J_n} \|\tilde{u} - u\|^2 \, dt \right)^{1/2} 
\leq C k^{q+1} |u|_{H^{q+1}(H)}.
\]

Since for the projection error the jumps \( [R_h u - u]_n \) vanish for \( n = 1, \ldots, N - 1 \), we have from (22) and (13)

\[
\|R_h u - u\|_s = \left( \int_0^T \|R_h u - u\|^2 \, ds + \frac{1}{2} \| (R_h u - u)_0^+ \|^2 + \frac{1}{2} \| (R_h u - u)^-_N \|^2 \right)^{1/2} 
\leq C (\varepsilon^{1/2} + h^{1/2} h^r \left( \|u\|_{L^2(H^{q+1})} + \|u\|_{C(H^{q+1})} \right))
\]

which completes the proof of the lemma.

**Theorem 7.** Let \( U_h \) and \( u \) be the solutions of fully discrete problem (20) and the continuous problem (1), respectively. If \( \tau_K \sim h_K \) for all \( K \in T_h \), then there exists a positive constant \( C \) independent of \( h \), \( k \) and \( \varepsilon \), such that the following error estimates

\[
\|U_h - u\|_s \leq \|R_h u_0^+ - u_{h,0}\| + C k^{q+1/2} |u|_{H^{q+1}(H^1)} 
+ C (\varepsilon^{1/2} + h^{1/2} h^r \left( \|u\|_{H^{1}(H^{q+1})} + \|u\|_{C(H^{q+1})} \right))
\]

and

\[
\|U_h - u\|_w \leq \|R_h u_0^+ - u_{h,0}\| + C k^{q+1} |u|_{H^{q+1}(H^1)} 
+ C (\varepsilon^{1/2} + h^{1/2} h^r \left( \|u\|_{H^{1}(H^{q+1})} + \|u\|_{C(H^{q+1})} \right))
\]

hold true.

**Proof.** The proof follows from the triangle inequality applied to the splitting

\[ U_h - u = (U_h - R_h \tilde{u}) + (R_h \tilde{u} - R_h u) + (R_h u - u) \]

and using Lemmas 5 and 6.

5. **Numerical results**

In this section, we will present some numerical results for the discontinuous Galerkin and LPS method applied to time dependent convection-diffusion-reaction problems. All numerical calculations were performed with the finite element package MooNMD [40].
Appropriate finite element spaces which fulfill the assumptions A1 and A2 are given in [4]. In our numerical computations we use mapped finite element spaces [41] where on the reference cell \( \hat{K} \) the enriched spaces are given by

\[
P_s^{\text{bubble}}(\hat{K}) = P_s(\hat{K}) + \hat{b}_{\Delta} P_{s-1}(\hat{K})
\]

\[
Q_s^{\text{bubble}}(\hat{K}) = Q_s(\hat{K}) + \text{span}\{\hat{b}_{\square}^{i-1}, \, i = 1, 2\}
\]

Here, \( \hat{b}_{\Delta} \) and \( \hat{b}_{\square} \) are the cubic bubble on the reference triangle and biquadratic bubble on the reference square, respectively.

The numerical tests are performed using for \((V_h, D_h)\) the pairs \((P_1^{\text{bubble}}, P_0^{\text{disc}})\), \((P_2^{\text{bubble}}, P_1^{\text{disc}})\), \((Q_1^{\text{bubble}}, P_0^{\text{disc}})\) and \((Q_2^{\text{bubble}}, P_1^{\text{disc}})\). The stabilization parameters \( \tau_K \) have been chosen as

\[
\tau_K := \tau_0 h_K \quad \forall K \in T_h
\]

where \( \tau_0 \) denotes a constant which will be given later for each of the test calculation. We used \( u_{h,0} := j_h u_0 \) as discrete initial condition. In order to compare our results with those in the literature, the first example is taken from [26] and the second from [32].

