On Multivariate Bernstein Polynomials

Merlin MOUAFO WOUODJIÉ (merlin@aims-cameroon.org)
African Institute for Mathematical Sciences (AIMS)
Cameroon

Supervised by: Professor Mama Foupouagnigni
University of Yaoundé I, Cameroon and AIMS-Cameroon

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Abstract

In this master thesis, we first present Bernstein polynomials with one variable as a proof of the Weierstrass theorem. These polynomials are special, for providing an explicit representation for polynomials of uniform approximation. In the second step, we review the main properties of the Bernstein polynomials of one variable; namely the point-wise convergence, the uniform convergence and the convergence of the derivatives. In the third step, as our main contribution, we state and prove these main properties for the multivariate Bernstein polynomials.

Keywords: Polynomials of one and several variables, Bernstein polynomials, multivariate Bernstein polynomials, approximation, summation, difference, and convergence.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Merlin MOUAFO WOUODJIÉ, July 7, 2014

Contents

1. Introduction

Interpolation and approximation are very important concepts in various domains of applied sciences such as Computer science, geological science, numerical analysis, statistics and with its large used in modern days computers. Due to the relevance of these areas of mathematics, more interest had made the study matter demand driven. The advancement or breakthrough in field had been due to simple theorems of polynomial interpolations upon which much practical numerical analysis do rest or rely upon.

Most often, it is not easy to find exacts solutions to most equations governing real life problems. Even when such solutions can be found, for their practical use, then is a need to find their approximation by polynomials, when possible.

Weierstrass's theorem asserts the possibility of uniform approximation by polynomials to continuous functions over a closed interval. An analytic function can be expanded in a uniformly convergent power series, and a continuous but non analytic function can be expanded in a uniformly convergent series of general polynomials, with no possibility of rearranging its terms so as to produce a convergent power series [Dav75]. There are many proofs of the Weierstrass theorem. In this thesis, we shall present one called "S. Bernstein's proofs" since it give also an explicit representation of the polynomials for uniform approximation. Bernstein polynomial of one variable, which are defined explicitly as follow for a function f

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} ,$$

are know to approximate uniformly f on the compact interval $[0,1]$, provided that f is continuous on $[0,1]$ [Dav75]. Moreover, the n^{th} derivative of $B_n(f;.)$ converges uniformly to $f^{(n)}$, provided that $f \in \mathcal{C}^n[0,1]$.

> "*In contrast to other modes of approximation, in particular to Tchebyschev or best uniform approximation, the Bernstein polynomials yields smooth approximants; but there is a price that must be paid for these beautiful approximations properties: the convergence of the Bernstein polynomials is very slow.*" [Dav75].

Many people like Richard V.Kadison [KAD66], George M.Phillips [Phi03], J. Davis [Dav75] and others, have worked on this Bernstein polynomial for function with one variable, and have derived some beautiful properties with major's ones: uniform convergence, uniform convergence of derivatives, fixed sign for the p^{th} derivative, deduction of upper and lower bounds of the of the Bernstein polynomial from those of the corresponding function.

The main problem here is to extend those properties of Bernstein polynomials of one variable by generalizing them for functions of several real variables. To solve this problem, some people like Mehemet Acikgoz and Serkan Araci [AA12], have tried to generalize, but not all, those properties of Bernstein polynomial with one variable to two variables.

In this thesis, to achieve our aim which is to generalize those beautiful properties for functions with one variable to functions with several variables, we have the following plan: chapter 1 is introduction, chapter 2 recall some definitions and properties needed, and present also the main properties of Bernstein polynomials of one variable such as

1)* If $f \in \mathcal{C}[0,1]$, the Bernstein polynomial $B_n(f;\cdot)$ converges uniformly to f, 2)* If $f \in \mathcal{C}^p[0,1]$, then $B_n^p(f;\cdot)$ converges uniformly to $f^{(p)}$,

3)* If $f \in \mathcal{C}^p[0,1]$, for some $0 \le p \le n$ and $A \le f^{(p)}(x) \le B, \quad 0 \le x \le 1$, then

$$A \le \frac{n^p}{n(n-1)\cdots(n-p+1)} B_n^{(p)}(f;x) \le B \ , \quad 0 \le x \le 1,$$

4)* If f is convex in $[0,1]$, then for $n = 2, 3, \cdots$

$$B_{n-1}(f;x) \ge B_n(f;x) \ , \quad 0 < x < 1,$$

5)* If $f(x)$ is bounded in $[0,1]$ and x_0 is a point of $[0,1]$ at which f'' exists and is continuous, then

$$\lim_{n\to\infty} n\left[B_n(f;x_0) - f(x_0)\right] = \frac{1}{2} x_0(1-x_0) f''(x_0) \ .$$

In chapter 3, which is the main part of this thesis and our main contribution, we extend to functions of several variables the main properties of Bernstein polynomial of one variable. More precisely, we define the generalized Bernstein polynomial as

$$B_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} f\left(\frac{k_1}{n_1},\cdots,\frac{k_m}{n_m}\right) \prod_{i=1}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i},$$

and prove the following:

1)* If $f \in \mathcal{C}[0,1]^m$, the Bernstein polynomial $B_{n_1,\cdots,n_m}(f;\cdot)$ converges uniformly to f,

2)* If $f \in \mathcal{C}^{p_1\cdots p_m}[0,1]^m$, then $B_{n_1,\cdots,n_m}^{p_1,\cdots,p_m}(f;\cdot)$ converges uniformly to $f^{(p_1)\cdots,(p_m)}$, where $f \in \mathcal{C}^{p_1,\cdots,p_m}[0,1]^m$ means f is continuous on $[0,1]^m$ and $\dfrac{\partial^{p_i} f}{\partial x^{p_i}}$ is continuous on $[0,1]^m$, $0 \le p_i \le n_i, \ i = 1,\cdots,m$,

3)* If $f \in \mathcal{C}^{p_1\cdots p_m}[0,1]^m$, $0 \le p_i \le n_i$, $i = 1,\cdots,m$, and $A \le f^{(p_1)\cdots(p_m)}(x_1,\cdots,x_m) \le B, \quad (x_1,\cdots,x_m) \in [0,1]^m$, then

$$A \le \prod_{i=1}^{m} \frac{n_i^{p_i}}{n_i(n_i-1)\cdots(n_i-p_i+1)} B_{n_1,\cdots,n_m}^{p_1,\cdots,p_m}(f;x_1,\cdots,x_m) \le B, \ (x_1,\cdots,x_m) \in [0,1]^m,$$

4)* If $f(x_1,\cdots,x_m)$ is convex in $[0,1]^m$, then, for $n_i = 2,3,\cdots, \quad i = 1,\cdots,m$, we have

$$B_{n_1-1,\cdots,n_m-1}(f;x_1,\cdots,x_m) \ge B_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m), \quad (x_1,\cdots,x_m) \in [0,1]^m,$$

5)* If $f(x_1,\cdots,x_m)$ is bounded in $[0,1]^m$ and let $(a_1,\cdots,a_m)$ is a point of $[0,1]^m$ at which $f^{(2)\cdots(2)}(a_1,\cdots,a_m)$ exists and is continuous, then, for $n_1 = n_2 = \cdots = n_m = n$

$$\lim_{n\to\infty} n\left[B_{n,\cdots,n}(f;a_1,\cdots,a_m) - f(a_1,\cdots,a_m)\right] = \frac{1}{2}\sum_{i=1}^{m} a_i(1-a_i)\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m).$$

2. Bernstein Polynomials with one Variable

2.1 Recalls and Weierstrass Approximation Theorem

This section contains material from analysis that will be of used in the later parts of this thesis. It is presented here for ready reference and review, and it is mainly taken from the book of Davis, James [Dav75], Phillips [Phi03].

2.1.1 Recalls.

Forward Differences

Let us discretize the interval $[a,b]$: $a = x_0 < x_1 < \cdots < x_n = b$ with $x_i = x_0 + ih$, $h = x_{i+1} - x_i$ is a constant, $i = 0, 1, \cdots, n$. We write

$$f(x_{i+1}) - f(x_i) = f(x_i + h) - f(x_i) = \triangle f(x_i)$$

which is called a **first difference**. The symbol $\triangle$ denote difference. Hence

$\dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \dfrac{\triangle f(x_i)}{h}$. We define

$$\triangle^2 f(x_i) = \triangle\left(\triangle f(x_i)\right) = \triangle f(x_{i+i}) - \triangle f(x_i) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) \ ,$$

and call $\triangle^2 f(x_i)$ a **second-order difference**. Then it seems natural to define **higher-order forward differences** recursively as

$$\triangle^{k+1} f(x_i) = \triangle\left(\triangle^k f(x_i)\right) = \triangle^k f(x_{i+1}) - \triangle^k f(x_i) \ , \quad k \geq 0$$

where $\triangle^1 f(x_i) = \triangle f(x_i)$ and $\triangle^0 f(x_i) = f(x_i)$.

Lemma 2.1.2. [Phi03]

Let m, n, p be positive integers, $f \in \mathcal{C}^m[a,b]$, and $a = x_0 < x_1 < \cdots < x_n = b$ with $x_i = x_0 + ih$, $h = x_{i+1} - x_i$, a subdivision of $[a,b]$. Then

$$\exists \varepsilon_0 \in]x_0, x_p[, \quad \frac{\triangle^p f(x_0)}{h^p} = f^{(p)}(\varepsilon_0) \ . \tag{2.1.1}$$

Here, $f^{(p)}(\varepsilon_0)$ means the p^{th} derivative of f at ε_0, and if f is a function of m variables then $f^{(p_1)\cdots(p_m)}(\varepsilon_1, \cdots, \varepsilon_m)$ means the p_i^{th} partial derivatives of f with respect to each of the m several variables at $(\varepsilon_1, \cdots, \varepsilon_m)$.

For integers $t, k, n, r \geq 0$ such that $n \neq 0,\ r \neq 0$, we deduce from [Phi03] this result

$$\triangle^t f\left(\frac{k}{n}\right) = \sum_{r=0}^{t} (-1)^{t-r} \binom{t}{r} f\left(\frac{k+r}{n}\right) \ , \quad \text{with } h = \frac{1}{n} \ .$$

This result can be generalized for a function f of m variables

$$\triangle_1^{t_1} \triangle_2^{t_2} \cdots \triangle_m^{t_m} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \cdots, \frac{k_m}{n_m}\right) = \sum_{r_1=0}^{t_1} \sum_{r_2=0}^{t_2} \cdots \sum_{r_m=0}^{t_m} \left[\prod_{i=1}^{m} (-1)^{t_i - r_i} \binom{t_i}{r_i}\right] f\left(\frac{k_1+r_1}{n_1}, \frac{k_2+r_2}{n_2}, \cdots, \frac{k_m+r_m}{n_m}\right)$$

with $h_i = \dfrac{1}{n_i}, t_i, k_i, n_i, r_i, n \geq 0$ integers, $n_i \neq 0,\ n \neq 0$. Here, $\triangle_j^{t_j}$ means partial difference of order t_j on the variable number j, $j = 1, \cdots, m$.

