Finite element methods of an operator splitting applied to population balance equations

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In population balance equations, the distribution of the entities depends not only on space and time but also on their own properties referred to as internal coordinates. The operator splitting method is used to transform the whole time-dependent problem into two unsteady subproblems of a smaller complexity. The first subproblem is a time-dependent convection-diffusion problem while the second one is a transient transport problem with pure advection. We use the backward Euler method to discretize the subproblems in time. Since the first problem is convection-dominated, the local projection method is applied as stabilization in space. The transport problem in the one-dimensional internal coordinate is solved by a discontinuous Galerkin method. The unconditional stability of the method will be presented. Optimal error estimates are given. Numerical results confirm the theoretical predictions.

Keywords: Operator splitting; discontinuous Galerkin; stabilized finite elements; population balance equations.

AMS Subject Classification: 65M12, 65M15, 65M60

1. Introduction

In this paper, we advocate the operator splitting method to approximate the solutions of population balance equations (PBE). This type of problems arises e.g. from models in the simulation of industrial crystallization process.[20] In PBE, the distribution of entities depends not only on space and time but also on their own properties referred to as internal coordinates and the source term usually involves an integral operator. For efficient methods to handle integral operators we refer to

Ref. 29. In this work we consider the source as a known function but still we have a high dimensional system of equations which is challenging from the computational point of view.

In order to overcome the difficulty of higher dimensions, the operator splitting method is used to decompose the original problem into two unsteady subproblems of smaller complexity. The first subproblem is a time-dependent convection-diffusion problem while the second one is a transport problem with pure advection. Operator splitting methods are widely used for time integration of unsteady problems. The basic theory of operator splitting for one-dimensional problems can be found in Ref. 41, 45. The concept of operator splitting for time-dependent problems is to decompose the spatial operator into a sum of two or more operators. For example in Ref. 33, the decomposition of convection-diffusion-reaction problem into pure convection and diffusion-reaction problems was studied. For more details about operator splitting methods for linear and non-linear convection-diffusion problems, see Ref. 22, 23, 24, 25, 30.

A detailed analysis of an alternating direction implicit (or operator-splitting) scheme is demonstrated in Ref. 26 for the Fokker-Planck equation. The basic idea in Ref. 26 is to split the high dimensional problem into two low dimensional problems corresponding to the configuration and the physical spaces. The solution of the convection-diffusion type problem in configuration space is obtained by a Galerkin spectral method at each quadrature point corresponding to the physical domain. Furthermore, a type of L^2 projection is used to update the right-hand side vector at the second stage where the solution of advection equation in physical space is approximated by a finite element method.

The main advantage of such splitting is that each subproblem can be discretized and stabilized separately by the best suitable method independently of the other subproblem(s). For example in Ref. 13 the Streamline-Upwind Petrov-Galerkin method (SUPG) has been combined with the standard Galerkin method. The main disadvantage of SUPG scheme, in particular for unsteady problems, is that several terms which include the time derivative, the source term, and second order derivatives have to be added into the stabilizing term in order to ensure the consistency of the resulting method.

There are several other stabilization techniques as alternative to SUPG. We mention the local projection stabilization (LPS),[3,4,34] the continuous interior penalty method (CIP),[5,6,7] the subgrid scale modeling (SGS),[15,32] and the orthogonal subscales method (OSS).[10,11] The LPS method was originally proposed for the Stokes problem in Ref. 2 as a two-level approach and extended to transport problems in Ref. 3. An analysis of the local projection stabilization applied to Oseen problems can be found in Ref. 4, 34 and for scalar convection-diffusion problems with mixed boundary condition in Ref. 35. The LPS in space combined with a discontinuous Galerkin (dG) method in time for transient convection-diffusion-reaction equations was studied in Ref. 1. A comparison of one- and two-level approaches of local projection stabilization for linear advection-diffusion-reaction problem is presented in

Ref. 27. For more details about local projection stabilization we refer to Ref. 43 where an overview on recent development for this class of stabilization method applied to scalar convection-diffusion, Stokes and linearized Navier-Stokes problems is given. In this article, we will concentrate on the one-level local projection stabilization technique.

The second subproblem in our splitting method is a transport problem with pure advection, so one suitable choice is to approximate it by the discontinuous Galerkin (dG) method. The dG method was first introduced for the neutron transport problem in Ref. 39 and then analyzed in Ref. 31. The theoretical analysis of the dG method for scalar hyperbolic equations can be found in Ref. 21 and for the space-time dG method in Ref. 12. For an introduction to discontinuous Galerkin method we refer to Ref. 9. The application of dG method makes the mass matrix corresponding to the internal coordinate diagonal which leads to the feasibility of parallel implementation without any projection steps between the two sub-steps during the computing process.

The aim of the present paper is to combine the local projection stabilization method in space with discontinuous Galerkin method in internal coordinate. We will give the stability and convergence estimates for fully discrete two-step scheme based on an operator splitting method.

The format of the paper is as follows: Section 2 introduces the model problem under consideration and defines basic notations. In Section 3, the operator splitting technique is applied to decompose the problem into two simpler ones. We shall formulate the backward Euler discretization and derive the weak form of the two subproblems. Further, we derive the unconditional stability of the two-step method. We then discretize the subproblems in space and internal coordinate using local projection stabilization and discontinuous Galerkin methods, respectively, in Section 4. We show the unconditional stability of the fully discrete two-step method. Section 5 presents the error analysis of the fully discrete scheme. Some implementation issues of the method are discussed in Section 6. Finally, we present in Section 7 some computational results supporting our theoretical results.

2. Model problem

Let Ω_x be a domain in $\mathbb{R}^d$ $(d = 2 \,\text{or}\, 3)$ with boundary $\partial\Omega_x$, $\Omega_\ell = [\ell_{\min}, \ell_{\max}] \subset \mathbb{R}$ and $T > 0$. The state of individual particle in population balance equation may consists of external coordinate x, referring to its position in the physical space, and internal coordinate ℓ, representing the properties of particles, such as size, temperature, volume etc. A population balance for a solid process such as crystallization with one internal coordinate can be described by the following partial differential equation:

Find $z : (0,T) \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial (Gz)}{\partial \ell} - \varepsilon \Delta_x z + \mathbf{b}(x)\cdot\nabla_x z = f & \text{in } (0,T]\times\Omega_\ell\times\Omega_x, \\ z(0,\cdot) = z_0 & \text{in } \Omega_\ell\times\Omega_x, \\ z|_{\ell_{\min}} = z_{\min} & \text{on } (0,T]\times\Omega_x, \\ z = 0 & \text{on } (0,T]\times\Omega_\ell\times\partial\Omega_x, \end{cases} \tag{2.1}$$ {ss1_e1}

where the diffusion coefficient $\varepsilon > 0$ is a given constant, Δ_x and ∇_x represent the Laplacian and gradient with respect to x, respectively, $\mathbf{b}$ is a given velocity field satisfying $\nabla_x \cdot \mathbf{b} = 0$, and f is a source function. Here $G > 0$ represents the growth rate of the particles that depends on ℓ but is independent of x and t, we also assume that $\partial_\ell G \geq 0$, see Ref. 36, 37. Furthermore, let us consider the data of the problem G, $\mathbf{b}$, f, z_0 and $z_{\min}$ to be sufficiently smooth functions.

Let us introduce some standard notations. Let $H^m(\Omega)$ denote the Sobolev space of functions with derivatives up to order m in $L^2(\Omega)$. We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega_\ell\times\Omega_x)$ and by $\|\cdot\|_0$ the corresponding L^2-norm defined by

$$(v,w) = \int_{\Omega_\ell\times\Omega_x} vw \, d\ell dx \qquad \text{and} \qquad \|v\|_0^2 = (v,v).$$

To distinguish the inner products and the corresponding norms with respect to the internal coordinate and the space variable we need some more notations. For this, let us denote by $(\cdot,\cdot)_\ell$ and $\|\cdot\|_{L^2(\Omega_\ell)}$ the L^2-inner product and the associated norm in Ω_ℓ, respectively, and by $(\cdot,\cdot)_x$ and $\|\cdot\|_{L^2(\Omega_x)}$ the L^2-inner product and the associated norm in Ω_x, i.e.,

$$\big(v,w\big)_\ell = \int_{\Omega_\ell} vw \,\mathrm{d}\ell \quad \text{and} \quad \|v\|^2_{L^2(\Omega_\ell)} = (v,v)_\ell,$$
$$\big(v,w\big)_x = \int_{\Omega_x} vw \,\mathrm{d}x \quad \text{and} \quad \|v\|^2_{L^2(\Omega_x)} = (v,v)_x.$$

The norm in the Sobolev space $H^m(\Omega_x)$ is defined as

$$\|v\|_m = \left(\sum_{|\alpha|\leq m} \|D^\alpha v\|^2_{L^2(\Omega_x)} \right)^{1/2}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ is a multi-index. We also consider certain Bochner spaces. For this, let X be a Banach space with norm $\|\cdot\|_X$. Then we define

$$C(\Omega_\ell; X) = \Big\{ v : \Omega_\ell \to X, \quad v \text{ continuous} \Big\},$$
$$L^2(\Omega_\ell; X) = \Big\{ v : \Omega_\ell \to X, \quad \int_{\Omega_\ell} \|v(\ell)\|_X^2 d\ell < \infty \Big\},$$
$$H^m(\Omega_\ell; X) = \Big\{ v \in L^2(\Omega_\ell; X) : \frac{\partial^j v}{\partial \ell^j} \in L^2(\Omega_\ell; X), \quad 1 \leq j \leq m \Big\},$$

where the derivatives $\partial^j v/\partial\ell^j$ are understood in the sense of distribution on Ω_ℓ. For spaces X and Y we use the short notation $Y(X) := Y(\Omega_\ell; X)$. The norms in the above defined spaces are given as follows

$$\|v\|_{C(X)} = \sup_{\ell\in\Omega_\ell} \|v(\ell)\|_X, \qquad \|v\|_{L^2(X)} = \left(\int_{\Omega_\ell} \|v(\ell)\|_X^2 d\ell\right)^{1/2},$$

$$\|v\|_{H^m(X)} = \left(\int_{\Omega_\ell} \sum_{j=0}^{m} \left\|\frac{\partial^j v}{\partial\ell^j}\right\|_X^2 d\ell\right)^{1/2}.$$

3. Operator splitting method

The numerical method for solving (2.1) in $d+1$ variable is based on an operator splitting with respect to (ℓ, t) and (x, t) in Ω_ℓ and Ω_x direction, respectively. We consider a uniform partition of the time interval $\tau = T/N$, i.e. $t^n = \tau n$, $n = 1, \ldots, N$. Then starting with $u(t^0) = z_0$, two subproblems are sequentially solved on the sub-intervals $(t^n, t^{n+1}]$, $n = 0, 1, \ldots, N-1$:

Given $u(t^n)$ find $\tilde{u} : (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial\tilde{u}}{\partial t} + L_x\tilde{u} = f & \text{in } (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \\ \tilde{u} = 0 & \text{on } (t^n, t^{n+1}] \times \Omega_\ell \times \partial\Omega_x \\ \tilde{u}(t^{n+}) = u(t^n). & \end{cases} \tag{3.1}$$

{spl1}

Find $u : (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + L_\ell u = 0 & \text{in } (t^n, t^{n+1}] \times \Omega_\ell \times \Omega_x \\ u|_{\ell_{\min}} = z_{\min} & \text{on } (t^n, t^{n+1}] \times \Omega_x \\ u(t^{n+}) = \tilde{u}(t^{n+1}), & \end{cases} \tag{3.2}$$

{spl2}

where

$$L_\ell z = \frac{\partial(Gz)}{\partial\ell}, \qquad L_x z = -\varepsilon\Delta_x z + \mathbf{b}\cdot\nabla_x z. \tag{3.3}$$

{ss1_e2}

This two-steps operator splitting scheme defines $u(t^n)$, $n = 1, \ldots, N$, as an approximation of $z(t^n)$.