**Example 1.** In this example, we consider a pure transport problem in two dimension given by \( \varepsilon = \sigma = f = 0, \, b = (-y, x)^T, \, \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \) with a Gaussian initial condition centered at \((0.3, 0.3)\) given by

\[
u_0(x, y) = e^{-10[(x-0.3)^2 + (y-0.3)^2]}.
\]

The calculations have been performed on triangular meshes which are obtained from an initial triangulation by successive refinement with boundary adaption due to the curved boundary. The initial mesh (level 0) and the mesh on level 3 are shown in Fig. 1.

![Figure 1: Triangular meshes for Example 1: coarsest mesh (left) and mesh after three refinement steps (right).](image-url)
To find the errors in space and time, we will use the standard strategy that is consider the time-step size small enough to find the convergence order in space and vice versa. In Tables 1–4, we show the convergence results in the strong and weak norms \( \| \cdot \|_s \) and \( \| \cdot \|_w \), respectively, defined in Section 4. For the time discretization, the discontinuous Galerkin methods of first and second order are used with the final time \( T = 2\pi \). Table 1 shows the error in space in the strong and weak norms with stabilizing parameter \( \tau_0 = 0.1 \) and time step length \( h = 2\pi \times 10^{-3} \) for \( (P_{\text{bubble}}^1, P_{\text{disc}}^0) \) and dG(1) in time. In Table 2, the convergence results for \( (P_{\text{bubble}}^2, P_{\text{disc}}^1) \) and dG(2) in time are listed. We see that the expected convergence order can be obtained.

**Table 1:** Errors and rates of convergence in space for \((P_{\text{bubble}}^1, P_{\text{disc}}^0)\), dG(1), \( h = 2\pi \times 10^{-3} \) and \( \tau_K = 0.1 \).

<table>
<thead>
<tr>
<th>level</th>
<th>( | u - u_h |_s )</th>
<th>( | u - u_h |_w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.22864e-01</td>
<td>5.312611e-01</td>
</tr>
<tr>
<td>1</td>
<td>1.044748e-01</td>
<td>1.983524e-01</td>
</tr>
<tr>
<td>2</td>
<td>3.902876e-02</td>
<td>6.334997e-02</td>
</tr>
<tr>
<td>3</td>
<td>1.3990926e-02</td>
<td>2.099493e-02</td>
</tr>
<tr>
<td>4</td>
<td>4.960714e-03</td>
<td>7.262014e-03</td>
</tr>
</tbody>
</table>

**Table 2:** Errors and rates of convergence in space for \((P_{\text{bubble}}^2, P_{\text{disc}}^1)\), dG(2), \( h = 2\pi \times 10^{-3} \) and \( \tau_K = 0.1 \).

<table>
<thead>
<tr>
<th>level</th>
<th>( | u - u_h |_s )</th>
<th>( | u - u_h |_w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.041856e-01</td>
<td>1.042458e-01</td>
</tr>
<tr>
<td>1</td>
<td>1.773761e-02</td>
<td>1.774158e-02</td>
</tr>
<tr>
<td>2</td>
<td>3.454257e-03</td>
<td>3.445058e-03</td>
</tr>
<tr>
<td>3</td>
<td>7.150589e-04</td>
<td>6.843577e-04</td>
</tr>
<tr>
<td>4</td>
<td>2.225006e-04</td>
<td>2.2784</td>
</tr>
</tbody>
</table>

The numerical errors and convergence order in time are listed in Table 3 for dG(1) and \((P_{\text{bubble}}^1, P_{\text{disc}}^0)\) on level 7. The error for dG(2) in time with \((P_{\text{bubble}}^2, P_{\text{disc}}^1)\) on level 6 are presented in Table 4. We see from the results of weaker norm in Table 3 that the expected rate of convergence are achieved for the two largest time step lengths. For smaller time step length the order starts decreasing. This is because of the error in space dominates, i.e., the mesh size \( h \) is not small enough so that one can see the corresponding convergence rate in time.