For $f \in \mathbb{C}_k[x]$ and $k < t$ we have $\triangle^t f(0) = 0$. Here $\mathbb{C}_k[x]$ is the linear space of polynomials with complex coefficients of degree maximum k.

We call the i^{th} **divided difference** the quantity

$$\frac{f(x_i) - f(x_0)}{x_i - x_0} .$$

Leibniz Rule

Theorem 2.1.3. *[Phi03]*

If the k^{th} derivatives of f and g both exist, then

$$\frac{d^k}{dx^k}\left(f(x)g(x)\right) = \sum_{r=0}^{k} \binom{k}{r} \frac{d^r}{dx^r} f(x) \frac{d^{k-r}}{dx^{k-r}} g(x) .$$

Theorem 2.1.4. *[Phi03] For any integer $i, k \geq 0$, we have*

$$\triangle^k \left(f(x_i)g(x_i)\right) = \sum_{r=0}^{k} \binom{k}{r} \triangle^r f(x_i) \triangle^{k-r} g(x_{i+r}) .$$

Convex functions

Definition 2.1.5. [Phi03]

A function f is said to be convex on $[a, b]$ if for any $x_1, x_2 \in [a, b]$,

$$\lambda f(x_1) + (1 - \lambda) f(x_2) \geq f(\lambda x_1 + (1 - \lambda) x_2)$$

for any $\lambda \in [0, 1]$.

Definition 2.1.6. [Phi03]

A function f is convex on $[a, b]$ if and only if all second order divided differences of f are non-negative.

The Weierstrass Approximation Theorem

The Weierstrass approximation theorem is a famous theorem in mathematical analysis. It asserts that every continuous function defined on a closed interval can be uniformly approximated as closely as desired by a polynomial [Dav75].

Theorem 2.1.7. *[Dav75]*

Let $f(x) \in \mathcal{C}[a, b]$. Given $\varepsilon > 0$, we can find a polynomial $P_n(x)$ (of sufficiently high degree) for which

$$|f(x) - P_n(x)| < \varepsilon , \quad a \leq x \leq b .$$

An analytic function can be expanded in a uniformly convergent power series, and a continuous but non analytic function can be expanded in a uniformly convergent series of general polynomials, with no possibility of rearranging its terms so as to produce a convergent power series [Dav75].

2.2 Definitions and Properties

Definition 2.2.1. [WQ11]

A Bernstein polynomial of index n is a polynomial expressed in the following form:

$$\sum_{k=0}^{n} \beta_{k,n} b_{k,n}(x) \ , \quad 0 \leq x \leq 1$$

where each $\beta_{k,n},\ k = 0, \cdots, n,$ is a real number and $b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

The coefficients $\beta_{k,n}$ are called **Bernstein coefficients** and the polynomials $b_{k,n}(x)$ are called **Bernstein basis of degree** n.

Definition 2.2.2. [WQ11]

Let $f(x)$ be defined on $[0, 1]$. The n^{th} $(n \geq 1)$ Bernstein polynomial for $f(x)$ is given by

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{k,n}(x) \ .$$

For positive integer n, we have the following relation between the n^{th} $(n \geq 1)$ Bernstein polynomial for $f(x)$ and the difference $\triangle^t$ of f at 0:

Theorem 2.2.3. *[Dav75]*

Let $f(x)$ be defined on $[0, 1]$. We have

$$B_n(f; x) = \sum_{t=0}^{n} \triangle^t f(0) \binom{n}{t} x^t$$

where $\triangle$ is applied with step size $h = \frac{1}{n}$.

We list some pertinent properties of Bernstein polynomials.

Properties 2.2.4. (a)- **A recursive definition and degree elevation** [WQ11], [Mul12] :

$$b_{k,n}(x) = (1-x) b_{k,n-1}(x) + x b_{k-1,n-1}(x) \ ;$$
$$\forall k = 0, \cdots, n \quad b_{k,m}(x) = \frac{m+1-k}{m+k} b_{k,m+1}(x) + \frac{k+1}{m+1} b_{k+1,m+1}(x).$$

(b)- **Derivative** [Mul12] :

$$\frac{d}{dx} b_{n,k}(x) = n\left(b_{k-1,n-1}(x) + b_{k,n-1}(x)\right) \quad \text{for} \ \ 0 \leq x \leq 1.$$

(c)- **Bernstein polynomials as a basis** [Mul12], [Hal47], [WQ11] :

The Bernstein polynomials of order maximum n form a basis for the space of polynomials of degree less than or equal to n. Therefore, we have

$$x^j = \sum_{k=0}^{j} \sigma_{j,k} b_{k,j}(x) \ ,$$

where each $\sigma_{j,k},\ k = 0, \cdots, j$, is real number.

(d)- **Identities** [Phi03] :

* Let $n \geq 1$ be an integer.

$$\begin{cases} B_n(1;x) = 1, \quad B_n(x;x) = x, \quad B_n(x^2;x) = \dfrac{n-1}{n}x^2 + \dfrac{1}{n}x; \\ B_n(x^3;x) = \dfrac{(n-1)(n-2)}{n^2}x^3 + \dfrac{3(n-1)}{n^2}x^2 + \dfrac{1}{n^2}x; \\ B_n(x^4;x) = \dfrac{(n-1)(n-2)(n-3)}{n^3}x^4 + \dfrac{6(n-1)(n-2)}{n^3}x^3 \\ \qquad\qquad\qquad + \dfrac{7(n-1)}{n^3}x^2 + \dfrac{1}{n^3}x \ . \end{cases} \tag{2.2.1}$$

*

$$B_n(\exp(\alpha x);x) = [x\exp(\alpha/n) + (1-x)]^n \ , \quad \text{for} \ \ \alpha \in \mathbb{C}.$$

* From (2.2.1), we have

$$\begin{cases} \sum_{k=0}^{n} \binom{n}{k} \left(\dfrac{k}{n} - x\right) x^k (1-x)^{n-k} = 0; \\ \sum_{k=0}^{n} \binom{n}{k} \left(\dfrac{k}{n} - x\right)^2 x^k (1-x)^{n-k} = \dfrac{x(1-x)}{n}; \\ \sum_{k=0}^{n} \binom{n}{k} \left(\dfrac{k}{n} - x\right)^3 x^k (1-x)^{n-k} = x(1-x)\dfrac{-2x+1}{n^2}; \\ \sum_{k=0}^{n} \binom{n}{k} \left(\dfrac{k}{n} - x\right)^4 x^k (1-x)^{n-k} = x(1-x)\dfrac{(3n-6)x(1-x)+1}{n^3}. \end{cases} \tag{2.2.2}$$

2.3 Approximation by Bernstein Polynomials

Here we recall the results showing:

- how a bounded function f on $[0,1]$ can be approximated by Bernstein polynomial at a point x on $[0,1]$ where f is continuous;
- how a function in $\mathcal{C}[0,1]$ can be approximated uniformly by Bernstein polynomials in $[0,1]$;
- how the p^{th} derivative of a function in $\mathcal{C}^p[0,1]$ can be approximate uniformly by the p^{th} derivative of Bernstein polynomials on $[0,1]$.

To achieve this goal, we will need the following intermediate results.

2.3.1 Approximation. The following intermediate result is useful in the proof of the next theorem.

Lemma 2.3.2. ([Dav75], page 110)

For a given $\delta > 0$ and $0 \leq x \leq 1$, we have

$$\sum_{\left|\frac{k}{n}-x\right|\geq\delta} \binom{n}{k} x^k(1-x)^{n-k} \leq \frac{1}{4n\delta^2}$$

$\Big($with the sum taken over those values of $k = 0, \cdots, n$ for which $\left|\frac{k}{n} - x\right| \geq \delta\Big)$.

The following theorem show how a bounded function f on $[0,1]$ can be approximated by Bernstein polynomial at a point on $[0,1]$ where f is continuous.

Theorem 2.3.3 (Bermstein). *([Dav75], page 109)*

Let $f(x)$ be bounded on $[0,1]$. Then

$$\lim_{n\to\infty} B_n(f;x) = f(x)$$

at any point $x \in [0,1]$ at which f is continuous. If $f \in \mathbb{C}[0,1]$, the limit holds uniformly in $[0,1]$.

Remark 2.3.4. [Dav75]

The Bernstein's theorem can be taken as a proof of the existence of polynomials of uniform approximation (the Weierstrass's theorem), and in addition, it also provides a simple explicit representation for them.

Lemma 2.3.5. ([Dav75], page 112)

For any integer $p \geq 0$, the p^{th} derivative of $B_n(f;x)$ may be expressed in term of p^{th} difference of f as

$$B^{(p)}_{n+p}(f;x) = \frac{(n+p)!}{n!} \sum_{t=0}^{n} \triangle^p f\left(\frac{t}{n+p}\right) \binom{n}{t} x^t(1-x)^{n-t}$$

for all $n \geq 0$, where $\triangle$ is applied with step size $h = \dfrac{1}{n+p}$.

Using the connection between differences and derivatives, we can deduce the following valuable result from the previous lemma.

Theorem 2.3.6. *([Dav75], page 114)*

Let n be a non negative integer and $f \in \mathcal{C}^p[0,1]$, for some $0 \leq p \leq n$. If $A \leq f^{(p)}(x) \leq B$, $0 \leq x \leq 1$, then

$$A \leq \frac{n^p}{n(n-1)\cdots(n-p+1)} B^{(p)}_n(f;x) \leq B\ , \quad 0 \leq x \leq 1.$$

For $p = 0$, the multiplier of $B^{(p)}_n$ is to be interpreted as 1.

- *If $f^{(p)}(x) \geq 0,\ 0 \leq x \leq 1$ then $B^{(p)}_n(f;x) \geq 0,\ 0 \leq x \leq 1$,*

- *If $f(x)$ is nondecreasing on $0 \leq x \leq 1$, then $B_n(f;x)$ is nondecreasing there,*
- *If $f(x)$ is convex on $0 \leq x \leq 1$, then $B_n(f;x)$ is convex there.*

Remark 2.3.7. [Dav75]

One consequence of this result is that if $f^{(p)}(x)$ is of fixed sign on $[0,1]$, then $B_n^{(p)}(f;x)$ also has this sign on $[0,1]$.

We shall now deal with some further properties of the Bernstein polynomials.

Theorem 2.3.8. *([Dav75], page 113)*

Let $f(x) \in \mathcal{C}^p[0,1]$, where p is a positive integer. Then

$$\lim_{n\to\infty} B_n^{(p)}(f;x) = f^{(p)}(x) \quad \text{uniformly on} \quad [0,1] \ .$$

Remark 2.3.9. [Dav75]

As we have just seen that the Bernstein polynomial for f converges to f, in addition, the derivatives of the Bernstein polynomial for f converge to the corresponding derivatives of f.