In the framework of PBE, the first subproblem (3.1) is a time-dependent convection-diffusion equation posed on Ω_x parameterized by the variable ℓ in internal coordinate and the second subproblem (3.2) is a one-dimensional transport problem on Ω_ℓ parameterized by the space variable x.

Let us consider the spaces $V = H_0^1(\Omega_x)$ and $W = H^1(\Omega_\ell)$. We introduce the space

$$\mathcal{P} = \Big\{v \in L^2(\Omega_\ell \times \Omega_x) : v \in L^2(\Omega_x; W) \cap L^2(\Omega_\ell; V)\Big\}.$$

A variational form of (3.1) and (3.2) reads as follows:

First step: Find $\tilde{u} : (t^n, t^{n+1}] \to \mathcal{P}$ with $\tilde{u}(t^{n+}) = u(t^n)$ such that

$$\int_{\Omega_\ell} \big(\tilde{u}_t, v\big)_x \, d\ell + \int_{\Omega_\ell} a(\tilde{u}, v)\, d\ell = \int_{\Omega_\ell} \big(f, v\big)_x \, d\ell \qquad \forall v \in \mathcal{P}, \tag{3.4}$$ {s2_e1}

where the bilinear form a is defined as

$$a(u, v) = \varepsilon(\nabla_x u, \nabla_x v)_x + (\mathbf{b} \cdot \nabla_x u, v)_x.$$

Second step: Find $u : (t^n, t^{n+1}] \to \mathcal{P}$ with $u(t^{n+}) = \tilde{u}(t^{n+1})$ such that

{s2_e2} $$\int_{\Omega_\ell} \big(u_t, v\big)_x \, d\ell + b\big(u, v\big) = \big((Gz)_{\min}, v(\ell_{\min})\big)_x \qquad \forall v \in \mathcal{P}, \tag{3.5}$$

where $w_{\min} = w(\ell_{\min})$ and the bilinear form b is defined as

$$b(u, v) = \int_{\Omega_\ell} \Big(\frac{\partial(Gu)}{\partial \ell}, v\Big)_x \, d\ell + \Big((Gu)(\ell_{\min}), v(\ell_{\min})\Big)_x.$$

Note that we have imposed the boundary condition $(u|_{\ell_{\min}} = z_{\min})$ in ℓ-direction in weak sense.

After discretizing in time by the backward Euler method, the first order accurate implicit scheme is considered as two-step method:

First step: Given $u^n \in \mathcal{P}$, find $\tilde{u}^{n+1} \in \mathcal{P}$ such that

{s2_e2a} $$\int_{\Omega_\ell} \Big(\frac{\tilde{u}^{n+1} - u^n}{\tau}, v\Big)_x \, d\ell + \int_{\Omega_\ell} a(\tilde{u}^{n+1}, v)\, d\ell = \int_{\Omega_\ell} (f^{n+1}, v)_x \, d\ell \tag{3.6}$$

for all $v \in \mathcal{P}$.

Second step: Update $\tilde{u}^{n+1}$ from the first step and find the solution $u^{n+1} \in \mathcal{P}$ such that

{s2_e5} $$\int_{\Omega_\ell} \Big(\frac{u^{n+1} - \tilde{u}^{n+1}}{\tau}, v\Big)_x \, d\ell + b(u^{n+1}, v) = \Big(G_{\min} z_{\min}^{n+1}, v(\ell_{\min})\Big)_x \tag{3.7}$$

for all $v \in \mathcal{P}$, where $z_{\min}^{n+1} = z_{\min}(t^{n+1}, \cdot)$.

The next paragraph gives the stability of the two-step method (3.6) and (3.7).

Lemma 3.1 (Stability). *Assume that $\tilde{u}^n$, u^n , $n = 1, 2 \ldots, N$, is the solution obtained from the two-step algorithm (3.6) and (3.7). If $\partial_\ell G \geq 0$ and $\tau \leq \frac{1}{4}$, then the following estimate shows the stability*

{s2_e6} $$\begin{aligned} &\big\|u^N\big\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big\{2\varepsilon \big\|\tilde{u}^{n+1}\big\|_{H^1(\Omega_x)}^2 + \partial_\ell G \big\|u^{n+1}\big\|_{L^2(\Omega_x)}^2\Big\} \, d\ell \\ &\qquad \leq \exp(3T/2) \left\{ \big\|u^0\big\|_0^2 + \tau \sum_{n=0}^{N-1} \Big(2\big\|f^{n+1}\big\|_0^2 + \big\|G_{\min}^{1/2} z_{\min}^{n+1}\big\|_{L^2(\Omega_x)}^2\Big) \right\}. \end{aligned} \tag{3.8}$$

Proof. Setting $v = \tilde{u}^{n+1}$ in (3.6), yields

$$\int_{\Omega_\ell} (\tilde{u}^{n+1} - u^n, \tilde{u}^{n+1})_x \, d\ell + \tau \int_{\Omega_\ell} a(\tilde{u}^{n+1}, \tilde{u}^{n+1}) \, d\ell = \tau \int_{\Omega_\ell} (f^{n+1}, \tilde{u}^{n+1})_x \, d\ell.$$

Using the relation $2(a-b)a = a^2 - b^2 + (a-b)^2$, one can write

$$\int_{\Omega_\ell} (\tilde{u}^{n+1} - u^n, \tilde{u}^{n+1})_x \, d\ell = \frac{1}{2}\|\tilde{u}^{n+1}\|_0^2 - \frac{1}{2}\|u^n\|_0^2 + \frac{1}{2}\|\tilde{u}^{n+1} - u^n\|_0^2.$$

Integrating by parts with respect to x the second term in the bilinear form $a(\cdot,\cdot)$, one obtains

$$\int_{\Omega_\ell} a(\tilde{u}^{n+1}, \tilde{u}^{n+1}) \, d\ell = \varepsilon \int_{\Omega_\ell} \|\tilde{u}^{n+1}\|_{H^1(\Omega_x)}^2 \, d\ell$$

since $\tilde{u}^{n+1}$ vanishes on the boundary $\partial\Omega_x$ and $\nabla_x \cdot \mathbf{b} = 0$. Hence by using Cauchy-Schwarz inequality for the right-hand side, we have for the first step

$$\|\tilde{u}^{n+1}\|_0^2 - \|u^n\|_0^2 + \|\tilde{u}^{n+1} - u^n\|_0^2 + 2\tau\varepsilon \int_{\Omega_\ell} \|\tilde{u}^{n+1}\|_{H^1(\Omega_x)}^2 d\ell \\ \leq \tau\|f^{n+1}\|_0^2 + \tau\|\tilde{u}^{n+1}\|_0^2. \qquad (3.9) \quad \{s2\_e7\}$$

Substituting $v = u^{n+1}$ in the second step (3.7) gives

$$\int_{\Omega_\ell} (u^{n+1} - \tilde{u}^{n+1}, u^{n+1})_x d\ell + \tau b(u^{n+1}, u^{n+1}) = \tau \Big(G_{\min} z_{\min}^{n+1}, u^{n+1}(\ell_{\min})\Big)_x. \qquad (3.10) \quad \{s2\_e7a\}$$

Starting from

$$b(u^{n+1}, u^{n+1}) = \int_{\Omega_\ell} \Big(\frac{\partial(Gu^{n+1})}{\partial\ell}, u^{n+1}\Big)_x d\ell + \Big(G_{\min} u^{n+1}(\ell_{\min}), u^{n+1}(\ell_{\min})\Big)_x$$

an integration by parts twice with respect to ℓ gives

$$b(u^{n+1}, u^{n+1}) = \frac{1}{2}\int_{\Omega_\ell} \partial_\ell G \big\|u^{n+1}\big\|_{L^2(\Omega_x)}^2 \, d\ell + \frac{1}{2}\big\|G_{\max}^{1/2} u^{n+1}(\ell_{\max})\big\|_{L^2(\Omega_x)}^2 \\ + \frac{1}{2}\big\|G_{\min}^{1/2} u^{n+1}(\ell_{\min})\,\big\|_{L^2(\Omega_x)}^2.$$

where $G_{\max} = G(\ell_{\max})$. Cauchy-Schwarz inequality gives for the right-hand side in (3.10)

$$\Big(G_{\min} z_{\min}^{n+1}, u^{n+1}(\ell_{\min})\Big)_x \leq \frac{1}{2}\big\|G_{\min}^{1/2} z_{\min}^{n+1}\big\|_{L^2(\Omega_x)}^2 + \frac{1}{2}\big\|G_{\min}^{1/2} u^{n+1}(\ell_{\min})\big\|_{L^2(\Omega_x)}^2.$$

Combining these two results in (3.10) and using the same relation $2(a-b)a = a^2 - b^2 + (a-b)^2$ for first term, we get for second step

$$\big\|u^{n+1}\big\|_0^2 - \big\|\tilde{u}^{n+1}\big\|_0^2 + \big\|u^{n+1} - \tilde{u}^{n+1}\big\|_0^2 + \tau \int_{\Omega_\ell} \partial_\ell G \big\|u^{n+1}\big\|_{L^2(\Omega_x)}^2 \\ \leq \tau \big\|G_{\min}^{1/2} z_{\min}^{n+1}\big\|_{L^2(\Omega_x)}^2. \qquad (3.11) \quad \{s2\_e8\}$$

Adding (3.9) and (3.11), neglecting some contribution of positive terms, and summing over $n = 0, \dots, N-1$, we obtain

$$\|u^N\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\{ 2\varepsilon \|\tilde{u}^{n+1}\|^2_{H^1(\Omega_x)} + \partial_\ell G \|u^{n+1}\|^2_{L^2(\Omega_x)} \right\} d\ell$$
$$\le \|u^0\|_0^2 + \tau \sum_{n=0}^{N-1} \left\{ \|f^{n+1}\|_0^2 + \|G_{\min}^{1/2} z_{\min}^{n+1}\|^2_{L^2(\Omega_x)} \right\} + \tau \sum_{n=0}^{N-1} \|\tilde{u}^{n+1}\|_0^2.$$

From (3.9) we have

{s2_e8a} $$\|\tilde{u}^{n+1}\|_0^2 \le \frac{\tau}{1-\tau} \|f^{n+1}\|_0^2 + \frac{1}{1-\tau} \|u^n\|_0^2. \tag{3.12}$$

Using this estimate in the last inequality, we get

$$\|u^N\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\{ 2\varepsilon \|\tilde{u}^{n+1}\|^2_{H^1(\Omega_x)} + \partial_\ell G \|u^{n+1}\|^2_{L^2(\Omega_x)} \right\} d\ell$$
$$\le \|u^0\|_0^2 + \tau \sum_{n=0}^{N-1} \left\{ \frac{4}{3} \|f^{n+1}\|_0^2 + \|G_{\min}^{1/2} z_{\min}^{n+1}\|^2_{L^2(\Omega_x)} \right\} + \frac{4\tau}{3} \sum_{n=0}^{N-1} \|u^n\|_0^2,$$

where we have used $1/(1-\tau) \le 4/3$ for $\tau \le 1/4$. We conclude the statement by using Gronwall's lemma. This completes the proof. □

The critical issue of the operator splitting method is the overall accuracy of the two-step method. Using Taylor series expansions first order accuracy of the two-step method (3.1) and (3.2) can be shown. A detail error analysis for the first order Lie operator splitting of the sum of two elliptic operators can be found in Ref. 16, 17. Unfortunately, we can't use these results due to the hyperbolic nature of the operator L_ℓ.