**Example 2.** The second example is the three body rotation used as a test case for advection problem from [32]. We choose \( \Omega = (0,1)^2 \) and the coefficients \( \varepsilon = 10^{-20} \), \( b = (0.5 - y, x - 0.5)^2 \), \( c = f = 0 \). The initial condition consists of three disjoint bodies: a slotted cylinder, a cone and smooth hump, see Fig. 2. The position of each body is given by its center \((x_0, y_0)\). Each of the bodies lies within a circle of radius \( r_0 = 0.15 \) with center \((x_0, y_0)\). The initial condition is
Table 3: Errors and rates of convergence in time for dG(1) and ($P_{1}^{\text{bubble}}, P_{0}^{\text{disc}}$) on level = 7 with $\tau_K = 0.1\ h_K$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$|u - u_h|_s$</th>
<th>$|u - u_h|_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\pi/10$</td>
<td>2.006905e-01</td>
<td>4.650374e-02</td>
</tr>
<tr>
<td>$2\pi/20$</td>
<td>9.357820e-02</td>
<td>1.198127e-02</td>
</tr>
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<td>$2\pi/40$</td>
<td>3.614780e-02</td>
<td>2.614660e-03</td>
</tr>
<tr>
<td>$2\pi/80$</td>
<td>1.337951e-02</td>
<td>8.423939e-04</td>
</tr>
<tr>
<td>$2\pi/160$</td>
<td>4.956802e-03</td>
<td>6.429738e-04</td>
</tr>
</tbody>
</table>

Table 4: Errors and rates of convergence in time for dG(2) and ($P_{2}^{\text{bubble}}, P_{1}^{\text{disc}}$) on level = 7 with $\tau_K = 0.1\ h_K$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$|u - u_h|_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\pi/10$</td>
<td>4.240334e-02</td>
</tr>
<tr>
<td>$2\pi/20$</td>
<td>8.815994e-03</td>
</tr>
<tr>
<td>$2\pi/40$</td>
<td>1.660338e-03</td>
</tr>
<tr>
<td>$2\pi/80$</td>
<td>4.149388e-04</td>
</tr>
<tr>
<td>$2\pi/160$</td>
<td>1.589712e-04</td>
</tr>
</tbody>
</table>

Figure 2: Initial condition for rotating body problem.

zero outside the three bodies. Let

$$r(x, y) = \frac{1}{r_0} \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$  

The center of the slotted cylinder is in $(x_0, y_0) = (0.5, 0.75)$ and its geometry is given by

$$u_0(x, y) = \begin{cases} 
1 & \text{if } r(x, y) \leq 1, \ |x - x_0| \geq 0.0225 \\
0 & \text{or } y \geq 0.85, \\
0 & \text{else.}
\end{cases}$$
The conical body at the bottom side is described by its center \((x_0, y_0) = (0.5, 0.25)\) and
\[ u_0(x, y) = 1 - r(x, y). \]
Finally, the hump at the left side is given by \((x_0, y_0) = (0.25, 0.5)\) and
\[ u_0(x, y) = \frac{1}{4}(1 + \cos(\pi \min\{r(x, y), 1\})). \]

The rotation of the body occurs counter-clockwise and the first full revolution takes place at \(T = 2\pi\) which is consider as final time. In the original example, see [42], the pure transport problem was considered and after each revolution one obtain the initial condition. In our numerical studies we have used the case of very small diffusion \((\varepsilon = 10^{-20})\). Hence, the results obtained by our method are very closed to the initial condition. The numerical solutions were compared with the initial condition \(u_0\). We present \(\|U - u\|_{L^2(L^2)}\) and
\[ \text{var}(t) := \max_{(x,y)\in\Omega} U_h(t; x, y) - \min_{(x,y)\in\Omega} U_h(t; x, y), \]
where the maximum and the minimum were computed in the vertices of the mesh cells. The values \(\|U_h - u\|_{L^2(L^2)}\) give some indication of the accuracy of the method and the smearing in the numerical solution whereas \(\text{var}(t)\) measures the size of the spurious oscillations. The optimal value is \(\text{var}(t) = 1\) for all \(t \in [0, T]\).