Theorem 2.3.10. *([Dav75], page 115)*

Let $f(x)$ be convex in $[0,1]$. Then for $n = 2, 3, \cdots$

$$B_{n-1}(f;x) \geq B_n(f;x) \ , \quad 0 < x < 1 \ .$$

If $f \in \mathcal{C}[0,1]$, the strict inequality holds unless f is linear in each of the intervals $\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$, $j = 1, 2, \cdots, n$. In this case $B_{n-1}(f;x) = B_n(f;x)$.

2.3.11 Order of convergence.

We will now study the speed of the convergence of Bernstein polynomials by stating one theorem in relation with the estimating of the error $f(x) - B_n(f;x)$. Before that let us give this lemma which should help us [Dav75].

Lemma 2.3.12. ([Dav75], page 117)

There is a constant C independent of a given positive integer n such that for all x in $[0,1]$,

$$\sum_{\left|\frac{k}{n}-x\right| \geq n^{-1/4}} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{C}{n^{3/2}} \ .$$

For a function twice differentiable, we have an asymptotic error term for its Bernstein polynomial by the following statement:

Theorem 2.3.13 (Voronovsky)**.** *([Dav75], page 117)*

Let $f(x)$ be bounded in $[0,1]$ and let x_0 be a point of $[0,1]$ at which f'' exists and is continuous. Then,

$$\lim_{n\to\infty} n\left[B_n(f;x_0) - f(x_0)\right] = \frac{1}{2} x_0(1-x_0) f''(x_0) \ .$$

Conclusion : The convergence of the Bernstein polynomials is slow.

Remark 2.3.14. [Dav75]

By a linear substitution:

$$x = \varphi(t) = \frac{t-a}{b-a}$$

the interval $[a, b]$ can be transformed into the interval unit $[0, 1]$. Thus, all the results stated above for a function f defined on $[0, 1]$ can easily be extended on any compact interval $[a, b]$.

The Bernstein polynomial on [a,b]

The Bernstein polynomial for a function f defined on $[a, b]$, denoted $B_n^{a,b}(f; .)$ is defined as follows:

$$\begin{aligned} B_n^{a,b}(f;t) &= \sum_{k=0}^{n} \binom{n}{k} f\left((b-a)\frac{k}{n} + a\right) \left(\frac{t-a}{b-a}\right)^k \left(\frac{b-t}{b-a}\right)^{n-k} \\ &= \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} f\left((b-a)\frac{k}{n} + a\right) (t-a)^k (b-t)^{n-k}, \quad t \in [a, b]. \end{aligned}$$

3. Bernstein Polynomials of Several Variables

Here we generalize all the results for the Bernstein polynomials with one variable to several variables. The proofs use many materials from analysis, algebra, probabilities which can be found from [Guz03],[Jel12],[BE95],[JMR12],[Rud87],[Fel64],[CC71] and [CT09].

3.1 Definition and Properties

We shall now derive the properties of the m-dimensional generalization of the Bernstein polynomials. First we give m-dimensional generalization of the Bernstein polynomials.

Definition 3.1.1 (Generalization). ([Dav75], page 122)

Let $f(x_1, \cdots, x_m)$ be defined on $[0,1]^m$ and $n_i \geq 1,\ i = 1, \cdots, m,$ integers. The $(n_1, \cdots, n_m)^{\text{th}}$ Bernstein polynomial for $f(x_1, \cdots, x_m)$ is given by

$$B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} f\left(\frac{k_1}{n_1}, \cdots, \frac{k_m}{n_m}\right) \prod_{i=1}^{m} b_{k_i,n_i}(x)$$

where $\quad b_{k_i,n_i}(x) = \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i}, \quad i = 1, \cdots, m.$

Using the results obtained with one variable, it is easy to obtain some pertinent properties for the generalization of Bernstein polynomials.

Properties 3.1.2.

1- Let $m \geq 1,\ j_1, \cdots, j_m$ be integers. Then

$$x_1^{j_1} \cdots x_m^{j_m} = \sum_{k_1=0}^{j_1} \cdots \sum_{k_m=0}^{j_m} \prod_{i=1}^{m} \sigma_{j_i,k_i} b_{k_i,j_i}(x_i)$$

where $\sigma_{j_i,k_i},\ \ k_i = 0, \cdots, j_i,\ 1 \leq i \leq m$, are real numbers.

2- Let $n_i \geq 1, j_i,\ i = 1, \cdots, m$, be integer. Then

$$\begin{cases} B_{n_1,\cdots,n_m}(1; x_1, \cdots, x_m) = 1; \\ B_{n_1,\cdots,n_m}\left(\prod_{i=1}^{m} x_i^{j_i}; x_1, \cdots, x_m\right) = \prod_{i=1}^{m} B_{n_i}\left(x_i^{j_i}; x_i\right) \end{cases} . \tag{3.1.1}$$

3-

$$B_{n_1,\cdots,n_m}\left(\exp\left(\sum_{i=1}^{m} \alpha_i x_i\right); x_1, \cdots, x_m\right) = \prod_{i=1}^{m} \left[x_i \exp(\alpha_i/n_i) + (1-x_i)\right]^{n_i} .$$

4-

$$\sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} \prod_{i=1}^{m} \binom{n_i}{k_i} \left(\frac{k_i}{n_i} - x_i\right)^{j_i} x_i^{k_i}(1-x_i)^{n_i-k_i}$$
$$= \prod_{i=1}^{m} \sum_{k_i=0}^{n_i} \binom{n_i}{k_i} \left(\frac{k_i}{n_i} - x_i\right)^{j_i} x_i^{k_i}(1-x_i)^{n_i-k_i}, \quad x_i \in [0,1],\ 1 \leq i \leq m. \tag{3.1.2}$$

3.2 Approximation by Bernstein Polynomials with Several Variables

3.2.1 Approximation.

Theorem 3.2.2.

Let $f(x_1, \cdots, x_m)$ be defined on $[0,1]^m$. We have for positive integers $n_i,\ 1 \le i \le m$

$$B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) = \sum_{t_1=0}^{n_1} \cdots \sum_{t_m=0}^{n_m} \triangle_1^{t_1} \cdots \triangle_m^{t_m} f(0, \cdots, 0) \prod_{i=1}^{m} \binom{n_i}{t_i} x_i^{t_i}$$

where $\triangle_i$ is applied with step size $h_i = \dfrac{1}{n_i}, \quad i = 1, \cdots, m.$

Proof:

$$\begin{aligned} B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} f\left(\frac{k_1}{n_1}, \cdots, \frac{k_m}{n_m}\right) \prod_{i=1}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \\ &= \sum_{k_2=0}^{n_2} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{k_1=0}^{n_1} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \cdots, \frac{k_m}{n_m}\right) \binom{n_1}{k_1} x_1^{k_1}(1-x_1)^{n_1-k_1}\right] \\ &\qquad \cdot \prod_{i=2}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \ . \end{aligned}$$

We take the first variable of f and fix the others. Hence $f\left(\frac{k_1}{n_1}, \cdots, \frac{k_m}{n_m}\right)$ is seen as a function $g_1\left(\frac{k_1}{n_1}\right)$, and we apply the theorem 2.2.3 to get

$$\begin{aligned} B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) &= \sum_{k_2=0}^{n_2} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{t_1=0}^{n_1} \triangle_1^{t_1} g_1(0) x_1^{t_1} \binom{n_1}{t_1}\right] \prod_{i=2}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \\ &= \sum_{k_2=0}^{n_2} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{t_1=0}^{n_1} \triangle_1^{t_1} f\left(0, \frac{k_2}{n_2}, \cdots, \frac{k_m}{n_m}\right) x_1^{t_1} \binom{n_1}{t_1}\right] \\ &\qquad \cdot \prod_{i=2}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \\ &= \sum_{k_3=0}^{n_3} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{t_1=0}^{n_1} \left(\sum_{k_2=0}^{n_2} \triangle_1^{t_1} f\left(0, \frac{k_2}{n_2}, \frac{k_3}{n_3}, \cdots, \frac{k_m}{n_m}\right) \binom{n_2}{k_2} x_2^{k_2}(1-x_2)^{n_2-k_2}\right) \right. \\ &\qquad \left. \cdot x_1^{t_1} \binom{n_1}{t_1}\right] \prod_{i=3}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \ . \end{aligned}$$

We repeat the action with the second variable of the function $\triangle_1^{t_1} f\left(0, \dfrac{k_2}{n_2}, \cdots, \dfrac{k_m}{n_m}\right)$, to get

$$B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) = \sum_{k_3=0}^{n_3} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{t_1=0}^{n_1} \sum_{t_2=0}^{n_2} \triangle_1^{t_1} \triangle_2^{t_2} f\left(0, 0, \frac{k_3}{n_3}, \cdots, \frac{k_m}{n_m}\right) x_1^{t_1} x_2^{t_2} \binom{n_1}{t_1} \binom{n_2}{t_2}\right] \cdot \prod_{i=3}^{m} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} .$$

Then

$$B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) = \sum_{k_4=0}^{n_4} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{t_1=0}^{n_1} \sum_{t_2=0}^{n_2} \left(\sum_{k_3=0}^{n_3} \triangle_1^{t_1} \triangle_2^{t_2} f\left(0, 0, \frac{k_3}{n_3}, \cdots, \frac{k_m}{n_m}\right) \binom{n_3}{k_3} x_3^{k_3} \cdot (1-x_3)^{n_3-k_3}\right) x_1^{t_1} x_2^{t_2} \binom{n_1}{t_1} \binom{n_2}{t_2}\right] \prod_{i=4}^{m} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} .$$

Similarly with the third variable of $\triangle_1^{t_1} \triangle_2^{t_2} f\left(0, 0, \dfrac{k_3}{n_3}, \cdots, \dfrac{k_m}{n_m}\right)$, we get

$$B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) = \sum_{k_4=0}^{n_4} \cdots \sum_{k_m=0}^{n_m} \left[\sum_{t_1=0}^{n_1} \sum_{t_2=0}^{n_2} \sum_{t_3=0}^{n_3} \triangle_1^{t_1} \triangle_2^{t_2} \triangle_3^{t_3} f\left(0, 0, 0, \frac{k_4}{n_4}, \cdots, \frac{k_m}{n_m}\right) x_1^{t_1} x_2^{t_2} x_3^{t_3} \cdot \prod_{i=1}^{3} \binom{n_i}{t_i}\right] \prod_{i=4}^{m} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} .$$

By repeating successively the actions with the $4^{\text{th}}, 5^{\text{th}}, \cdots, m^{\text{th}}$ variables but with the functions $\triangle_1^{t_1} \triangle_2^{t_2} \triangle_3^{t_3} f\left(0, 0, 0, \dfrac{k_4}{n_4}, \cdots, \dfrac{k_m}{n_m}\right)$, $\triangle_1^{t_1} \triangle_2^{t_2} \triangle_3^{t_3} \triangle_4^{t_4} f\left(0, 0, 0, 0, \dfrac{k_5}{n_5}, \cdots, \dfrac{k_m}{n_m}\right), \cdots, \triangle_1^{t_1} \triangle_2^{t_2} \cdots \triangle_{m-1}^{t_{m-1}}$ $f\left(0, 0, 0, \cdots, 0, \dfrac{k_m}{n_m}\right)$ respectively, we obtain the result. □

Lemma 3.2.3.