4. Fully-discrete method

In view of different properties of operator L_ℓ and L_x, the operator splitting technique allows us to use different types of discretization methods to solve the problems in Ω_ℓ and Ω_x. Since the first subproblem (3.5) is convection-dominated, we use the local projection method to stabilize the space discretization. While the second subproblem (3.7) is a transport problem with pure advection, one suitable choice is the discontinuous Galerkin method for the discretization with respect to the internal coordinate.

4.1. *Local projection stabilization in space*

In this subsection, we discretize the subproblem in space. For this, let us denote by $\{\mathcal{T}_h\}$ a family of shape regular decompositions of Ω_x into d-simplices, quadrilateral or hexahedra. The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K and h describes the maximum diameter of cells K. We will consider the one-level LPS in which the approximation and projection space live on the same mesh. For other variants of LPS we refer to Ref. 18, 28, 34, 40, 43, 44.

Let $V_h \subset V$ denote the standard finite element space of continuous, piecewise polynomials of degree r. The Galerkin discretization of the problem (3.5) is generally unstable due to dominating advection when the diffusion coefficient is very small $\varepsilon \ll 1$. We will handle this difficulty by adding a stabilizing term based on local projection. Let $\mathcal{D}_h$ be the projection space of discontinuous, piecewise polynomials of degree $r-1$ with $r \geq 1$. Let $\mathcal{D}_h(K) = \{q_h|_K : q_h \in \mathcal{D}_h\}$ be the local projection space and $\pi_K : L^2(K) \to \mathcal{D}_h(K)$ the local L^2-projection onto $\mathcal{D}_h(K)$. Define the global projection $\pi_h : L^2(\Omega_x) \to \mathcal{D}_h$ by $(\pi_h v)|_K := \pi_K(v|_K)$. The fluctuation operator $\kappa_h : L^2(\Omega_x) \to L^2(\Omega_x)$ is given by $\kappa_h := id - \pi_h$, where $id : L^2(\Omega_x) \to L^2(\Omega_x)$ is the identity mapping.

We define the stabilizing term S_h

$$S_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \mu_K \Big(\kappa_h(\nabla_x u_h), \kappa_h(\nabla_x v_h) \Big)_K$$

with user chosen non-negative constant μ_K, $K \in \mathcal{T}_h$. It gives additional control over the fluctuations of gradients. Note that one can also replace the gradient $\nabla_x w_h$ by the derivative in streamline direction $\mathbf{b} \cdot \nabla_x w_h$ or (even better Ref. 27, 28) by $\mathbf{b}_K \cdot \nabla_x w_h$ where $\mathbf{b}_K$ is a piecewise constant approximation of $\mathbf{b}$, which leads to similar results.

The stabilized bilinear form is then defined as

$$a_h(u_h, v_h) = a(u_h, v_h) + S_h(u_h, v_h). \tag{4.1}$$ {ss3_1e3}

The bilinear form a_h is coercive on V_h with respect to the mesh dependent norm

$$|||v||| := \Big(\varepsilon |v|^2_{H^1(\Omega_x)} + \sum_{K \in \mathcal{T}_h} \mu_K \|\kappa_h(\nabla_x v)\|^2_{L^2(K)} \Big)^{1/2}, \tag{4.2}$$ {ss3_1e4}

that is $a_h(v_h, v_h) \geq |||v_h|||^2$ for all $v_h \in V_h$. The stability and convergence properties of the LPS method are based on the following assumptions with respect to the pair (V_h, D_h), see Ref. 34, 40.

Assumption A1 : There is an interpolation operator $j_h : H^2(\Omega) \to V_h$ such that the approximation properties,

$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \leq C h_k^l \|v\|_{l,K} \quad \forall v \in H^l(\Omega_x),\ 2 \leq l \leq r+1, \tag{4.3}$$ {ss3_1e6}

for all $K \in \mathcal{T}_h$ and the orthogonality

$$(v - j_h v, q_h) = 0 \qquad \forall q_h \in \mathcal{D}_h, \forall v \in H^2(\Omega) \tag{4.4}$$ {ss3_1e7}

hold true.

Assumption A2 : The fluctuation operator κ_h satisfies the following approximation property

$$\|\kappa_h q\|_{0,K} \le C h_K^l |q|_{l,K} \quad \forall K \in \mathcal{T}_h,\ \forall q \in H^l(K),\ 0 \le l \le r. \tag{4.5}$$ {ss3_1e8}

In numerical computations, we use mapped finite element spaces, see Ref. 8, where on the reference cell $\widehat{K}$ the enriched spaces are given by

$$P_r^{\text{bubble}}(\widehat{K}) = P_r(\widehat{K}) + \hat{b}_\triangle P_{r-1}(\widehat{K}),$$
$$Q_r^{\text{bubble}}(\widehat{K}) = Q_r(\widehat{K}) + \operatorname{span}\{\hat{b}_\Box \hat{x}_i^{r-1},\ i = 1, 2\}.$$

Here, $\hat{b}_\triangle$ and $\hat{b}_\Box$ are the cubic bubble on the reference triangle and the biquadratic bubble on the reference square, respectively. The pairs $(P_r^{\text{bubble}}, P_{r-1}^{\text{disc}})$, $r \ge 1$, on triangles and the pairs $(Q_r^{\text{bubble}}, P_{r-1}^{\text{disc}})$, $r \ge 1$, on quadrilaterals fulfill assumptions A1 and A2. Further examples of spaces (V_h, D_h) satisfying A1 and A2 are given in Ref. 34, 40.

4.2. *Discontinuous Galerkin method in internal coordinate*

To discretize (3.5) and (3.7) in internal coordinate ℓ, we apply a discontinuous Galerkin method. Let $M > 0$ be a given positive integer and $\ell_{\min} = \ell_0 < \ell_1 < \cdots < \ell_M = \ell_{\max}$ is a partition of Ω_ℓ with $I_i = (\ell_{i-1}, \ell_i]$, $k_i = \ell_i - \ell_{i-1}$, and $k = \max_i k_i$. Let us introduce the function space of discontinuous piecewise polynomials of degree $q \ge 1$ as

$$S_k^q = \Big\{ v : \Omega_\ell \to \mathbb{R} : v|_{I_i}(\ell) = \sum_{j=0}^{q} v_j \ell^j \quad \text{with} \quad v_j \in \mathbb{R},\ j = 0, \dots, q \Big\}.$$

Then we give the fully discrete space $S_{h,k}^{r,q}$ as follows

{ss3_1e10}
$$\begin{aligned} S_{h,k}^{r,q} &= V_h \otimes S_k^q \\ &= \Big\{ v : \Omega_\ell \times \Omega_x \to \mathbb{R} : v|_{I_i}(\ell) = \sum_{j=0}^{q} v_j \ell^j \quad \text{with} \quad v_j \in V_h,\ j = 0, \dots, q \Big\}. \end{aligned} \tag{4.6}$$

The functions in these spaces are allowed to be discontinuous at the nodes ℓ_i, $i = 1, \dots, M-1$. The jumps across the nodes are defined by $[\phi]_i = \phi(\ell_i^+) - \phi(\ell_i^-)$, where

$$\varphi_m^\pm = \varphi(\ell_m^\pm) = \lim_{\ell \to \ell_m \pm 0} \varphi(\ell).$$

In the next paragraph, we define the fully discrete scheme based on two-step method.

First step : For given $u_{h,k}^n \in S_{h,k}^{r,q}$, find $\tilde{u}_{h,k}^{n+1} \in S_{h,k}^{r,q}$ such that

{ss3_1e11}
$$\int_{\Omega_\ell} \Big(\frac{\tilde{u}_{h,k}^{n+1} - u_{h,k}^n}{\tau}, X \Big)_x d\ell + \int_{\Omega_\ell} a_h(\tilde{u}_{h,k}^{n+1}, X)\, d\ell = \int_{\Omega_\ell} (f^{n+1}, X)_x\, d\ell \tag{4.7}$$

for all $X \in S_{h,k}^{r,q}$ where $u_{h,k}^0$ is a suitable approximation of z_0 in $S_{h,k}^{r,q}$.

Second step : Update the solution $\tilde{u}_{h,k}^{n+1}$ from (4.7) and find $u_{h,k}^{n+1} \in S_{h,k}^{r,q}$ such that

{ss3_1e12}
$$\int_{\Omega_\ell} \Big(\frac{u_{h,k}^{n+1} - \tilde{u}_{h,k}^{n+1}}{\tau}, X \Big)_x \, d\ell + B(u_{h,k}^{n+1}, X) = \Big(G_{\min} z_{\min,\mathrm{h}}^{n+1}, X(\ell_0^+) \Big)_x \tag{4.8}$$

for all $X \in S_{h,k}^{r,q}$, where $z_{\min,\mathrm{h}}^{n+1} \in S_{h,k}^{r,q}$ is an approximation of $z_{\min}^{n+1}$ and the bilinear form B is defined as

$$B(u,v) = \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial (Gu)}{\partial \ell}, v \Big)_x \, d\ell + \sum_{i=1}^{M-1} \Big(\big[(Gu)\big]_i, v(\ell_i^+) \Big)_x + \Big(G_{\min} u(\ell_0^+), v(\ell_0^+) \Big)_x . \tag{4.9}$$
{b_uv_1}

Integrating by parts

$$\int_{I_i} \Big(\frac{\partial (Gu)}{\partial \ell}, v \Big)_x \, d\ell = \Big((Gu)(\ell_i^-), v(\ell_i^-) \Big)_x - \Big((Gu)(\ell_{i-1}^+), v(\ell_{i-1}^+) \Big)_x - \int_{I_i} \Big(Gu, \frac{\partial v}{\partial \ell} \Big)_x \, d\ell$$

leads to the representation

$$B(u,v) = -\sum_{i=1}^{M} \int_{I_i} \Big(Gu, \frac{\partial v}{\partial \ell} \Big)_x \, d\ell - \sum_{i=1}^{M-1} \Big(u(\ell_i^-), \big[(Gv)\big]_i \Big)_x + \Big(G_{\max} u(\ell_M^-), v(\ell_M^-) \Big)_x . \tag{4.10}$$
{b_uv_2}

We introduce the mesh dependent norm

$$\begin{aligned} \|v\|_{\mathrm{dG}}^2 = {} & \sum_{i=1}^{M} \int_{I_i} \partial_\ell G \|v\|_{L^2(\Omega_x)}^2 \, d\ell + \big\| G_{\min}^{1/2} v(\ell_0^+) \big\|_{L^2(\Omega_x)}^2 + \sum_{i=1}^{M-1} \big\| \big[(G^{1/2} v)\big]_i \big\|_{L^2(\Omega_x)}^2 \\ & + \big\| G_{\max}^{1/2} v(\ell_M^-) \big\|_{L^2(\Omega_x)}^2 . \end{aligned} \tag{4.11}$$
{dg_norm}

Lemma 4.1. *The bilinear form B is coercive with respect to the mesh dependent norm $\|\cdot\|_{\mathrm{dG}}$, i.e.,*

$$B(v,v) \geq \frac{1}{2} \|v\|_{\mathrm{dG}}^2 . \tag{4.12}$$
{B_coer}

holds for all $v \in S_{h,k}^{r,q}$.