We have used triangular and quadrilateral meshes which are generated by successive refinement starting from the coarsest meshes (level 0) which are shown in Fig. 3.

![Figure 3: Meshes on level 0 for Example 2.](image)

The results computed for the dG(1) in time with time step length \(k = 2\pi \times 10^{-3}\) and the pairs \((Q_1^{\text{bubble}}/P_0^{\text{disc}})\) and \((P_1^{\text{bubble}}/P_0^{\text{disc}})\) on level 7 are listed in Tables 5 and 6 and are plotted in Fig. 4. Note that the same meshes were used in [32]. For the higher order methods \((Q_2^{\text{bubble}}/P_1^{\text{disc}})\) or \((Q_2^{\text{bubble}}/P_1^{\text{disc}})\) with dG(2) using \(k = 8\pi \times 10^{-3}\) we list the results in Table 7 and 8 and plot them in Fig. 5.

From Tables 5 and 6 for the first order discretizations, we see that the \(L^2\)-error decreased initially since the oscillations becomes smaller. However,
Figure 4: Body rotation problem, the computed solution for $(Q_{\text{bubble}}/P_{\text{disc}})$ with $\tau_0 = 0.1$, $\tau_0 = 10.0$ and for $(P_{\text{bubble}}/P_{\text{disc}})$ with $\tau_0 = 0.1$ and $\tau_0 = 10.0$ at $t = 2\pi$; from top to bottom.
Figure 5: Body rotation problem, the computed solution for \( \frac{Q_{\text{bubble}}}{P_{\text{disc}}} \) with \( \tau_0 = 0.5 \), \( \tau_0 = 10.0 \) and for \( \frac{P_{\text{bubble}}}{P_{\text{disc}}} \) with \( \tau_0 = 0.05 \) and \( \tau_0 = 10.0 \) at \( t = 2\pi \); from top to bottom.
increasing $\tau_0$ further, $L^2$-error increases due to smearing. The results concerning
the variations differ on the underlying meshes. On quadrilateral, the variations
are decreasing by increasing $\tau_0$, see Table 5 and the second picture in Fig. 4.
On triangles, an increase of $\tau_0$ causes an increase of the variations, see Table 6
and last picture in Fig. 4.

In the second order discretization, the $L^2$-errors are increasing when $\tau_0$ be-
comes larger, see Tables 7 and 8 and Fig. 5.
Table 8: Body rotation ($P_{2}^{\text{bubble}}/P_{1}^{\text{disc}}$).

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$|U - u|_{L^2(L^2)}$</th>
<th>$\text{var}(2\pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.11584</td>
<td>1.30134</td>
</tr>
<tr>
<td>0.05</td>
<td>0.116975</td>
<td>1.29865</td>
</tr>
<tr>
<td>0.1</td>
<td>0.118591</td>
<td>1.33371</td>
</tr>
<tr>
<td>0.5</td>
<td>0.119912</td>
<td>1.40233</td>
</tr>
<tr>
<td>1.0</td>
<td>0.119735</td>
<td>1.41362</td>
</tr>
<tr>
<td>2.0</td>
<td>0.120005</td>
<td>1.40841</td>
</tr>
<tr>
<td>5.0</td>
<td>0.121273</td>
<td>1.41464</td>
</tr>
<tr>
<td>10.0</td>
<td>0.12278</td>
<td>1.42961</td>
</tr>
</tbody>
</table>

6. Conclusion

We have analyzed the error estimate for the time dependent convection-diffusion-reaction problem with local projection stabilization in space and discontinuous Galerkin method in time. The optimal error estimates in strong and weak norms have been obtained.

From the numerical results, we find that the parameters of LPS leads to different influences to first and second order schemes. The first order schemes are more sensitive for the parameters of LPS than the second order schemes.

Acknowledgments

The authors acknowledges the financial support from the Federal Ministry of Education and Research (BMBF) under grant 03TOPAA1, Germany, and Kohat University of Science and Technology (KUST-HEC), Pakistan.

References