For given $\delta_i > 0,\ i = 1, \cdots, m$ and $(x_1, \cdots, x_m) \in [0,1]^m$, We have

$$\sum_{\left|\frac{k_1}{n_1} - x_1\right| \geq \delta_1} \cdots \sum_{\left|\frac{k_m}{n_m} - x_m\right| \geq \delta_m} \prod_{i=1}^{m} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} \leq \frac{1}{4^m \prod_{i=1}^{m} n_i \delta_i^2}$$

$\Bigg($we sum over those value of $k_i = 0, \cdots, n_i$ for which $\left|\dfrac{k_i}{n_i} - x_i\right| \geq \delta_i\Bigg)$.

Proof:

$$\begin{aligned}
&\sum_{\left|\frac{k_1}{n_1}-x_1\right|\geq\delta_1}\cdots\sum_{\left|\frac{k_m}{n_m}-x_m\right|\geq\delta_m}\prod_{i=1}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}\\
&=\prod_{i=1}^{m}\sum_{\left|\frac{k_i}{n_i}-x_i\right|\geq\delta_i}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}\\
&\leq\prod_{i=1}^{m}\left(\frac{1}{4n_i\delta_i^2}\right)=\frac{1}{4^m\prod_{i=1}^{m}n_i\delta_i^2}\quad(\text{ using lemma 2.3.2})\ .
\end{aligned}$$

□

Remark 3.2.4. Lemma 3.2.3 is still valid if the summation is not on all the k_i, $1\leq i\leq m$ but on some of them. In this case, the upper bound will contain only terms involved in the summation.

Theorem 3.2.5.

Let $f(x_1,\cdots,x_m)$ be bounded in the m-dimensional cube $0\leq x_i\leq 1$, $i=1,\cdots,m$. Then $B_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m)$ converges towards $f(x_1,\cdots,x_m)$ at any point of continuity of this function, as $n_i\to\infty$ for all i. If $f\in\mathcal{C}[0,1]^m$, the limit holds uniformly in $[0,1]^m$.

Proof: Because of space limitation, we prove this theorem for two variables and provide clear indications for the proof for m variables (which we have done).

The function $f(x_1,x_2)$ is assumed bounded in $[0,1]^2$. Hence for some

$M>0,\quad |f(x_1,x_2)|\leq M,\quad\forall(x_1,x_2)\in[0,1]^2$, and for any two points (α_1,α_2), $(\beta_1,\beta_2)\in[0,1]^2$

$$|f(\alpha_1,\alpha_2)-f(\beta_1,\beta_2)|\leq 2M\ .$$

Let $(x_1,x_2)\in[0,1]^2$ be a point of continuity of f. Given $\varepsilon>0$, we can find δ_1,δ_2 depending on (x_1,x_2) and ε such that $|f(x_1,x_2)-f(y_1,y_2)|<\frac{\varepsilon}{2}$, whenever $|x_i-y_i|<\delta_i,\quad i=1,2$.

Since $B_{n_1,n_2}(1;x_1,x_2)=1$, we have

$$\begin{aligned}
f(x_1,x_2)=&f(x_1,x_2)B_{n_1,n_2}(1;x_1,x_2)\\
=&f(x_1,x_2)\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}x_1^{k_1}(1-x_1)^{n_1-k_1}\binom{n_2}{k_2}x_2^{k_2}(1-x_2)^{n_2-k_2}\\
=&\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}f(x_1,x_2)\binom{n_1}{k_1}x_1^{k_1}(1-x_1)^{n_1-k_1}\binom{n_2}{k_2}x_2^{k_2}(1-x_2)^{n_2-k_2}.
\end{aligned}$$

Let us consider the set $E=\{0,\cdots,n_1\}\times\{0,\cdots,n_2\}$, and for $j=1,2$, define the sets

$$\Omega_j=\left\{k_j\in\{0,\cdots,n_j\}:\ \left|\frac{k_j}{n_j}-x_j\right|<\delta_j\right\}\tag{3.2.1}$$

and

$$\begin{aligned} F &= \{0, \cdots, n_1\} \times \{0, \cdots, n_2\} \setminus (\Omega_1 \times \Omega_2) \\ &= (\Omega_1 \times \Omega_2)^c \quad \text{(the complement of } \ \Omega_1 \times \Omega_2 \ \text{ in } E) \\ &= \Omega_1^c \times \Omega_2^c \cup \Omega_1^c \times \Omega_2 \cup \Omega_1 \times \Omega_2^c \ , \end{aligned}$$

where the small c over the set means the complement of this set in E. We have

$$\begin{aligned} &|f(x_1, x_2) - B_{n_1,n_2}(f; x_1, x_m)| \\ &= \left| \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} f(x_1, x_2) \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2} \right. \\ &\qquad \left. - \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2} \right| \\ &= \left| \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \left[f(x_1, x_2) - f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) \right] \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2} \right| \\ &\le \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \left| f(x_1, x_2) - f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) \right| \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2} \ . \end{aligned}$$

Hence,

$$\begin{aligned} &|f(x_1, x_2) - B_{n_1,n_2}(f; x_1, x_2)| \\ &\le \sum_{\Omega_1} \sum_{\Omega_2} \left| f(x_1, x_2) - f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) \right| \prod_{i=1}^{2} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} \binom{n_i}{k_i} x_i^{k_i} (1-x_2)^{n_2-k_2} \\ &\quad + \sum_{F} \left| f(x_1, x_2) - f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right) \right| \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2}. \end{aligned}$$

Using the fact that f is continuous and bounded, we have

$$\begin{aligned} &|f(x_1, x_2) - B_{n_1,n_2}(f; x_1, x_2)| \\ &\le \frac{\varepsilon}{2} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2} \\ &\qquad + 2M \sum_{F} \binom{n_1}{k_1} x_1^{k_1} (1-x_1)^{n_1-k_1} \binom{n_2}{k_2} x_2^{k_2} (1-x_2)^{n_2-k_2} \\ &\le \frac{\varepsilon}{2} \prod_{i=1}^{2} \sum_{k_i=0}^{n_i} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} + 2M \sum_{F} \prod_{i=1}^{2} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} \\ &\le \frac{\varepsilon}{2} + 2M \sum_{F} \prod_{i=1}^{2} \binom{n_i}{k_i} x_i^{k_i} (1-x_i)^{n_i-k_i} \quad \text{(using equations (2.2.1))} \ . \end{aligned}$$

Since $(k_1,k_2) \notin \Omega_1 \times \Omega_2 \iff (k_1,k_2) \in (\Omega_1 \times \Omega_2)^c = \Omega_1^c \times \Omega_2^c \cup \Omega_1^c \times \Omega_2 \cup \Omega_1 \times \Omega_2^c$, we have $(k_1,k_2) \in \Omega_1^c \times \Omega_2^c$ or $(k_1,k_2) \in \Omega_1^c \times \Omega_2$ or $(k_1,k_2) \in \Omega_1 \times \Omega_2^c$. Hence, we split the sum over F as follows:

$$\begin{aligned}
&\sum_F \prod_{i=1}^2 \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \\
&= \left(\sum_{\Omega_1^c} \binom{n_1}{k_1} x_1^{k_1}(1-x_1)^{n_1-k_1}\right)\left(\sum_{\Omega_2^c} \binom{n_2}{k_2} x_2^{k_2}(1-x_2)^{n_2-k_2}\right) \\
&+ \left(\sum_{\Omega_1^c} \binom{n_1}{k_1} x_1^{k_1}(1-x_1)^{n_1-k_1}\right)\left(\sum_{\Omega_2} \binom{n_2}{k_2} x_2^{k_2}(1-x_2)^{n_2-k_2}\right) \\
&+ \left(\sum_{\Omega_1} \binom{n_1}{k_1} x_1^{k_1}(1-x_1)^{n_1-k_1}\right)\left(\sum_{\Omega_2^c} \binom{n_2}{k_2} x_2^{k_2}(1-x_2)^{n_2-k_2}\right) \\
&\le \frac{1}{4n_1\delta_1^2} \cdot \frac{1}{4n_2\delta_2^2} + \frac{1}{4n_1\delta_1^2} + \frac{1}{4n_2\delta_2^2} \quad \text{(using the lemma 2.3.2 and bounding for} \\
&\quad i=1,2 \text{ the expression } \sum_{\Omega_i^c} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \le \sum_{k_i=0}^{n_i} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} = 1 \Bigg).
\end{aligned}$$

Let us assume $n_i, i = 1,2$ are such that $4n_i\delta_i^2 > 1$. Then

$$\begin{aligned}
\sum_F \prod_{i=1}^2 \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} &\le \max_{1\le i\le 2}\left\{\frac{1}{4n_i\delta_i^2}\right\} + \max_{1\le i\le 2}\left\{\frac{1}{4n_i\delta_i^2}\right\} + \max_{1\le i\le 2}\left\{\frac{1}{4n_i\delta_i^2}\right\} \\
&= 3\left(\max_{1\le i\le 2}\left\{\frac{1}{4n_i\delta_i^2}\right\}\right)
\end{aligned}$$

Therefore,

$$|f(x_1,x_2) - B_{n_1,n_2}(f;x_1,x_2)| \le \frac{\varepsilon}{2} + 6M\left(\max_{1\le i\le 2}\left\{\frac{1}{4n_i\delta_i^2}\right\}\right).$$

For n_1, n_2 sufficiently large, we have $6M\left(\max_{1\le i\le 2}\left\{\frac{1}{4n_i\delta_i^2}\right\}\right) < \frac{\varepsilon}{2}$. Therefore

$$|f(x_1,x_2) - B_{n_1,n_2}(f;x_1,x_2)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Here, it should be noticed that the minimum integer n_{10} and n_{20} such that

$$|f(x_1,x_2) - B_{n_1,n_2}(f;x_1,x_2)| \le \varepsilon \quad \forall n_1 \ge n_{10},\ n_2 \ge n_{20}$$

depend on δ_1 and δ_2 which themselves depend on x_1 and x_2. This is why we have only point-wise convergence.

Suppose now that $f \in \mathcal{C}[0,1]^2$, then f is uniformly continuous on $[0,1]^2$ i.e. given $\varepsilon > 0$, we can find δ_1, δ_2 depending just on ε, but no on any element in $[0,1]$ such that $|f(x_1,x_2) - f(y_1,y_2)| < \frac{\varepsilon}{2}$ for all

$(x_1, x_2), (y_1, y_2) \in [0,1]^2$ satisfying $|x_i - y_i| < \delta_i,\ i = 1, 2$. By following the same process, we obtain that

$$|f(x_1,x_2) - B_{n_1,n_2}(f; x_1, x_2)| \leq \frac{\varepsilon}{2} + 6M\left(\max_{1\leq i\leq 2}\left\{\frac{1}{4n_i\delta_i^2}\right\}\right), \quad \forall x_1, x_2 \in [0,1].$$

Hence, there exist n_{01} and n_{02} depending only on ε and not on x_1 and x_2 such that

$$|f(x_1,x_2) - B_{n_1,n_2}(f; x_1, x_2)| < \varepsilon, \quad x_1, x_2 \in [0,1],\ n_1 > n_{01},\ n_2 > n_{02}.$$

Therefore, the convergence is uniform in $[0,1]^2$.