Proof. Setting $u = v$ in (4.9) and (4.10), then adding them together we conclude the statement of the lemma. □

The next lemma provides a stability result of the fully discrete two-step method (4.7) and (4.8).

Lemma 4.2 (Stability). *Let $\partial_\ell G \geq 0$ and $\tau \leq 1/2$, then the solution $\tilde{u}_{h,k}^n$ and $u_{h,k}^n$, $n = 1, 2, \ldots, N$, of (4.7) and (4.8),respectively, satisfies*

$$\begin{aligned} \|u_{h,k}^N\|_0^2 + 2\tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{u}_{h,k}^{n+1}|||^2 \, d\ell + \tau \sum_{n=0}^{N-1} \|u_{h,k}^{n+1}\|_{\mathrm{dG}}^2 \\ \leq \exp(3T/2) \left\{ \|u_{h,k}^0\|_0^2 + \tau \sum_{n=0}^{N-1} \left(\frac{4}{3} \|f^{n+1}\|_0^2 + \|(G_{\min}^{1/2} z_{\mathrm{min,h}}^{n+1})\|_{L^2(\Omega_x)}^2 \right) \right\}. \end{aligned} \tag{4.13}$$ {ss3_1e19}

Proof. Following the similar derivation steps as in Lemma 3.1, we get the proof of lemma. □

5. Error analysis

In this section, we derive the error estimates of the fully discrete two-step scheme (4.7) and (4.8). First we define a special interpolant $\Pi_k w(t, \cdot, x) \in S_k^q$ of a function $w(t, \ell, x)$ by

{int_pi1} $$\Pi_k w(\ell_i^-) = w(\ell_i^-), \qquad i = 1, \ldots, M-1, \tag{5.1}$$

{int_pi2} $$\int_{I_i} (\Pi_k w - w) \ell^s \, d\ell = 0, \qquad s \leq q-1, \, i \geq 1, \tag{5.2}$$

i.e., $\Pi_k w$ interpolates at the nodal points and the interpolation error is orthogonal to the space of polynomials of degree $q-1$ on I_i. For this type of interpolant we have the following error estimates

{int_pi3} $$\sup_{0 \leq \ell \leq \ell_M} |\Pi_k w(\ell) - w(\ell)|_j \leq C k^{q+1} \sup_{0 \leq \ell \leq \ell_M} |w^{(q+1)}(\ell)|_j, \qquad j = 0, 1, \tag{5.3}$$

{int_pi4} $$\int_{I_i} |\Pi_k w^{(i)}(\ell) - w^{(i)}(\ell)|_j^2 \, d\ell \leq C k^{2(q+1-i)} \int_{I_i} |w^{(q+1)}(\ell)|_j \, d\ell, \qquad i, j = 0, 1, \tag{5.4}$$

see Ref. 38, 42. In order to obtain the error estimate for the splitting method in space and internal coordinate, we define a projection operator R_h which maps onto the tensor product space $S_{h,k}^{r,q}$. It is defined as follows

{R} $$R_h w = j_h \Pi_k w = \Pi_k j_h w \qquad \forall w \in \mathcal{P}, \tag{5.5}$$

where j_h is the special interpolant in space satisfying Assumption A1. In addition, we have the stability property of interpolant Π_k given by

{stb_jpi} $$\int_{\Omega_\ell} \|\Pi_k u\|_{H^{r+1}(\Omega_x)}^2 \, d\ell \leq C \int_{\Omega_\ell} \|u\|_{H^{r+1}(\Omega_x)}^2 \, d\ell \tag{5.6}$$

since Π_k acts in ℓ-direction and the norms are with respect to the space direction. Let us consider $\xi^n := u(t^n) - R_h u(t^n)$ and $\eta^n := R_h u(t^n) - u_{h,k}^n$. We also denote $\tilde{\xi}^n := \tilde{u}(t^n) - R_h \tilde{u}(t^n)$ and $\tilde{\eta}^n := R_h \tilde{u}(t^n) - \tilde{u}_{h,k}^n$, then the error $u(t^n) - u_{h,k}^n$ can be decomposed as follows

$$e^n = u(t^n) - u_{h,k}^n = \xi^n + \eta^n$$

where $u^n_{h,k}$ is the solution for fully discrete scheme (4.7) and (4.8) and $u(t^n)$ is the solution of (3.1) and (3.2). Furthermore, to obtain the separate estimates in space and internal coordinate we use the following decomposition of errors

$$R_h w - w = \big(R_h w - \Pi_k w\big) + \big(\Pi_k w - w\big) = \vartheta + \varphi. \tag{5.7}$$ {sp_sp_int}

Assumption A3 : Let $u,\ u_t,\ u_{tt},\ \tilde u,\ \tilde u_t,\ \tilde u_{tt},\ z_{\min}$ and z_0 satisfy the following regularity conditions

$$\begin{aligned}
&u,\, \tilde u \in H^1\big(L^2(H^{r+1})\big) \cap H^1\big(H^{q+1}(H^1)\big), u_t,\, \tilde u_t \in L^2\big(L^2(H^{r+1})\big) \cap L^2\big(H^{q+1}(L^2)\big),\\
&u_{tt},\, \tilde u_{tt} \in L^2\big(L^2(L^2)\big), \qquad z_0 \in L^2\big(\Omega_\ell; H^{r+1}(\Omega_x)\big) \cap H^{q+1}\big(\Omega_\ell; L^2(\Omega_x)\big),\\
&z_{\min} \in H^1\big(0,T; H^{r+1}(\Omega_x)\big).
\end{aligned}$$

Lemma 5.1. *Let the assumptions A1-A3 be fulfilled. Then for all $t \in (0,T]$, we have the following estimates for the interpolation error*

$$\begin{aligned}
\|\vartheta(t)\|_{\mathrm{dG}} &\le C\, h^{r+1}\Big\{ \|u(t)\|_{L^2(H^{r+1})} + \|u(t)\|_{C(H^{r+1})}\Big\},\\
\|\varphi(t)\|_{\mathrm{dG}} &\le C\, k^{q+1/2}\|u(t)\|_{H^{q+1}(L^2)}.
\end{aligned}$$

Proof. For simplicity we skip the dependency of t within the proof. Since for the interpolation error the jumps $[j_h u - u]_i$, $i = 1, \dots, M-1$, vanishes due to the continuity of $j_h u$ in internal coordinate, we have from (4.11), the interpolation error estimates (4.3) and condition (5.6)

$$\begin{aligned}
\|\vartheta\|^2_{\mathrm{dG}} &\le \sum_{i=1}^{M-1} \int_{I_i} \partial_\ell G \|\vartheta\|^2_{L^2(\Omega_x)}\, d\ell + \|G^{1/2}_{\min}\vartheta(\ell_0^+)\|^2_{L^2(\Omega_x)} + \|G^{1/2}_{\max}\vartheta(\ell_M^-)\|^2_{L^2(\Omega_x)}\\
&\le C\, h^{2r+2}\Big\{ \|u\|^2_{L^2(H^{r+1})} + \|u\|^2_{C(H^{r+1})}\Big\}.
\end{aligned}$$

For the second estimate with respect to the internal coordinate, we use the definition of interpolant $\Pi_k u$, i.e., the interpolation $\Pi_k u$ satisfies $\Pi_k u(\ell_i^-) = u(\ell_i)$, $i = 1, \dots, M$, thus from the second representation (4.10) of the bilinear form B and interpolation estimates (5.3), (5.4), we have

$$\begin{aligned}
\|\varphi\|^2_{\mathrm{dG}} \le B(\varphi,\varphi) &= \sum_{i=1}^{M} \int_{I_i} -\Big(G\varphi, \frac{\partial\varphi}{\partial\ell}\Big)_x\, d\ell\\
&\le \sum_{i=1}^{M} \int_{I_i} \|G\varphi\|_{L^2(\Omega_x)} \|\partial_\ell\varphi\|_{L^2(\Omega_x)}\, d\ell\\
&\le C\, k^{2q+1} \sum_{i=1}^{M} \int_{I_i} \|u^{q+1}\|^2_{L^2(\Omega_x)}\, d\ell \le C\, k^{2q+1}\|u\|^2_{H^{q+1}(L^2)}
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 5.2. *Let the assumptions A1-A3 be fulfilled and $\tau_K \sim h_K$. Then for all $t \in (0,T]$, the following estimates hold*

$$\int_{\Omega_\ell} a_h\big(\vartheta(t),\eta(t)\big)\,d\ell \le C\,(\varepsilon^{1/2}+h^{1/2})h^r\|u(t)\|_{L^2(H^{r+1})}\left(\int_{\Omega_\ell}|||\eta(t)|||^2\,d\ell\right)^{1/2} + C\,h^{r+1}\|u(t)\|_{L^2(H^{r+1})}\,\|\eta(t)\|_0, \tag{5.8}$$ {c4:lem7:atheta}

$$\int_{\Omega_\ell} a_h\big(\varphi(t),\eta(t)\big)\,d\ell \le C\,(\varepsilon^{1/2}+h^{1/2})\,k^{q+1}\|u(t)\|_{H^{q+1}(H^1)}\left(\int_{\Omega_\ell}|||\eta(t)|||^2\,d\ell\right)^{1/2} + C\,k^{q+1}\|u(t)\|_{H^{q+1}(H^1)}\|\eta(t)\|_0, \tag{5.9}$$ {c4:lem7:aphi}

$$B\big(\vartheta(t),\eta(t)\big) \le C\,h^{r+1}\Big\{\|u(t)\|_{H^1(H^{r+1})}\|\eta(t)\|_0 + \|u(t)\|_{C(H^{r+1})}\|\eta(t)\|_{\mathrm{dG}}\Big\}, \tag{5.10}$$ {c4:lem7:btheta}

$$B\big(\varphi(t),\eta(t)\big) \le C\,k^{q+1}\|u(t)\|_{H^{q+1}(L^2)}\|\eta(t)\|_0. \tag{5.11}$$ {c4:lem7:bphi}

Proof. For simplicity of the presentation we again skip the dependency of the time within the proof. From the definition of the stabilized bilinear form a_h, we have

{c4:lem7:e1}
$$\int_{\Omega_\ell} a_h\big(\vartheta,\eta\big)\,d\ell = \varepsilon\int_{\Omega_\ell}\big(\nabla_x\vartheta,\nabla_x\eta\big)_x + \int_{\Omega_\ell}\big(\mathbf{b}\cdot\nabla_x\vartheta,\eta\big)_x\,d\ell + \int_{\Omega_\ell} S_h\big(\vartheta,\eta\big)\,d\ell = I_1 + I_2 + I_3. \tag{5.12}$$

We start by estimating the first term on the right-hand side. Using Cauchy-Schwarz inequality, the interpolation estimates (4.3) of j_h and the condition (5.6), it follows that

$$\begin{aligned}|I_1| &\le \varepsilon\int_{\Omega_\ell}||\vartheta||_{H^1(\Omega_x)}||\eta||_{H^1(\Omega_x)}\,d\ell \le C\,\varepsilon^{1/2}h^r\int_{\Omega_\ell}\|\Pi_k u\|_{H^{r+1}}\,|||\eta|||\,d\ell\\ &\le C\varepsilon^{1/2}h^r\left(\int_{\Omega_\ell}\|u\|^2_{H^{r+1}}d\ell\right)^{1/2}\left(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\right)^{1/2}\\ &\le C\,\varepsilon^{1/2}h^r\|u\|_{L^2(H^{r+1})}\left(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\right)^{1/2}.\end{aligned}$$