The proof for m variables is obtained in the same way by representing F as union of m different disjoint sets F_k:

$$F = \left(\prod_{i=1}^{m}\Omega_i\right)^c = \bigcup_{k=1}^{m} F_k, \quad \text{with } F_k = \prod_{i=1}^{m}\Omega_i^{[\alpha_i]}, \quad 0 \leq \alpha_i \leq 1 \text{ and } \sum_{i=1}^{m}\alpha_i = k$$

$$\text{where } \Omega_i^{[\alpha_i]} = \begin{cases} \Omega_i & \text{if } \alpha_i = 0 \\ \Omega_i^c & \text{if } \alpha_i = 1. \end{cases}$$

□

Lemma 3.2.6.

For $p_1, \cdots, p_m \geq 0$ with $0 \leq p_i \leq n_i,\ i = 1, 2, \cdots, m$, the $(p_1, \cdots, p_m)^{\text{th}}$ derivative of $B_{n_1+p_1,\cdots,n_m+p_m}(f; x_1, \cdots, x_m)$ may be expressed in terms of $(p_1, \cdots, p_m)^{\text{th}}$ differences of f as

$$B^{p_1,\cdots,p_m}_{n_1+p_1,\cdots,n_m+p_m}(f; x_1, \cdots, x_m) = \left[\prod_{i=1}^{m}\frac{(n_i+p_i)!}{n_i!}\right]\sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m}\triangle_1^{p_1}\cdots\triangle_m^{p_m} f\left(\frac{t_1}{n_1+p_1}, \cdots, \frac{t_m}{n_m+p_m}\right)\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}. \tag{3.2.2}$$

Proof: We write

$$\begin{aligned} B_{n_1+p_1,\cdots,n_m+p_m}(f; x_1, \cdots, x_m) &= \sum_{k_1=0}^{n_1+p_1}\cdots\sum_{k_m=0}^{n_m+p_m} f\left(\frac{k_1}{n_1+p_1}, \cdots, \frac{k_m}{n_m+p_m}\right)\prod_{i=1}^{m}\binom{n_i}{k_i} \cdot x_i^{k_i}(1-x_i)^{n_i-k_i} \\ &= \sum_{k_2=0}^{n_2+p_2}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\sum_{k_1=0}^{n_1+p_1} f\left(\frac{k_1}{n_1+p_1}, \cdots, \frac{k_m}{n_m+p_m}\right)\binom{n_1}{k_1} \cdot x_1^{k_1}(1-x_1)^{n_1-k_1}\right]\prod_{i=2}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}. \end{aligned}$$

We take the first variable of f and fix the others. Hence $f\left(\frac{k_1}{n_1+p_1}, \cdots, \frac{k_m}{n_m+p_m}\right)$ is seen as a function $g_1\left(\frac{k_1}{n_1+p_1}\right)$:

$$B_{n_1+p_1,\cdots,n_m+p_m}(f;x_1,\cdots,x_m) = \sum_{k_2=0}^{n_2+p_2}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\sum_{k_1=0}^{n_1+p_1} g_1\left(\frac{k_1}{n_1+p_1}\right)\binom{n_1}{k_1}x_1^{k_1}(1-x_1)^{n_1-k_1}\right]$$
$$\prod_{i=2}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}$$
$$= \sum_{k_2=0}^{n_2+p_2}\cdots\sum_{k_m=0}^{n_m+p_m}[B_{n_1+p_1}(g_1,x_1)]\prod_{i=2}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}.$$

Hence

$$B^{p_1}_{n_1+p_1,\cdots,n_m+p_m}(f;x_1,\cdots,x_m)$$
$$= \sum_{k_2=0}^{n_2+p_2}\cdots\sum_{k_m=0}^{n_m+p_m}[B^{p_1}_{n_1+p_1}(g_1,x_1)]\prod_{i=2}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}$$
$$= \sum_{k_2=0}^{n_2+p_2}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\frac{(n_1+p_1)!}{n!}\sum_{t_1=0}^{n_1}\triangle_1^{p_1}g_1\left(\frac{t_1}{n_1+p_1}\right)\binom{n_1}{t_1}x_1^{t_1}(1-x_1)^{n_1-t_1}\right]\prod_{i=2}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}$$
$$= \sum_{k_2=0}^{n_2+p_2}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\frac{(n_1+p_1)!}{n!}\sum_{t_1=0}^{n_1}\triangle_1^{p_1}f\left(\frac{t_1}{n_1+p_1},\frac{k_2}{n_2+p_2},\cdots,\frac{k_m}{n_m+p_m}\right)\binom{n_1}{t_1}x_1^{t_1}(1-x_1)^{n_1-t_1}\right]$$
$$\cdot\prod_{i=2}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}\quad (\text{ using lemma 2.3.5})$$
$$= \sum_{k_3=0}^{n_3+p_3}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\frac{(n_1+p_1)!}{n!}\sum_{t_1=0}^{n_1}\left(\sum_{k_2=0}^{n_2+p_2}\triangle_1^{p_1}f\left(\frac{t_1}{n_1+p_1},\frac{k_2}{n_2+p_2},\cdots,\frac{k_m}{n_m+p_m}\right)\right.\right.$$
$$\left.\left.\cdot\binom{n_2}{k_2}x_2^{k_2}(1-x_2)^{n_2-k_2}\right)\binom{n_1}{t_1}x_1^{t_1}(1-x_1)^{n_1-t_1}\right]\prod_{i=3}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}\ .$$

We repeat the action with the second variable of the function

$\triangle_1^{p_1}f\left(\frac{t_1}{n_1+p_1},\frac{k_2}{n_2+p_2},\cdots,\frac{k_m}{n_m+p_m}\right)$, we get

$$B^{p_1,p_2}_{n_1+p_1,\cdots,n_m+p_m}(f;x_1,\cdots,x_m)$$
$$= \sum_{k_3=0}^{n_3+p_3}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\left(\prod_{i=1}^{2}\frac{(n_i+p_i)!}{n_i!}\right)\sum_{t_1=0}^{n_1}\sum_{t_2=0}^{n_2}\triangle_1^{p_1}\triangle_2^{p_2}f\left(\frac{t_1}{n_1+p_1},\frac{t_2}{n_2+p_2},\frac{k_3}{n_3+p_3},\cdots,\right.\right.$$
$$\left.\left.\frac{k_m}{n_m+p_m}\right)\prod_{i=1}^{2}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}\right]\times\prod_{i=3}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}$$
$$= \sum_{k_3=0}^{n_3+p_3}\cdots\sum_{k_m=0}^{n_m+p_m}\left[\left(\prod_{i=1}^{2}\frac{(n_i+p_i)!}{n_i!}\right)\sum_{t_1=0}^{n_1}\sum_{t_2=0}^{n_2}\left(\sum_{k_3=0}^{n_3+p_3}\triangle_1^{p_1}\triangle_2^{p_2}f\left(\frac{t_1}{n_1+p_1},\frac{t_2}{n_2+p_2},\frac{k_3}{n_3+p_3},\cdots,\right.\right.\right.$$
$$\left.\left.\left.\frac{k_m}{n_m+p_m}\right)\binom{n_3}{k_3}x_3^{k_3}(1-x_3)^{n_3-k_3}\right)\prod_{i=1}^{2}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}\right]\prod_{i=4}^{m}\binom{n_i}{k_i}x_i^{k_i}(1-x_i)^{n_i-k_i}\ .$$

After with the third variable of $\triangle_1^{t_1} \triangle_2^{t_2} f\left(\frac{t_1}{n_1+p_1}, \frac{t_2}{n_2+p_2}, \frac{k_3}{n_3+p_3}, \cdots, \frac{k_m}{n_m+p_m}\right)$, we get

$$\begin{aligned} & B^{p_1,p_2,p_3}_{n_1+p_1,\cdots,n_m+p_m}(f; x_1, \cdots, x_m) \\ = & \sum_{k_4=0}^{n_4+p_4} \cdots \sum_{k_m=0}^{n_m+p_m} \left[\left(\prod_{i=1}^{3} \frac{(n_i+p_i)!}{n_i!}\right) \sum_{t_2=0}^{n_2} \triangle_1^{p_1} \triangle_2^{p_2} f\left(\frac{t_1}{n_1+p_1}, \frac{t_2}{n_2+p_2}, \frac{t_3}{n_3+p_3}, \frac{k_4}{n_4+p_4}, \cdots, \right.\right. \\ & \left.\left. \frac{k_m}{n_m+p_m}\right) \prod_{i=1}^{3} \binom{n_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-k_i}\right] \prod_{i=4}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \ , \end{aligned}$$

which can be written as

$$\begin{aligned} & B^{p_1,p_2,p_3}_{n_1+p_1,\cdots,n_m+p_m}(f; x_1, \cdots, x_m) \\ = & \sum_{k_5=0}^{n_5+p_5} \cdots \sum_{k_m=0}^{n_m+p_m} \left[\left(\prod_{i=1}^{3} \frac{(n_i+p_i)!}{n_i!}\right) \sum_{t_1=0}^{n_1}\sum_{t_2=0}^{n_2}\sum_{t_3=0}^{n_3} \left(\sum_{k_4=0}^{n_4+p_4} \triangle_1^{p_1} \triangle_2^{p_2} \triangle_3^{p_3} f\left(\frac{t_1}{n_1+p_1}, \frac{t_2}{n_2+p_2}, \frac{t_3}{n_3+p_3}, \right.\right.\right. \\ & \left.\left.\left. \frac{k_4}{n_4+p_4}, \cdots, \frac{k_m}{n_m+p_m}\right) \binom{n_4}{t_4} x_4^{t_4}(1-x_4)^{n_4-k_4}\right) \prod_{i=1}^{3} \binom{n_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-k_i}\right] \\ & \cdot \prod_{i=5}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \ . \end{aligned}$$

By repeating successively the actions with the $4^{\text{th}}, 5^{\text{th}}, \cdots, m^{\text{th}}$ variables but with the functions $\triangle_1^{t_1} \triangle_2^{t_2} \triangle_3^{t_3} f\left(\frac{t_1}{n_1+p_1}, \frac{t_2}{n_2+p_2}, \frac{t_3}{n_3+p_3}, \frac{k_4}{n_4+p_4}, \cdots, \frac{k_m}{n_m+p_m}\right)$, $\triangle_1^{t_1} \triangle_2^{t_2} \triangle_3^{t_3} \triangle_4^{t_4} f\left(\frac{t_1}{n_1+p_1}, \frac{t_2}{n_2+p_2}, \frac{t_3}{n_3+p_3}, \frac{t_4}{n_4+p_4}, \frac{k_5}{n_5+p_5}, \cdots, \frac{k_m}{n_m+p_m}\right), \cdots, \triangle_1^{t_1} \triangle_2^{t_2} \cdots \triangle_{m-1}^{t_{m-1}} f\left(\frac{t_1}{n_1+p_1}, \frac{t_2}{n_2+p_2}, \cdots, \frac{t_{m-1}}{n_{m-1}+p_{m-1}}, \frac{k_m}{n_m+p_m}\right)$ respectively, we obtain the result. □

Theorem 3.2.7.