Integrating I_2 by parts with respect to the space variable x, using the orthogonality property of interpolant j_h and Cauchy-Schwarz inequality to get

$$\begin{aligned}|I_2| &= \Big|\int_{\Omega_\ell}\big(\mathbf{b}\cdot\nabla_x\vartheta,\eta\big)_x\,d\ell\Big| = \Big|\int_{\Omega_\ell}\big(\vartheta,\mathbf{b}\cdot\nabla_x\eta\big)_x\,d\ell\Big|\\ &\le \Big|\int_{\Omega_\ell}\big(\vartheta,\kappa_h(\mathbf{b}\cdot\nabla_x\eta)\big)_x\,d\ell\Big|\\ &\le \int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}\|\vartheta\|_{L^2(K)}\|\kappa_h(\mathbf{b}\cdot\nabla_x\eta)\|_{L^2(K)}\,d\ell.\end{aligned}$$

Let $\overline{\mathbf{b}}$ be the L^2-projection of $\mathbf{b}$ in the space of piecewise constant functions with respect to $\mathcal{T}_h$. Using the L^2-stability of the fluctuation operator κ_h, inverse inequality and $\kappa_h(\overline{\mathbf{b}}\cdot\nabla_x)\eta = \overline{\mathbf{b}}\cdot\kappa_h(\nabla_x\eta)$, we get in a same fashion as in Ref. 34 the following estimate

{C4_th1_14a} $$\big\|\kappa_h(\mathbf{b}\cdot\nabla_x)\eta\big\|_{L^2(K)} \le C|\mathbf{b}|_{1,\infty,K}\|\eta\|_{L^2(K)} + \|\mathbf{b}\|_{0,\infty,K}\big\|\kappa_h(\nabla_x\eta)\big\|_{L^2(K)}. \qquad (5.13)$$

Thus inserting this in the previous estimate, using (4.3), $\mu_K \sim h_K$, and (5.6) to get

$$\begin{aligned}
|I_2| &\le C\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}|\mathbf{b}|_{1,\infty,K}\|\vartheta\|_{L^2(K)}\|\eta\|_{L^2(K)}\,d\ell\\
&\quad + C\int_{\Omega_\ell}\sum_{K\in\mathcal{T}_h}|\mathbf{b}|_{0,\infty,K}\|\vartheta\|_{L^2(K)}\big\|\kappa_h(\nabla_x\eta)\big\|_{L^2(K)}\,d\ell\\
&\le C\,h^{r+1}\int_{\Omega_\ell}\|u\|_{H^{r+1}(\Omega_x)}\|\eta\|_{L^2(\Omega_x)}\,d\ell + C\,h^{r+1/2}\int_{\Omega_\ell}\|u\|_{H^{r+1}(\Omega_x)}\,|||\eta|||\,d\ell\\
&\le C\,h^{r+1/2}\Big\{h^{1/2}\|\eta\|_0 + \Big(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\Big)^{1/2}\Big\}\|u\|_{L^2(H^{r+1})}.
\end{aligned}$$

For I_3, the Cauchy-Schwarz inequality and interpolation error estimates give

$$\begin{aligned}
|I_3| &= \Big|\int_{\Omega_\ell}S_h\big(\vartheta,\eta\big)\,d\ell\Big| \le \int_{\Omega_\ell}S_h\big(\vartheta,\vartheta\big)^{1/2}S_h\big(\eta,\eta\big)^{1/2}\,d\ell\\
&\le Ch^{r+1/2}\int_{\Omega_\ell}\|u\|_{H^{r+1}(\Omega_x)}\,|||\eta|||\,d\ell \le Ch^{r+1/2}\|u\|_{L^2(H^{r+1})}\Big(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\Big)^{1/2}.
\end{aligned}$$

Combining I_1, I_2 and I_3, we get the desired estimate

$$\begin{aligned}
\int_{\Omega_\ell}a_h\big(\vartheta,\eta\big)\,d\ell &\le C\,(\varepsilon^{1/2}+h^{1/2})\|u\|_{L^2(H^{r+1})}\Big(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\Big)^{1/2}\\
&\quad + C\,h^{r+1}\|u\|_{L^2(H^{r+1})}\,\|\eta\|_0.
\end{aligned}$$

Next, we find the estimates in internal coordinate. From the definition, we have

$$\begin{aligned}
\int_{\Omega_\ell}a_h(\varphi,\eta)\,d\ell &= \varepsilon\int_{\Omega_\ell}\big(\nabla_x\varphi,\nabla_x\eta\big)_x\,d\ell + \int_{\Omega_\ell}\big(\mathbf{b}\cdot\nabla_x\varphi,\eta\big)_x\,d\ell + \int_{\Omega_\ell}S_h\big(\varphi,\eta\big)\,d\ell\\
&= I_4 + I_5 + I_6.
\end{aligned}$$

Then by using the Cauchy-Schwarz inequality, the stability property of the fluctuation operator κ_h, the approximation properties (5.3) of interpolant Π_k and the parameter choice $\mu_K \sim h_K$, we get for I_4, I_5, and I_6 the following estimates

$$\begin{aligned}
|I_4| &\le \varepsilon\int_{\Omega_\ell}\|\Pi_k u - u\|_{H^1(\Omega_x)}\|\eta\|_{H^1(\Omega_x)}\,d\ell\\
&\le \varepsilon^{1/2}\int_{\Omega_\ell}\|\Pi_k u - u\|_{H^1(\Omega_x)}|||\eta|||\,d\ell\\
&\le \varepsilon^{1/2}\Big(\int_{\Omega_\ell}\|\Pi_k u - u\|^2_{H^1(\Omega_x)}\,d\ell\Big)^{1/2}\Big(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\Big)^{1/2}\\
&\le C\,\varepsilon^{1/2}k^{q+1}\|u\|_{H^{q+1}(H^1)}\Big(\int_{\Omega_\ell}|||\eta|||^2\,d\ell\Big)^{1/2}.\\
|I_5| &\le C\,k^{q+1}\|u\|_{H^{q+1}(H^1)}\|\eta\|_0.
\end{aligned}$$

$$
\begin{aligned}
|I_6| &\le \int_{\Omega_\ell} \sum_{K\in\mathcal{T}_h} \mu_K \big\|\kappa_h(\nabla_x(\Pi_k u - u))\big\|_{L^2(K)} \big\|\kappa_h(\nabla_x \eta)\big\|_{L^2(K)}\, d\ell \\
&\le C\, h^{1/2} \int_{\Omega_\ell} \big\|\nabla_x(\Pi_k u - u)\big\|_{L^2(\Omega_x)} |||\eta|||\, d\ell \\
&\le C\, h^{1/2} k^{q+1} \|u\|_{H^{q+1}(H^1)} \Big(\int_{\Omega_\ell} |||\eta|||^2\, d\ell\Big)^{1/2}.
\end{aligned}
$$

Hence, combining these estimates we get the second statement of the lemma

$$
\begin{aligned}
\int_{\Omega_\ell} a_h\big(\varphi,\eta\big)\, d\ell \le C\,(\varepsilon^{1/2} + h^{1/2})\, k^{q+1} \|u\|_{H^{q+1}(H^1)} \Big(\int_{\Omega_\ell} |||\eta|||^2\, d\ell\Big)^{1/2} \\
+ C\, k^{q+1} \|u\|_{H^{q+1}(H^1)} \|\eta\|_0.
\end{aligned}
$$

To obtain the last two estimates, we use the two different representations (4.9) and (4.10) of B. Note that the jump terms $[j_h u - u]_i$, $i = 1, \ldots, M-1$, vanishes due to the continuity of the interpolant $j_h u$ in ℓ-direction. We have from (4.9), (4.3), and (5.6)

$$
\begin{aligned}
B(\vartheta,\eta) &= \sum_{i=1}^{M} \int_{I_i} \Big(\frac{\partial(G\vartheta)}{\partial\ell}, \eta\Big)_x d\ell + \Big(G_{\min}\vartheta(\ell_0^+), \eta(\ell_0^+)\Big)_x \\
&\le \sum_{i=1}^{M} \int_{I_i} \big\|\partial_\ell(G\vartheta)\big\|_{L^2(\Omega_x)} \|\eta\|_{L^2(\Omega_x)}\, d\ell + \big\|G_{\min}^{1/2}\vartheta(\ell_0^+)\big\|_{L^2(\Omega_x)} \big\|G_{\min}^{1/2}\eta(\ell_0^+)\big\|_{L^2(\Omega_x)} \\
&\le C\, h^{r+1} \Big\{ \|u\|_{H^1(H^{r+1})} \|\eta\|_0 + \|u\|_{C(H^{r+1})} \|\eta\|_{\mathrm{dG}} \Big\}.
\end{aligned}
$$

The interpolation $\Pi_k u$ satisfies $\Pi_k u(\ell_i^-) = u(\ell_i)$, $i = 1, \ldots, M$. Hence, we get from (4.10) the relation

$$
B(\varphi,\eta) = \sum_{i=1}^{M} \int_{I_i} -\Big(G\varphi, \frac{\partial\eta}{\partial\ell}\Big)_x d\ell.
$$

Let $\Pi_0 G$ be the L^2-projection of G in a space of piecewise constant functions in ℓ-direction. Using the orthogonality (5.2) of the interpolant Π_k, we get

$$
\begin{aligned}
B(\varphi,\eta) &= \sum_{i=1}^{M} \int_{I_i} \Big(\varphi, (G - \Pi_0 G)\frac{\partial\eta}{\partial\ell}\Big)_x d\ell \\
&\le \sum_{i=1}^{M} \int_{I_i} \|\varphi\|_{L^2(\Omega_x)} \big\|(G - \Pi_0 G)\partial_\ell \eta\big\|_{L^2(\Omega_x)}\, dx \\
&\le C\, k^{q+1} \|u\|_{H^{q+1}(L^2)} \|\eta\|_0.
\end{aligned}
$$

Here, we used the Cauchy-Schwarz inequality, the inverse inequality and the interpolation error estimates (5.3). This complete the proof. □

Theorem 5.1. *Let $\tilde{u}(t^n)$, $u(t^n)$ and $\tilde{u}^n_{h,k}$, $u^n_{h,k}$, be the solutions of two-step method (3.1), (3.2) and (4.7), (4.8), respectively. Under the assumptions A1-A3 and $\mu_K \sim h_K$ there holds for $\eta^n = R_h u(t^n) - u^n_{h,k}$ and $\tilde{\eta}^n = R_h u(t^n) - u^n_{h,k}$*

{err_e23}
$$\begin{aligned} &\|\eta^N\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{\eta}^{n+1}|||^2\, d\ell + \frac{\tau}{2} \sum_{n=0}^{N-1} \|\eta^{n+1}\|^2_{\mathrm{dG}} \\ &\qquad \le C_u e^{9T/2} \Big[\|R_h z_0 - u^0_{h,k}\|_0^2 + \tau^2 + (\varepsilon + h)\, h^{2r} + k^{2q+2} \Big] \end{aligned} \tag{5.14}$$

and for $e^n = u(t^n) - u^n_{h,k}$ and $\tilde{e}^n = \tilde{u}(t^n) - \tilde{u}^n_{h,k}$

$$\begin{aligned} &\|e^N\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{e}^{n+1}|||^2\, d\ell + \frac{\tau}{2} \sum_{n=0}^{N-1} \|e^{n+1}\|^2_{\mathrm{dG}} \\ &\qquad \le C_u e^{9T/2} \Bigg[\|R_h z_0 - u^0_{h,k}\|_0^2 + \tau^2 + (\varepsilon + h)\, h^{2r} + k^{2q+1} \Bigg] \end{aligned} \tag{5.15}$$ {err_e23a}

where C_u depends on u, u_t, u_{tt}, $\tilde{u}, \tilde{u}_t$, $\tilde{u}_{tt}$ and $z_{\min}$.