Let $0 \leq p_i \leq n_i, \ i = 1, \cdots, m$, be fixed an integers and the restriction of f to each of its variables i belong to $\mathcal{C}^{p_i}[0,1], \ i = 1, \cdots, m$. If

$$A \leq f^{(p_1)\cdots(p_m)}(x_1, \cdots, x_m) \leq B, \quad (x_1, \cdots, x_m) \in [0,1]^m$$

then $$A \leq \prod_{i=1}^{m} \frac{n_i^{p_i}}{n_i(n_i-1)\cdots(n_i-p_i+1)} B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) \leq B, \ (x_1, \cdots, x_m) \in [0,1]^m \ .$$

For all $p_i = 0, \ i = 1, \cdots, m$, the multiplier of $B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}$ is to be interpreted as 1.

- *If $f^{(p_1)\cdots(p_m)}(x_1, \cdots, x_m) \geq 0, \quad (x_1, \cdots, x_m) \in [0,1]^m$ then*
$$B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) \geq 0, \quad (x_1, \cdots, x_m) \in [0,1]^m$$
- *If $f(x_1, \cdots, x_m)$ is convex on $(x_1, \cdots, x_m) \in [0,1]^m$ then $B_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m)$ is convex there.*

Proof: We begin with equation (3.2.2) and replace n_i by $n_i - p_i,\ i = 1, \cdots, m$. We have

$$B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) = \left[\prod_{i=1}^{m} \frac{n_i!}{(n_i - p_i)!}\right] \sum_{t_1=0}^{n_1-p_1} \cdots \sum_{t_m=0}^{n_m-p_m} \triangle_1^{p_1} \cdots \triangle_m^{p_m} f\left(\frac{t_1}{n_1}, \cdots, \frac{t_m}{n_m}\right) \prod_{i=1}^{m} \binom{n_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-p_i-t_i} .$$

Then, using (2.1.1) with $h_i = \dfrac{1}{n_i},\ i = 1, \cdots, m$, we write

$$\triangle_1^{p_1} \cdots \triangle_m^{p_m} f\left(\frac{t_1}{n_1}, \cdots, \frac{t_m}{n_m}\right) = \frac{f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1}, \cdots, \varepsilon_{t_m})}{n_1^{p_1} \cdots n_m^{p_m}} ,$$

where $\dfrac{t_i}{n_i} < \varepsilon_{t_i} < \dfrac{t_i + p_i}{n_i},\ i = 1, \cdots, m$. Thus

$$\begin{aligned} &B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) \\ &= \left[\prod_{i=1}^{m} \frac{n_i!}{(n_i - p_i)!}\right] \sum_{t_1=0}^{n_1-p_1} \cdots \sum_{t_m=0}^{n_m-p_m} \frac{f^{(p_1)\cdots(p_m)}(x_1, \cdots, x_m)}{n_1^{p_1} \cdots n_m^{p_m}} \qquad (3.2.3)\\ &\quad \times \prod_{i=1}^{m} \binom{n_i - p_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-p_i-t_i} \\ &= \left[\prod_{i=1}^{m} \frac{n_i(n_i-1)\cdots(n_i-p_i+1)}{n_i^{p_i}}\right] \sum_{t_1=0}^{n_1-p_1} \cdots \sum_{t_m=0}^{n_m-p_m} f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1}, \cdots, \varepsilon_{t_m}) \\ &\quad \cdot \prod_{i=1}^{m} \binom{n_i - p_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-p_i-t_i} . \qquad (3.2.4) \end{aligned}$$

$$\begin{aligned} &A \le f^{(p_1)\cdots(p_m)}(x_1, \cdots, x_m) \le B, \quad (x_1, \cdots, x_m) \in [0,1]^m \\ &\Longrightarrow A \le f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1}, \cdots, \varepsilon_{t_m}) \le B \\ &\Longrightarrow A \le f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1}, \cdots, \varepsilon_{t_m}) \sum_{t_1=0}^{n_1-p_1} \cdots \sum_{t_m=0}^{n_m-p_m} \prod_{i=1}^{m} \binom{n_i - p_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-p_i-t_i} \le B \\ &\qquad \left(\text{since} \sum_{t_1=0}^{n_1-p_1} \cdots \sum_{t_m=0}^{n_m-p_m} \prod_{i=1}^{m} \binom{n_i - p_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-p_i-t_i} = 1\right) \\ &\Longrightarrow A \le \sum_{t_1=0}^{n_1-p_1} \cdots \sum_{t_m=0}^{n_m-p_m} f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1}, \cdots, \varepsilon_{t_m}) \prod_{i=1}^{m} \binom{n_i - p_i}{t_i} x_i^{t_i}(1-x_i)^{n_i-p_i-t_i} \le B \\ &\Longrightarrow A \le \left[\prod_{i=1}^{m} \frac{n_i^{p_i}}{n_i(n_i-1)\cdots(n_i-p_i+1)}\right] B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f; x_1, \cdots, x_m) \le B \quad \text{for all} \\ &\qquad (x_1, \cdots, x_m) \in [0,1]^m \quad (\text{using } (3.2.4)) . \end{aligned}$$

When all $p_i = 0$, the multiplier of $B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}$ is

$$\prod_{i=1}^{m} \frac{n_i^{p_i}}{n_i(n_i-1)\cdots(n_i-p_i+1)} = \prod_{i=1}^{m} \frac{(n_i-p_i)!n^{p_i}}{n_i!} = 1 \quad (p_i = 0\ \forall i = 1, \cdots, m) .$$

* If $f^{(p_1)\cdots(p_m)}(x_1,\cdots,x_m)\geq 0,\quad (x_1,\cdots,x_m)\in[0,1]^m$, we set $A=0$ and obtain

$$0\leq\left[\prod_{i=1}^{m}\frac{n_i^{p_i}}{n_i(n_i-1)\cdots(n_i-p_i+1)}\right]B_{n_1,\cdots,n_m}^{p_1,\cdots,p_m}(f;x_1,\cdots,x_m)\ \text{ for all } (x_1,\cdots,x_m)\in[0,1]^m\ ,$$

that means $B_{n_1,\cdots,n_m}^{p_1,\cdots,p_m}(f;x_1,\cdots,x_m)\geq 0,\quad (x_1,\cdots,x_m)\in[0,1]^m$.

* If f is convex, $\triangle_1^2\cdots\triangle_m^2\, f\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)\geq 0$ and hence we have $\frac{f^{(2)\cdots(2)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})}{n_1^2\cdots n_m^2}\geq 0$ since $\frac{f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})}{n_1^{p_1}\cdots n_m^{p_m}}=\triangle_1^{p_1}\cdots\triangle_m^{p_m}\, f\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)$, that means $f^{(2)\cdots(2)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})\geq 0$. Thus from (3.2.4) with $p_i=2\ \forall i=1,\cdots,m$,

$$B_{n_1,\cdots,n_m}^{2,\cdots,2}(f;x_1,\cdots,x_m)\geq 0,\ (x_1,\cdots,x_m)\in[0,1]^m.$$

That implies $B_{n_1,\cdots,n_m}$ is convex in every closed interval of $(0,1)^m$. since $B_{n_1,\cdots,n_m}$ is continuous, it is convex in $[0,1]^m$.

□

Theorem 3.2.8.

Let $0\leq p_i\leq n_i,\ i=0,\cdots,m$, be fixed integers, f continuous on $[0,1]^m$ and $\frac{\partial^{p_i}f}{\partial x_i^{p_i}}$ is continuous on $[0,1]^m,\ i=1\cdots m$. Then $B_{n_1,\cdots,n_m}^{p_1,\cdots,p_m}(f;x_1,\cdots,x_m)$ converges towards $f^{(p_1)\cdots(p_m)}(x_1,\cdots,x_m)$ uniformly on $[0,1]^m$.

Proof: We have already shown that the the above result holds for $p_i=0,\forall i=0,\cdots,m$. We have to see the case when p_i are not all zero.

We begin with the expression for $B_{n_1+p_1,\cdots,n_m+p_m}^{p_1,\cdots,p_m}(f;x_1,\cdots,x_m)$ given in (3.2.2) and write (using (2.1.1) with $h_i=\frac{1}{n_i+p_i},\ i=1,\cdots,m$)

$$\triangle_1^{p_1}\cdots\triangle_m^{p_m}\, f\left(\frac{t_1}{n_1+p_1},\cdots,\frac{t_m}{n_m+p_m}\right)=\frac{f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})}{\prod_{i=1}^{m}(n_i+p_i)^{p_i}}$$

where $\frac{t_i}{n_i+p_i}<\varepsilon_{t_i}<\frac{t_i+p_i}{n_i+p_i},\quad i=1,\cdots,m$. We get

$$\begin{aligned}B_{n_1+p_1,\cdots,n_m+p_m}^{p_1,\cdots,p_m}(f;x_1,\cdots,x_m)&=\left[\prod_{i=1}^{m}\frac{(n_i+p_i)!}{n_i!}\right]\sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m}\triangle_1^{p_1}\cdots\triangle_m^{p_m}\, f\left(\frac{t_1}{n_1+p_1},\cdots\right.\\&\qquad\left.,\frac{t_m}{n_m+p_m}\right)\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}\\&=\left[\prod_{i=1}^{m}\frac{(n_i+p_i)!}{n_i!(n_i+p_i)^{p_i}}\right]\sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m}f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})\\&\qquad\cdot\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}\ ,\end{aligned}$$

which give us

$$\left[\prod_{i=1}^{m}\frac{n_i!(n_i+p_i)^{p_i}}{(n_i+p_i)!}\right]B^{p_1,\cdots,p_m}_{n_1+p_1,\cdots,n_m+p_m}(f;x_1,\cdots,x_m) = \sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m} f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}\ . \tag{3.2.5}$$

We then approximate $f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})$ writing

$$\begin{aligned} & f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m}) \\ =& f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)+\left[f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})-f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)\right]\ . \end{aligned}$$

We thus obtain

$$\begin{aligned} &\left[\prod_{i=1}^{m}\frac{n_i!(n_i+p_i)^{p_i}}{(n_i+p_i)!}\right]B^{p_1,\cdots,p_m}_{n_1+p_1,\cdots,n_m+p_m}(f;x_1,\cdots,x_m) \\ &= \sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m}\left(f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)+\left[f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})\right.\right. \\ &\qquad \left.\left. -f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)\right]\right)\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i} \\ &= T_1(x_1,\cdots,x_m)+T_2(x_1,\cdots,x_m)\ \text{ where} \end{aligned}$$

$$\begin{cases} T_1(x_1,\cdots,x_m) = \displaystyle\sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m} f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i} \\ T_2(x_1,\cdots,x_m) = \displaystyle\sum_{t_1=0}^{n_1}\cdots\sum_{t_m=0}^{n_m}\left[f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})-f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)\right] \\ \qquad\qquad\qquad \cdot\displaystyle\prod_{i=1}^{m}\binom{n_i}{t_i}x_i^{t_i}(1-x_i)^{n_i-t_i}. \end{cases}$$