Note that the error to the interpolant $R_h u$ is superclose with respect to the internal coordinate (order $k+1$ instead of $k+1/2$).

Proof. From the result of the Lemma 4.2, we can write for $\eta^n = R_h u(t^n) - u^n_{h,k}$

$$\frac{1}{2}\|\eta^N\|_0^2 - \frac{1}{2}\|\eta^0\|_0^2 + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{\eta}^{n+1}|||^2\, d\ell + \frac{\tau}{2} \sum_{n=0}^{N-1} \|\eta^{n+1}\|^2_{\mathrm{dG}} \le T_1 + T_2 \tag{5.16}$$ {err_e24}

where

$$T_1 = \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big\{ \Big(\frac{\tilde{\eta}^{n+1} - \eta^n}{\tau}, \tilde{\eta}^{n+1} \Big)_x + a_h(\tilde{\eta}^{n+1}, \tilde{\eta}^{n+1}) \Big\}\, d\ell, \tag{5.17}$$

$$T_2 = \tau \sum_{n=0}^{N-1} \Big\{ \int_{\Omega_\ell} \Big(\frac{\eta^{n+1} - \tilde{\eta}^{n+1}}{\tau}, \eta^{n+1} \Big)_x\, d\ell + B(\eta^{n+1}, \eta^{n+1}) \Big\}. \tag{5.18}$$

We first consider T_1. Using (4.7), we obtain

$$\begin{aligned} T_1 &= \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big\{ \Big(\frac{R_h \tilde{u}(t^{n+1}) - R_h u(t^n)}{\tau}, \tilde{\eta}^{n+1} \Big)_x + a_h\big(R_h \tilde{u}(t^{n+1}), \tilde{\eta}^{n+1}\big) \\ &\qquad - \Big(\frac{\tilde{u}^{n+1}_{h,k} - u^n_{h,k}}{\tau}, \tilde{\eta}^{n+1} \Big)_x - \int_{\Omega_\ell} a_h(\tilde{u}^{n+1}_{h,k}, \tilde{\eta}^{n+1}) \Big\}\, d\ell \\ &= \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big\{ \Big(\frac{R_h \tilde{u}(t^{n+1}) - R_h u(t^n)}{\tau}, \tilde{\eta}^{n+1} \Big)_x + a_h\big(R_h \tilde{u}(t^{n+1}), \tilde{\eta}^{n+1}\big) \\ &\qquad - \big(f^{n+1}, \tilde{\eta}^{n+1}\big)_x \Big\}\, d\ell. \end{aligned}$$

For the last term on the right-hand side of the above equation, using (3.4) at $t = t^{n+1}$, we get for the first term

$$\begin{aligned} T_1 &= \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big(\frac{R_h \tilde{u}(t^{n+1}) - R_h u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}), \tilde{\eta}^{n+1} \Big)_x \, d\ell \\ &+ \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} a\big(R_h \tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1}), \tilde{\eta}^{n+1}\big) \, d\ell + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} S_h\big(R_h \tilde{u}(t^{n+1}), \tilde{\eta}^{n+1}\big) \, d\ell \\ &= \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big(\frac{R_h \tilde{u}(t^{n+1}) - R_h u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}), \tilde{\eta}^{n+1} \Big)_x \, d\ell \\ &+ \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} a_h\big(R_h \tilde{u}(t^{n+1}) - \tilde{u}(t^{n+1}), \tilde{\eta}^{n+1}\big) \, d\ell + \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} S_h\big(\tilde{u}(t^{n+1}), \tilde{\eta}^{n+1}\big) \, d\ell \\ &= T_{1,1} + T_{1,2} + T_{1,3}. \end{aligned} \tag{5.19}$$ {T_1}

We treat the contribution of the terms on the right-hand side of (5.19) separately. For the first term, using Cauchy-Schwarz inequality, the Young's inequality and the initial condition $\tilde{u}(t^n) = u(t^n)$ for first step

$$\begin{aligned} |T_{1,1}| &\le \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\| \frac{R_h \tilde{u}(t^{n+1}) - R_h u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}) \right\|_{L^2(\Omega_x)} \|\tilde{\eta}^{n+1}\|_{L^2(\Omega_x)} \, d\ell \\ &\le \frac{\tau}{2} \sum_{n=0}^{N-1} \int_{\Omega_\ell} \left\| \frac{R_h \tilde{u}(t^{n+1}) - R_h u(t^n)}{\tau} - \tilde{u}_t(t^{n+1}) \right\|^2_{L^2(\Omega_x)} d\ell \\ &\quad + \frac{\tau}{2} \sum_{n=0}^{N-1} \int_{\Omega_\ell} \|\tilde{\eta}^{n+1}\|^2_{L^2(\Omega_x)} \, d\ell \\ &\le \tau \sum_{n=0}^{N-1} \left\| \frac{R_h \tilde{u}(t^{n+1}) - R_h \tilde{u}(t^n)}{\tau} - R_h \tilde{u}_t(t^{n+1}) \right\|^2_0 \\ &\quad + \tau \sum_{n=0}^{N-1} \big\| R_h \tilde{u}_t(t^{n+1}) - \tilde{u}_t(t^{n+1}) \big\|_0^2 + \frac{\tau}{2} \sum_{n=0}^{N-1} \big\| \tilde{\eta}^{n+1} \big\|_0^2. \end{aligned}$$

For first term, applying Taylor's theorem with integral remainder term and for second term the approximation properties of interpolant j_h and Π_k with the stability property (5.6) yields

$$\begin{aligned} |T_{1,1}| &\le \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \big\| \tilde{u}_{tt} \big\|_0^2 \, dt + \frac{\tau}{2} \sum_{n=0}^{N-1} \big\| \tilde{\eta}^{n+1} \big\|_0^2 \\ &\quad + C\tau \sum_{n=0}^{N-1} \Big[h^{2r+2} \big\| \tilde{u}_t(t^{n+1}) \big\|^2_{L^2(H^{r+1})} + k^{2q+2} \big\| \tilde{u}_t(t^{n+1}) \big\|^2_{H^{q+1}(L^2)} \Big]. \end{aligned}$$

To find the estimates for $T_{1,2}$, we use the decomposition (5.7) of errors into space and internal coordinate and get

$$T_{1,2} = \tau \sum_{n=0}^{N-1} \Big\{ a_h\big(\tilde{\vartheta}^{n+1}, \tilde{\eta}^{n+1}\big) + a_h\big(\tilde{\varphi}^{n+1}, \tilde{\eta}^{n+1}\big) \Big\}.$$

Then from the results (5.8) and (5.9) of Lemma 5.2, we obtain

$$\begin{aligned} |T_{1,2}| \le{}& C\,(\varepsilon + h)\,\tau \sum_{n=0}^{N-1} \Big[h^{2r} \big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})} + k^{2q+2} \big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^1)} \Big] \\ &+ C\,\tau \sum_{n=0}^{N-1} \Big[h^{2r+2} \big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})} + k^{2q+2} \big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^1)} \Big] \\ &+ \frac{\tau}{4} \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{\eta}^{n+1}|||^2\, d\ell + \frac{\tau}{2} \sum_{n=0}^{N-1} \|\tilde{\eta}^{n+1}\|_0^2. \end{aligned}$$

The estimate for $T_{1,3}$ follows from the approximation properties of the fluctuation operator κ_h and the choice of the stabilizing parameter $\mu_K \sim h_K$. We have

$$\begin{aligned} |T_{1,3}| &\le \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} S_h\big(\tilde{u}(t^{n+1}), \tilde{u}(t^{n+1})\big)\, d\ell + \frac{\tau}{4} \sum_{n=0}^{N-1} \int_{\Omega_\ell} S_h\big(\tilde{\eta}^{n+1}, \tilde{\eta}^{n+1}\big)\, d\ell \\ &\le C\, h^{2r+1} \tau \sum_{n=0}^{N-1} \big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+1})} + \frac{\tau}{4} \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{\eta}^{n+1}|||^2\, d\ell. \end{aligned}$$

Finally, by inserting the estimates $T_{1,1}$, $T_{1,2}$, and $T_{1,3}$ into (5.19), we obtain

$$\begin{aligned} |T_1| \le{}& \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \|\tilde{u}_{tt}\|_0^2\, dt + \frac{\tau}{2} \sum_{n=0}^{N-1} \int_{\Omega_\ell} |||\tilde{\eta}^{n+1}|||^2\, d\ell + \tau \sum_{n=0}^{N-1} \|\tilde{\eta}^{n+1}\|_0^2 \\ &+ C\, h^{2r} \tau \sum_{n=0}^{N-1} \Big[(\varepsilon + h)\, \big\|\tilde{u}(t^{n+1})\big\|^2_{L^2(H^{r+2})} + h^2 \big\|\tilde{u}_t(t^{n+1})\big\|^2_{L^2(H^{r+1})} \Big] \\ &+ C\, k^{2q+2}\, \tau \sum_{n=0}^{N-1} \Big[(\varepsilon + h + 1)\, \big\|\tilde{u}(t^{n+1})\big\|^2_{H^{q+1}(H^1)} + \big\|\tilde{u}_t(t^{n+1})\big\|^2_{H^{q+1}(L^2)} \Big]. \end{aligned} \tag{5.20}$$

{T_1_n}

Now we estimate the second term T_2. Using (4.8) and (3.5) we obtain the following error equation for second step

$$\begin{aligned} T_2 ={}& \tau \sum_{n=0}^{N-1} \int_{\Omega_\ell} \Big(\frac{R_h u(t^{n+1}) - R_h \tilde{u}(t^{n+1})}{\tau} - u_t(t^{n+1}), \eta^{n+1} \Big)_x\, d\ell \\ &+ \tau \sum_{n=0}^{N-1} B\big(R_h u(t^{n+1}) - u(t^{n+1}), \eta^{n+1}\big) \\ &- \tau \sum_{n=0}^{N-1} \Big(G_{\min} z_{\min}^{n+1} - G_{\min} z_{\min,\mathrm{h}}^{n+1}, \eta^{n+1}(\ell_0^+) \Big)_x \\ ={}& T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned} \tag{5.21}$$

{T_2}

The estimates for the first term can be obtained by using the same procedure as for $T_{1,1}$ and get

$$
\begin{aligned}
|T_{2,1}| &\le \tau \sum_{n=0}^{N-1} \left\| \frac{R_h u(t^{n+1}) - R_h \tilde{u}(t^{n+1})}{\tau} - R_h u_t(t^{n+1}) \right\|_0^2 \\
&\quad + \tau \sum_{n=0}^{N-1} \left\| R_h u_t(t^{n+1}) - u_t(t^{n+1}) \right\|_0^2 + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\| \eta^{n+1} \right\|_0^2 \\
&\le \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left\| u_{tt} \right\|_0^2 dt + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\| \eta^{n+1} \right\|_0^2 \\
&\quad + C\tau \sum_{n=0}^{N-1} \left[h^{2r+2} \left\| u_t(t^{n+1}) \right\|_{L^2(H^{r+1})}^2 + k^{2q+2} \left\| u_t(t^{n+1}) \right\|_{H^{q+1}(L^2)}^2 \right].
\end{aligned}
$$