Since $\dfrac{t_i}{n_i+p_i}<\dfrac{t_i}{n_i}<\dfrac{t_i+p_i}{n_i+p_i},\quad i=1,\cdots,m,\ t_i=0,\cdots,n_i$ it follows from the bounds on $\varepsilon_{t_i},\ i=1,\cdots,m$, that

$$\left|\varepsilon_{t_i}-\frac{t_i}{n_i}\right|<\frac{t_i+p_i}{n_i+p_i}-\frac{t_i}{n_i+p_i}=\frac{p_i}{n_i+p_i}\ ,\quad i=1,\cdots,m.$$

From the uniform continuity of $f^{(p_1)\cdots(p_m)}(x_1,\cdots,x_m)$, given $\varepsilon>0$, we can find n_{i0} such that for all $n_i\geq n_{i0}$ and all t_i,

$$\left|f^{(p_1)\cdots(p_m)}(\varepsilon_{t_1},\cdots,\varepsilon_{t_m})-f^{(p_1)\cdots(p_m)}\left(\frac{t_1}{n_1},\cdots,\frac{t_m}{n_m}\right)\right|<\varepsilon.$$

Thus $T_2(x_1,\cdots,x_m)$ converges uniformly to zero on $[0,1]^m$. We have

$$\prod_{i=1}^{m}\frac{n_i!(n_i+p_i)^{p_i}}{(n_i+p_i)!}\longrightarrow 1\ \text{ as }\ n_i\to\infty,\ \forall i=1,\cdots,m\ ,$$

and we see from theorem (3.2.5) with $f^{(p_1)\cdots(p_m)}$ in place of f that $T_1(x_1,\cdots,x_m)$ converges uniformly to $f^{(p_1)\cdots(p_m)}(x_1,\cdots,x_m)$. This completes the proof. □

Theorem 3.2.9.

Let $f(x_1,\cdots,x_m)$ *be convex in* $[0,1]^m$. *Then, for* $n_i = 2,3,\cdots$,

$i = 1,\cdots,m$, *we have*

$$B_{n_1-1,\cdots,n_m-1}(f;x_1,\cdots,x_m) \geq B_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m), \quad (x_1,\cdots,x_m) \in [0,1]^m.$$

Proof: We have already solve the case for one variable in appendix; now because of long calculations, space limitation, we will just show how this theorem work for two variables.

Set $t_i = \dfrac{x_i}{1-x_i}$, $\quad i = 1,2$. Then

$$\begin{cases} x_i^{k_i}(1-x_i)^{-k_i} = t_i^{k_i} \\ 1+t_i = (1-x_i)^{-1} \end{cases} \qquad i.e. \qquad \begin{cases} x_i^{k_i}(1-x_i)^{-k_i} = t_i^{k_i} \\ x_i^{k_i}(1-x_i)^{-k_i-1} = t_i^{k_i}(1+t_i) \end{cases} \tag{3.2.6}$$

$$\begin{aligned}
&(1-x_1)^{-n_1}(1-x_2)^{-n_2}\left[B_{n_1-1,n_2-1}(f;x_1,x_2) - B_{n_1,n_2}(f;x_1,x_2)\right] \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)\binom{n_1-1}{k_1}\binom{n_2-1}{k_2} x_1^{k_1}(1-x_1)^{-k_1-1}x_2^{k_2}(1-x_2)^{-k_2-1} \\
&\quad - \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\binom{n_1}{k_1}\binom{n_2}{k_2} x_1^{k_1}(1-x_1)^{-k_1}x_2^{k_2}(1-x_2)^{-k_2} \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)\binom{n_1-1}{k_1}\binom{n_2-1}{k_2} t_1^{k_1}(1+t_1)t_2^{k_2}(1+t_2) \\
&\quad - \sum_{k_1=0}^{n_1}\cdots\sum_{k_m=0}^{n_m} f\left(\frac{k_1}{n_1},\cdots,\frac{k_m}{n_m}\right)\binom{n_1}{k_1}\binom{n_2}{k_2} t_1^{k_1}t_2^{k_2} \quad \text{(using (3.2.6))} \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2} t_1^{k_1}t_2^{k_2}\right](1+t_1+t_2+t_1t_2) \\
&\quad - \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\left[\binom{n_1}{k_1}\binom{n_2}{k_2} t_1^{k_1}t_2^{k_2}\right] \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2} t_1^{k_1}t_2^{k_2}\right] \\
&\quad + \sum_{k_1=1}^{n_1}\sum_{k_2=0}^{n_2-1} f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2} t_1^{k_1}t_2^{k_2}\right] \\
&\quad + \sum_{k_1=0}^{n_1-1}\sum_{k_2=1}^{n_2} f\left(\frac{k_1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2-1} t_1^{k_1}t_2^{k_2}\right] \\
&\quad + \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} f\left(\frac{k_1-1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\left[\binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2-1} t_1^{k_1}t_2^{k_2}\right] \\
&\quad - \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\left[\binom{n_1}{k_1}\binom{n_2}{k_2} t_1^{k_1}t_2^{k_2}\right] \quad \text{(using some change of variables).}
\end{aligned}$$

Then

$$
\begin{aligned}
&(1-x_1)^{-n_1}(1-x_2)^{-n_2}\left[B_{n_1-1,n_2-1}(f;x_1,x_2)-B_{n_1,n_2}(f;x_1,x_2)\right]\\
&= \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2}t_1^{k_1}t_2^{k_2}\right]\\
&\quad + \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2}t_1^{k_1}t_2^{k_2}\right]\\
&\quad + \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2-1}t_1^{k_1}t_2^{k_2}\right]\\
&\quad + \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1-1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\left[\binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2-1}t_1^{k_1}t_2^{k_2}\right]\\
&\quad - \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\left[\binom{n_1}{k_1}\binom{n_2}{k_2}t_1^{k_1}t_2^{k_2}\right]\\
&\quad + \sum_{k_2=0}^{n_2-1} f\left(0,\frac{k_2}{n_2-1}\right)\left[\binom{n_2-1}{k_2}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1-1},0\right)\left[\binom{n_1-1}{k_1}t_1^{k_1}\right]\\
&\quad + \sum_{k_2=0}^{n_2-1} f\left(1,\frac{k_2}{n_2-1}\right)\left[\binom{n_2-1}{k_2}t_1^{n_1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1-1}{n_1-1},0\right)\left[\binom{n_1-1}{k_1-1}t_1^{k_1}\right]\\
&\quad + \sum_{k_2=1}^{n_2} f\left(0,\frac{k_2-1}{n_2-1}\right)\left[\binom{n_2-1}{k_2-1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1-1},1\right)\left[\binom{n_1-1}{k_1}t_1^{k_1}t_2^{n_2}\right]\\
&\quad + \sum_{k_2=1}^{n_2} f\left(1,\frac{k_2-1}{n_2-1}\right)\left[\binom{n_2-1}{k_2-1}t_1^{n_1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1-1}{n_1-1},1\right)\left[\binom{n_1-1}{k_1-1}t_1^{k_1}t_2^{n_2}\right]\\
&\quad - \sum_{k_2=0}^{n_2} f\left(0,\frac{k_2}{n_2}\right)\left[\binom{n_2}{k_2}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1},0\right)\left[\binom{n_1}{k_1}t_1^{k_1}\right]\\
&\quad - \sum_{k_2=0}^{n_2} f\left(1,\frac{k_2}{n_2}\right)\left[\binom{n_2}{k_2}t_1^{n_1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1},1\right)\left[\binom{n_1}{k_1}t_1^{k_1}t_2^{n_2}\right]
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(1-x_1)^{-n_1}(1-x_2)^{-n_2}\left[B_{n_1-1,n_2-1}(f;x_1,x_2)-B_{n_1,n_2}(f;x_1,x_2)\right]\\
&= \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2}t_1^{k_1}t_2^{k_2}\right]\\
&\quad + \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2-1}\right)\left[\binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2}t_1^{k_1}t_2^{k_2}\right]\\
&\quad + \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2-1}t_1^{k_1}t_2^{k_2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1-1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\left[\binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2-1}t_1^{k_1}t_2^{k_2}\right]\\
&- \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-1} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\left[\binom{n_1}{k_1}\binom{n_2}{k_2}t_1^{k_1}t_2^{k_2}\right]\\
&+ \sum_{k_2=1}^{n_2-1} f\left(0,\frac{k_2}{n_2-1}\right)\left[\binom{n_2-1}{k_2}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1-1},0\right)\left[\binom{n_1-1}{k_1}t_1^{k_1}\right]\\
&+ \sum_{k_2=1}^{n_2-1} f\left(1,\frac{k_2}{n_2-1}\right)\left[\binom{n_2-1}{k_2}t_1^{n_1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1-1}{n_1-1},0\right)\left[\binom{n_1-1}{k_1-1}t_1^{k_1}\right]\\
&+ \sum_{k_2=1}^{n_2-1} f\left(0,\frac{k_2-1}{n_2-1}\right)\left[\binom{n_2-1}{k_2-1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1-1},1\right)\left[\binom{n_1-1}{k_1}t_1^{k_1}t_2^{n_2}\right]\\
&+ \sum_{k_2=1}^{n_2-1} f\left(1,\frac{k_2-1}{n_2-1}\right)\left[\binom{n_2-1}{k_2-1}t_1^{n_1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1-1}{n_1-1},1\right)\left[\binom{n_1-1}{k_1-1}t_1^{k_1}t_2^{n_2}\right]\\
&- \sum_{k_2=1}^{n_2-1} f\left(0,\frac{k_2}{n_2}\right)\left[\binom{n_2}{k_2}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1},0\right)\left[\binom{n_1}{k_1}t_1^{k_1}\right]\\
&- \sum_{k_2=1}^{n_2-1} f\left(1,\frac{k_2}{n_2}\right)\left[\binom{n_2}{k_2}t_1^{n_1}t_2^{k_2}\right] + \sum_{k_1=1}^{n_1-1} f\left(\frac{k_1}{n_1},1\right)\left[\binom{n_1}{k_1}t_1^{k_1}t_2^{n_2}\right] + f(0,0) + f(1,0)t_1^{n_1}\\
&+ f(0,1)t_2^{n_2} + f(1,1)t_1^{n_1}t_2^{n_2} - f(0,0) - f(0,1)t_2^{n_2} - f(1,0)t_1^{n_1} - f(1,1)t_1^{n_1}t_2^{n_2}\\
= &\sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-2}\left[\binom{n_1-1}{k_1}\binom{n_2-1}{k_2}f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right) + \binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2}f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2-1}\right)\right.\\
&+ \binom{n_1-1}{k_1}\binom{n_2-1}{k_2-1}f\left(\frac{k_1}{n_1-1},\frac{k_2-1}{n_2-1}\right) + \binom{n_1-1}{k_1-1}\binom{n_2-1}{k_2-1}f\left(\frac{k_1-1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\\
&\left. - \binom{n_1}{k_1}\binom{n_2}{k_2}f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\right]t_1^{k_1}t_2^{k_2}\\
&+ \sum_{k_1=1}^{n_1-1}\left[\binom{n_1-1}{k_1}f\left(\frac{k_1}{n_1-1},0\right) + \binom{n_1-1}{k_1-1}f\left(\frac{k_1-1}{n_1-1},0\right) - \binom{n_1}{k_1}f\left(\frac{k_1}{n_1},0\right)\right]t_1^{k_1}\\
&+ \sum_{k_1=1}^{n_1-1}\left[\binom{n_1-1}{k_1}f\left(\frac{k_1}{n_1-1},1\right) + \binom{n_1-1}{k_1-1}f\left(\frac{k_1-1}{n_1-1},1\right) - \binom{n_1}{k_1}f\left(\frac{k_1}{n_1},1\right)\right]t_1^{k_1}t_2^{n_2}\\
&+ \sum_{k_2=1}^{n_2-1}\left[\binom{n_2-1}{k_2}f\left(0,\frac{k_2}{n_2-1}\right) + \binom{n_2-1}{k_2-1}f\left(0,\frac{k_2-1}{n_2-1}\right) - \binom{n_2}{k_2}f\left(0,\frac{k_2}{n_2}\right)\right]t_2^{k_2}\\
&+ \sum_{k_2=1}^{n_2-1}\left[\binom{n_2-1}{k_2}f\left(1,\frac{k_2}{n_2-1}\right) + \binom{n_2-1}{k_2-1}f\left(1,\frac{k_2-1}{n_2-1}\right) - \binom{n_2}{k_2}f\left(1,\frac{k_2}{n_2}\right)\right]t_1^{n_1}t_2^{k_2}\ .
\end{aligned}
$$