Note that in above estimates we have used the initial condition $u(t^n) = \tilde{u}(t^{n+1})$ from (3.2). The bounds on the second term $T_{2,2}$ are obtained by using the error decomposition (5.7) and the estimates (5.10) and (5.11)

$$
\begin{aligned}
|T_{2,2}| &= \left| \tau \sum_{n=0}^{N-1} \left\{ B\big(\vartheta^{n+1}, \eta^{n+1}\big) + B\big(\varphi^{n+1}, \eta^{n+1}\big) \right\} \right| \\
&\le C\, h^{2r+2}\, \tau \sum_{n=0}^{N-1} \left[\left\| u(t^{n+1}) \right\|_{H^1(H^{r+1})}^2 + \left\| u(t^{n+1}) \right\|_{C(H^{r+1})}^2 \right] \\
&\quad + C\, k^{2q+2}\, \tau \sum_{n=0}^{N-1} \left\| u(t^{n+1}) \right\|_{H^{q+1}(L^2)}^2 + \frac{\tau}{2} \sum_{n=0}^{N-1} \left\| \eta^{n+1} \right\|_0^2 + \frac{\tau}{8} \sum_{n=0}^{N-1} \left\| \eta^{n+1} \right\|_{\mathrm{dG}}^2.
\end{aligned}
$$

Cauchy-Schwarz inequality and Young's inequality give for $T_{2,3}$

$$
\begin{aligned}
|T_{2,3}| &\le \tau \sum_{n=0}^{N-1} \left\| G_{\min}^{1/2} z_{\min}(t^{n+1}) - G_{\min}^{1/2} z_{\min,\mathrm{h}}^{n+1} \right\|_{L^2(\Omega_x)} \left\| G_{\min}^{1/2} \eta^{n+1}(\ell_0^+) \right\|_{L^2(\Omega_x)} \\
&\le C\, h^{2r+2}\, \tau \sum_{n=0}^{N-1} \left\| z_{\min}(t^{n+1}) \right\|_{H^{r+1}(\Omega_x)}^2 + \frac{\tau}{8} \sum_{n=0}^{N-1} \left\| \eta^{n+1} \right\|_{\mathrm{dG}}^2.
\end{aligned}
$$

Finally using these estimates in (5.21) we get for T_2

$$
\begin{aligned}
|T_2| &\le \tau^2 \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left\| u_{tt} \right\|_0^2 dt + \tau \sum_{n=0}^{N-1} \|\eta^{n+1}\|_0^2 + \frac{\tau}{4} \sum_{n=0}^{N-1} \left\| \eta^{n+1} \right\|_{\mathrm{dG}}^2 \\
&\quad + C\tau h^{2r+2} \sum_{n=0}^{N-1} \Big[\left\| u(t^{n+1}) \right\|_{H^1(H^{r+1})}^2 + \left\| z_{\min}(t^{n+1}) \right\|_{H^{r+1}(\Omega_x)}^2 + \left\| u_t(t^{n+1}) \right\|_{L^2(H^{r+1})}^2 \\
&\qquad + \left\| u(t^{n+1}) \right\|_{C(H^{r+1})}^2 \Big] \\
&\quad + C\tau k^{2q+2} \sum_{n=0}^{N-1} \left[\left\| u(t^{n+1}) \right\|_{H^{q+1}(L^2)}^2 + \left\| u_t(t^{n+1}) \right\|_{H^{q+1}(L^2)}^2 \right].
\end{aligned}
$$

Inserting T_1 and T_2 in (5.16), absorbing the triple norm and the dG norm contributions in the left-hand side and using (3.12), we get

$$\begin{aligned}
&\frac{1}{2}\|\eta^N\|_0^2 - \frac{1}{2}\|\eta^0\|_0^2 + \tau\sum_{n=0}^{N-1}\int_{\Omega_\ell} |||\tilde{\eta}^{n+1}|||^2\,d\ell + \frac{\tau}{2}\sum_{n=0}^{N-1}\|\eta^{n+1}\|_{\mathrm{dG}}^2 \\
&\quad\le \tau^2\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\|u_{tt}\|_0^2\,dt + \tau\sum_{n=0}^{N-1}\gamma_n\|\eta^n\|_0^2 + 2\tau\sum_{n=0}^{N-1}\|f^{n+1}\|_0^2 \\
&\qquad + C\,h^{2r}\,\tau\sum_{n=0}^{N-1}\Big[(\varepsilon+h)\,\|u(t^{n+1})\|_{H^1(H^{r+1})}^2 + h^2\|u_t(t^{n+1})\|_{L^2(H^{r+1})}^2 \\
&\qquad\qquad + h^2\|z_{\min}(t^{n+1})\|_{H^{r+1}(\Omega_x)}^2 + \|u(t^{n+1})\|_{C(H^{r+1})}^2\Big] \\
&\qquad + C\,k^{2q+2}\,\tau\sum_{n=0}^{N-1}\Big[(\varepsilon+h+1)\|u(t^{n+1})\|_{H^{q+1}(H^1)}^2 + \|u_t(t^{n+1})\|_{H^{q+1}(L^2)}^2\Big]
\end{aligned}$$

where $\gamma_0 = 2$, $\gamma_N = 1$ and $\gamma_n = 3$, $n = 1, \ldots, N-1$. We conclude by applying the Gronwall's Lemma in the same fashion as in Lemma 3.1. □

6. Implementation of numerical method

This section indicates the implementation of the operator splitting method in the context of finite element methods. For more details, we refer to Ref. 14.

Using the bases

$$V_h = \mathrm{span}\{\phi_i\},\ 1 \le i \le N_x, \quad S_k^q = \mathrm{span}\{\psi_k\},\ 1 \le k \le N_\ell,$$

the tensor product space $S_{h,k}^{r,q}$ is defined as follows

$$S_{h,k}^{r,q} = \left\{ v = \sum_{i=1}^{N_x}\sum_{k=1}^{N_\ell}\alpha_{ik}\phi_i(x)\psi_k(\ell),\ \alpha_{ik}\in\mathbb{R},\ \ 1\le i\le N_x,\ 1\le k\le N_\ell \right\}.$$

The finite element functions are represented as

$$u_{h,k}^n = \sum_{i=1}^{N_x}\sum_{k=1}^{N_\ell}\xi_{ik}^n\phi_i(x)\psi_k(\ell), \qquad X = \sum_{j=1}^{N_x}\sum_{l=1}^{N_\ell}x_{jl}\phi_j(x)\psi_l(\ell).$$

We define the matrices $M_x, T_x, D_x, S_x \in \mathbb{R}^{N_x\times N_x}$ by

$$\begin{aligned}
(M_x)_{ij} &= \big(\phi_i(x),\phi_j(x)\big)_x, & (D_x)_{ij} &= \varepsilon\big(\nabla_x\phi_i(x),\nabla_x\phi_j(x)\big)_x \\
(T_x)_{ij} &= \big(\mathbf{b}\cdot\nabla_x\phi_i(x),\phi_j(x)\big)_x, & (S_x)_{ij} &= S_h\big(\phi_i(x),\phi_j(x)\big).
\end{aligned}$$

Similarly we define the matrices $M_\ell, T_\ell \in \mathbb{R}^{N_\ell\times N_\ell}$ as

$$\begin{aligned}
(M_\ell)_{kl} &= \Big(\psi_k(\ell),\psi_l(\ell)\Big)_\ell, \\
(T_\ell)_{kl} &= \sum_{i=1}^{N_\ell}\Big(\partial_\ell(G\psi_k(\ell)),\psi_l(\ell)\Big)_{I_i} + \sum_{i=1}^{N_\ell-1}[G\psi_k(\ell)]_i\,\psi_l(\ell_i^+) + G\psi_k(\ell_0^+)\psi_l(\ell_0^+).
\end{aligned}$$

For the ease of presentation let us consider (2.1) with source term $f = 0$. Then the computing scheme for the operator splitting method described in (4.7) and (4.8) is as follows:

Within each time interval $(t^n, t^{n+1}]$, we begin with the x-direction step where we are looking for the solution of the time-dependent convection-diffusion equation (4.7). We compute $\tilde{\boldsymbol{\eta}}_j^{n+1} \in \mathbb{R}^{N_x}$, $j = 1, \ldots, N_\ell$, by solving the linear systems

$$(M_x + \tau D_x + \tau T_x + \tau S_x)\tilde{\boldsymbol{\eta}}_j^{n+1} = M_x \boldsymbol{\eta}_j^n, \qquad j = 1, \ldots, N_\ell.$$

With obtaining the solutions $\tilde{\boldsymbol{\eta}}_j^{n+1}$, $j = 1, \ldots, N_\ell$, the x-direction step is completed. Then, we continue with the ℓ-direction step where we update the solution from the first step and compute the solution of the one-dimensional transport problem (4.8) by a discontinuous Galerkin method. In this step we solve the linear systems

$$(M_\ell + \tau T_\ell)\boldsymbol{\eta}_j^{n+1} = M_\ell \tilde{\boldsymbol{\eta}}_j^{n+1}, \qquad j = 1, \ldots, N_x,$$

and the obtained solutions $\boldsymbol{\eta}_j^{n+1}$, $j = 1, \ldots, N_\ell$, are used as input for the time interval $(t^{n+1}, t^{n+1}]$.

7. Numerical tests

We report in this section the numerical computations illustrating the theoretical results obtained in the previous section. The two-step method (4.7) and (4.8) in the context of finite element method in space and discontinuous Galerkin method in internal coordinate is implemented in the finite element package MooNMD.[19]

The tests are made in two plus one dimensions, i.e, we consider $\Omega_x = (0,1)\times(0,1)$ as two-dimensional domain in space and $\Omega_\ell = (0,1)$ as one-dimensional domain in the internal coordinate. We consider the velocity field $\mathbf{b}$ as $b_1 = b_2 = 0.1$, the growth rate $G(\ell) = 1$ and two different choices for the diffusion coefficient ε, $\varepsilon = 1$ and $\varepsilon \ll 1$. The source term f and the boundary and initial conditions are chosen such that the analytical solution of the problem (2.1) is

$$z(t, \ell, x, y) = e^{-0.1t} \sin(\pi\ell) \cos(\pi x) \cos(\pi y).$$

Let $e^n := z(t^n) - u_{h,k}^n$, where z is the exact solution of (2.1) and the numerical solution $u_{h,k}^n$ is obtained by two-step method (4.7) and (4.8). We use the following notations

$$\|e\|_0 = \left(\tau \sum_{n=1}^N \|e^n\|_{L^2(L^2)}^2 + \tau \sum_{n=1}^N \|e^n\|_{\mathrm{dG}}^2\right)^{1/2},$$

$$\|e\|_1 = \left(\tau \sum_{n=1}^N \|e^n\|_{L^2(H^1)}^2 + \tau \sum_{n=1}^N \|e^n\|_{\mathrm{dG}}^2\right)^{1/2},$$

$$\|e\|_{\mathrm{DG}} = \left(\tau \sum_{n=1}^N \int_{\Omega_\ell} |||e^n|||^2 \, d\ell + \tau \sum_{n=1}^N \|e^n\|_{\mathrm{dG}}^2\right)^{1/2}.$$

In order to illustrate the convergence order in time, internal coordinate and space, we use the well known strategy, i.e., the convergence order in time can be obtained by assuming that the mesh sizes k and h are small enough compared to the time-step size τ.

In the numerical computations, we have used triangular and quadrilateral meshes which are generated by successive refinement starting from coarsest meshes (level 0) as in Fig. 1 for two-dimensional domain Ω_x and a line divided into two cells for one-dimensional domain Ω_ℓ.

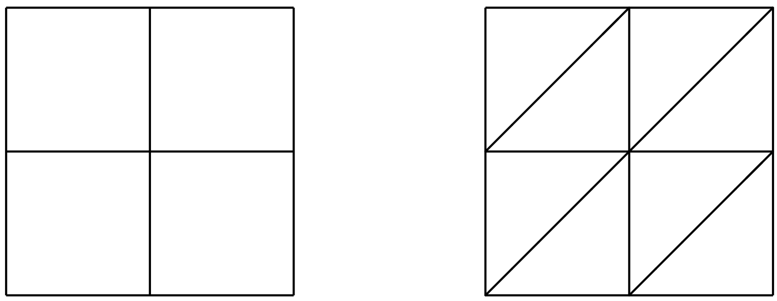

Fig. 1: Meshes for Ω_x on level 0.

Case $\varepsilon = 1$: In this case, the Galerkin finite element method in space is combined with a discontinuous Galerkin method in internal coordinate. For time discretization, the backward Euler time stepping scheme is used with final time $T = 1$. One can expect a convergence for $\|\cdot\|_0$-norm of order $\mathcal{O}(h^{r+1})$ and for $\|\cdot\|_1$-norms of order $\mathcal{O}(h^r)$ using Q_r and P_r finite elements in space with sufficiently small time step length τ and mesh size k. The results are presented in Tables 1–4.

Tables 1 and 2 show the second order convergence in the $\|\cdot\|_0$-norm and first order convergence in the $\|\cdot\|_1$-norm for both Q_1 and P_1 finite elements in space with dG(1) in internal coordinate. The length of the time step was set to be $\tau = 10^{-3}$ and mesh size to $k = 1/64$. For Q_2 and P_2 finite elements in space with dG(2) in internal coordinate, the time step length was set to $\tau = 10^{-4}$ and mesh size $k = 1/64$. The results of Tables 3 and 4 show third order convergence for the $\|\cdot\|_0$-norm and second order for the $\|\cdot\|_1$-norm.

In Tables 5 and 6, the errors and convergence orders for internal coordinate and time are listed. We expect a convergence of order $\mathcal{O}(k^{q+1/2})$ in the internal coordinate and a convergence of $\mathcal{O}(\tau)$ in time. The errors for dG(1) in internal coordinate with Q_1 on level 7 and time step length $\tau = 2.5 \cdot 10^{-4}$ are presented in Table 5. We see that the expected orders of convergence are achieved. The numerical errors and convergence orders in time are listed in Table 6 for dG(1) with $k = 1/32$ and Q_1 on level 6. The theoretically predicted convergence order is achieved.

Case $\varepsilon = 10^{-9}$**:** In the case of convection-dominated convection-diffusion, we

Table 1: Errors and rate of convergence in space for Q_1 and dG(1), $k = 1/64$ and $\tau = 10^{-3}$.

Level	$\|e\|_0$		$\|e\|_1$	
	error	order	error	order
0	1.719554e-01	——	1.006185	——
1	4.746460e-02	1.8571	4.892384e-01	1.0403
2	1.206219e-02	1.9764	2.412003e-01	1.0203
3	3.167958e-03	1.9289	1.201483e-01	1.0054

Table 2: Errors and rate of convergence in space for P_1 and dG(1), $k = 1/64$ and $\tau = 10^{-3}$.

Level	$\|e\|_0$		$\|e\|_1$	
	error	order	error	order
0	2.353104e-01	——	1.432599	——
1	7.412177e-02	1.6666	7.996426e-01	0.8413
2	1.981996e-02	1.9029	4.113880e-01	0.9589
3	5.144843e-03	1.9458	2.072235e-01	0.9893

Table 3: Errors and rate of convergence in space for Q_2 and dG(2), $k = 1/64$ and $\tau = 10^{-4}$.

Level	$\|e\|_0$		$\|e\|_1$	
	error	order	error	order
0	1.916287e-02	——	2.396151e-01	——
1	2.599528e-03	2.8820	6.137457e-02	1.9650
2	3.354662e-04	2.9540	1.561139e-02	1.9750

Table 4: Errors and rate of convergence in space for P_2 and dG(2), $k = 1/64$ and $\tau = 10^{-3}$.

Level	$\|e\|_0$		$\|e\|_1$	
	error	order	error	order
0	3.511498e-02	——	5.583590e-01	——
1	4.796648e-03	2.8720	1.526520e-01	1.8710
2	6.138514e-04	2.9661	3.929766e-02	1.9577

consider local projection as stabilization in space. Discontinuous Galerkin methods of first and second order are used for the discretization in internal coordinate. For time discretization, the backward Euler time stepping scheme is used.

The numerical tests are performed using for (V_h, D_h) the pairs $(P_1^{\text{bubble}}, P_0^{\text{disc}})$,

Table 5: Errors and rate of convergence in internal coordinate for dG(1), Q_1 on level 6 and $\tau = 2.5 \cdot 10^{-4}$.

k	$\|e\|_0$	
1/2	6.696513e-02	——
1/4	1.829413e-02	1.7398
1/8	6.521805e-03	1.4880

Table 6: Errors and rate of convergence in time for Q_1 and dG(1) on level = 6 and $k = 1/32$.

τ	$\|e\|_0$		$\|e\|_1$	
	error	order	error	order
1/10	1.815303e-01	——	4.027364	——
1/20	9.577853e-02	0.9224	2.170105	0.8921
1/40	4.983170e-02	0.9427	1.141479	0.9269
1/80	2.567753e-02	0.9566	5.869174e-01	0.9597

$(P_2^{\text{bubble}}, P_1^{\text{disc}})$, $(Q_1^{\text{bubble}}, P_0^{\text{disc}})$, and $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$. The stabilization parameters μ_K have been chosen as

$$\mu_K := \mu_0 h_K \qquad \forall K \in \mathcal{T}_h$$

where μ_0 denotes a constant which will be given for each of the test calculations.

In Tables 7 and 8 we show the convergence results for space in norm $\| \cdot \|_{\text{DG}}$. Table 7 shows the error in space with stabilizing parameter $\mu_0 = 5$, time step length $\tau = 10^{-3}$ and mesh size $k = 1/64$ for $(Q_1^{\text{bubble}}, P_0)$ and $(P_1^{\text{bubble}}, P_0)$ with dG(1) in internal coordinate. In Table 8, the convergence results for $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ and $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ with dG(2) in internal coordinate with $\mu_0 = 5$, $k = 1/64$ and $\tau = 10^{-4}$ are listed. We see that the expected orders of convergence $\mathcal{O}(h^{r+1/2})$ are achieved for quadrangles. For smaller mesh size h, the convergence order starts to decrease for triangles. This is because the influence of the error in internal coordinate increases, i.e., the mesh size k is not small enough that one can see the corresponding convergence rate in space for higher order elements.

The numerical errors and convergence orders in internal coordinate are listed in Table 9 for dG(1) and $(Q_1^{\text{bubble}}, P_0)$ with $\mu_0 = 5$ on level 7 and $\tau = 2.5 \cdot 10^{-4}$. The convergence order starts to decrease for small mesh size k since the errors in space have increasing influence. Finally, Table 10 shows the errors and convergence orders in time for $(Q_1^{\text{bubble}}, P_0)$ on level 6 with $\mu_0 = 2.5$ and dG(1) with $k = 1/32$. We see that the time stepping scheme is of first order convergent.

Table 7: Errors and rate of convergence in space for $(Q_1^{\text{bubble}}, P_0)$ and $(P_1^{\text{bubble}}, P_0)$ and dG(1), $k = 1/64$, $\tau = 10^{-3}$ and $\mu_K = 5h_K$.

	$(Q_1^{\text{bubble}}, P_0)$		$(P_1^{\text{bubble}}, P_0)$	
Level	$\|e^n\|_{\text{DG}}$		$\|e^n\|_{\text{DG}}$	
0	1.756772	——	1.93314	——
1	6.394630e-01	1.4580	7.247844e-01	1.4153
2	2.280495e-01	1.4875	2.661525e-01	1.4453
3	8.245890e-02	1.4678	1.086554e-01	1.2925

Table 8: Errors and rate of convergence in space for $(Q_2^{\text{bubble}}, P_1^{\text{disc}})$ and $(P_2^{\text{bubble}}, P_1^{\text{disc}})$ and dG(2), $k = 1/64$, $\tau = 10^{-4}$ and $\mu_K = 5h_K$.

	$(Q_2^{\text{bubble}}, P_1^{\text{disc}})$		$(P_2^{\text{bubble}}, P_1^{\text{disc}})$	
Level	$\|e^n\|_{\text{DG}}$		$\|e^n\|_{\text{DG}}$	
0	1.272972	——	1.234504	——
1	2.558153e-01	2.3151	2.352103e-01	2.3919
2	4.700162e-02	2.4443	5.094834e-02	2.2069
3	8.010563e-03	2.5527	1.222369e-02	2.0593

Table 9: Errors and rate of convergence in internal coordinate for dG(1) and $(Q_1^{\text{bubble}}, P_0)$ on level 7 with $\mu_K = 5h_K$ and $\tau = 2.5 \cdot 10^{-4}$.

k	$\|e^n\|_{\text{DG}}$	
1/2	2.493607e-01	——
1/4	9.283060e-02	1.4256
1/8	3.425394e-02	1.4383
1/16	1.446166e-02	1.2441

Table 10: Errors and rate of convergence in time for dG(1) and $(Q_1^{\text{bubble}}, P_0)$ on level with $\mu_K = 2.5h_K$ and $k = 1/32$.

τ	$\|e^n\|_{\text{DG}}$	
1/10	8.017623e-01	——
1/20	4.318566e-01	0.8926
1/40	2.270064e-01	0.9278
1/80	1.166372e-01	0.9607

8. Conclusion

In this paper we have been concerned with the numerical solution of the population balance equation with one internal coordinate posed on the domain $\Omega_\ell \times \Omega_x$ in $d+1$ dimension. We proposed an operator splitting method to reduce the original problem into two subproblems. The method combines the continuous finite element method (and local projection stabilization) in space with discontinuous Galerkin method in internal coordinate. We have considered first order backward Euler time stepping scheme. Under certain regularity of exact solution, we have derived error estimates for the two-step method, i.e., using polynomials of degree r in space and of degree q in internal coordinate the error is of order $\mathcal{O}(\tau + h^{r+1/2} + k^{q+1/2})$ when $\varepsilon \ll 1$ and $\mathcal{O}(\tau + h^r + k^{q+1/2})$ when $\varepsilon = 1$.

The application of discontinuous Galerkin method makes the mass matrix corresponding to the internal coordinate diagonal, which leads to the feasibility of the implementation without any projection between the two-steps in the computation process. Computational results shown in Section 7 confirms the theoretical prediction of error estimates.

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