It is obvious to show that

$$\binom{n-1}{k} = \binom{n}{k}\frac{n-k}{n} \quad \text{with } k,n \text{ positive integer } n \geq k.$$

So

$$
\begin{aligned}
&(1-x_1)^{-n_1}(1-x_2)^{-n_2}\left[B_{n_1-1,n_2-1}(f;x_1,x_2)-B_{n_1,n_2}(f;x_1,x_2)\right]\\
&= \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-2}\binom{n_1}{k_1}\binom{n_2}{k_2}\left[\frac{n_1-k_1}{n_1}\cdot\frac{n_2-k_2}{n_2}f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)+\frac{k_1}{n_1}\cdot\frac{n_2-k_2}{n_2}f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2-1}\right)\right.\\
&\qquad\left.+\frac{n_1-k_1}{n_1}\cdot\frac{k_2}{n_2}f\left(\frac{k_1}{n_1-1},\frac{k_2-1}{n_2-1}\right)+\frac{k_1}{n_1}\cdot\frac{k_2}{n_2}f\left(\frac{k_1-1}{n_1-1},\frac{k_2-1}{n_2-1}\right)-f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\right]t_1^{k_1}t_2^{k_2}\\
&\quad+\sum_{k_1=1}^{n_1-1}\binom{n_1}{k_1}\left[\frac{n_1-k_1}{n_1}f\left(\frac{k_1}{n_1-1},0\right)+\frac{k_1}{n_1}f\left(\frac{k_1-1}{n_1-1},0\right)-f\left(\frac{k_1}{n_1},0\right)\right]t_1^{k_1}\\
&\quad+\sum_{k_1=1}^{n_1-1}\binom{n_1}{k_1}\left[\frac{n_1-k_1}{n_1}f\left(\frac{k_1}{n_1-1},1\right)+\frac{k_1}{n_1}f\left(\frac{k_1-1}{n_1-1},1\right)-f\left(\frac{k_1}{n_1},1\right)\right]t_1^{k_1}t_2^{n_2}\\
&\quad+\sum_{k_2=1}^{n_2-1}\binom{n_2}{k_2}\left[\frac{n_2-k_2}{n_2}f\left(0,\frac{k_2}{n_2-1}\right)+\frac{k_2}{n_2}f\left(0,\frac{k_2-1}{n_2-1}\right)-f\left(0,\frac{k_2}{n_2}\right)\right]t_2^{k_2}\\
&\quad+\sum_{k_2=1}^{n_2-1}\binom{n_2}{k_2}\left[\frac{n_2-k_2}{n_2}f\left(1,\frac{k_2}{n_2-1}\right)+\frac{k_2}{n_2}f\left(1,\frac{k_2-1}{n_2-1}\right)-f\left(1,\frac{k_2}{n_2}\right)\right]t_1^{n_1}t_2^{k_2}\\
&= \sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2-2}\binom{n_1}{k_1}\binom{n_2}{k_2}\left[\frac{n_1-k_1}{n_1}\left(\frac{n_2-k_2}{n_2}f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2-1}\right)+\frac{k_2}{n_2}f\left(\frac{k_1}{n_1-1},\frac{k_2-1}{n_2-1}\right)\right.\right.\\
&\qquad\qquad\left.-f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2}\right)\right)\\
&\quad+\frac{k_1}{n_1}\left(\frac{n_2-k_2}{n_2}f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2-1}\right)+\frac{k_2}{n_2}f\left(\frac{k_1-1}{n_1-1},\frac{k_2-1}{n_2-1}\right)-f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2}\right)\right)\\
&\quad\left.+\left(\frac{n_1-k_1}{n_1}f\left(\frac{k_1}{n_1-1},\frac{k_2}{n_2}\right)+\frac{k_1}{n_1}f\left(\frac{k_1-1}{n_1-1},\frac{k_2}{n_2}\right)-f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\right)\right]t_1^{k_1}t_2^{k_2}\\
&\quad+\sum_{k_1=1}^{n_1-1}\binom{n_1}{k_1}\left[\frac{n_1-k_1}{n_1}f\left(\frac{k_1}{n_1-1},0\right)+\frac{k_1}{n_1}f\left(\frac{k_1-1}{n_1-1},0\right)-f\left(\frac{k_1}{n_1},0\right)\right]t_1^{k_1}\\
&\quad+\sum_{k_1=1}^{n_1-1}\binom{n_1}{k_1}\left[\frac{n_1-k_1}{n_1}f\left(\frac{k_1}{n_1-1},1\right)+\frac{k_1}{n_1}f\left(\frac{k_1-1}{n_1-1},1\right)-f\left(\frac{k_1}{n_1},1\right)\right]t_1^{k_1}t_2^{n_2}\\
&\quad+\sum_{k_2=1}^{n_2-1}\binom{n_2}{k_2}\left[\frac{n_2-k_2}{n_2}f\left(0,\frac{k_2}{n_2-1}\right)+\frac{k_2}{n_2}f\left(0,\frac{k_2-1}{n_2-1}\right)-f\left(0,\frac{k_2}{n_2}\right)\right]t_2^{k_2}\\
&\quad+\sum_{k_2=1}^{n_2-1}\binom{n_2}{k_2}\left[\frac{n_2-k_2}{n_2}f\left(1,\frac{k_2}{n_2-1}\right)+\frac{k_2}{n_2}f\left(1,\frac{k_2-1}{n_2-1}\right)-f\left(1,\frac{k_2}{n_2}\right)\right]t_1^{n_1}t_2^{k_2}
\end{aligned}
$$

For $i=1,2$, let us set:

$$
\lambda_i=\frac{n_i-k_i}{n_i},\quad x_1=\frac{k_i}{n_i-1},\quad x_2=\frac{k_i-1}{n_i-1}\ \text{ and }\ x_3=\frac{k_i}{n_i}\ .
$$

Then $1-\lambda_i = \dfrac{k_i}{n_i}$ and $\lambda_i x_1 + (1-\lambda_i)x_2 = \dfrac{k_i}{n_i} = x_3$. Hence, the fact that f is convex give us

$$\begin{cases} \dfrac{n_2-k_2}{n_2} f\left(\dfrac{k_1}{n_1-1}, \dfrac{k_2}{n_2-1}\right) + \dfrac{k_2}{n_2} f\left(\dfrac{k_1}{n_1-1}, \dfrac{k_2-1}{n_2-1}\right) - f\left(\dfrac{k_1}{n_1-1}, \dfrac{k_2}{n_2}\right) \geq 0 \\ \dfrac{n_2-k_2}{n_2} f\left(\dfrac{k_1-1}{n_1-1}, \dfrac{k_2}{n_2-1}\right) + \dfrac{k_2}{n_2} f\left(\dfrac{k_1-1}{n_1-1}, \dfrac{k_2-1}{n_2-1}\right) - f\left(\dfrac{k_1}{n_1-1}, \dfrac{k_2}{n_2}\right) \geq 0 \\ \dfrac{n_1-k_1}{n_1} f\left(\dfrac{k_1}{n_1-1}, \dfrac{k_2}{n_2}\right) + \dfrac{k_1}{n_1} f\left(\dfrac{k_1-1}{n_1-1}, \dfrac{k_2}{n_2}\right) - f\left(\dfrac{k_1}{n_1}, \dfrac{k_2}{n_2}\right) \geq 0 \\ \dfrac{n_1-k_1}{n_1} f\left(\dfrac{k_1}{n_1-1}, 0\right) + \dfrac{k_1}{n_1} f\left(\dfrac{k_1-1}{n_1-1}, 0\right) - f\left(\dfrac{k_1}{n_1}, 0\right) \geq 0 \\ \dfrac{n_1-k_1}{n_1} f\left(\dfrac{k_1}{n_1-1}, 1\right) + \dfrac{k_1}{n_1} f\left(\dfrac{k_1-1}{n_1-1}, 1\right) - f\left(\dfrac{k_1}{n_1}, 1\right) \geq 0 \\ \dfrac{n_2-k_2}{n_2} f\left(0, \dfrac{k_2}{n_2-1}\right) + \dfrac{k_2}{n_2} f\left(0, \dfrac{k_2-1}{n_2-1}\right) - f\left(0, \dfrac{k_2}{n_2}\right) \geq 0 \\ \dfrac{n_2-k_2}{n_2} f\left(1, \dfrac{k_2}{n_2-1}\right) + \dfrac{k_2}{n_2} f\left(1, \dfrac{k_2-1}{n_2-1}\right) - f\left(1, \dfrac{k_2}{n_2}\right) \geq 0. \end{cases}$$

Thus, we have $(1-x_2)^{-n_1}(1-x_2)^{-n_2}\left[B_{n_1-1,n_2-1}(f;x_1,x_2) - B_{n_1,n_2}(f;x_1,x_2)\right] \geq 0$. That means

$$B_{n_1-1,n_2-1}(f;x_1,x_2) \geq B_{n_1,n_2}(f;x_1,x_2), \ \forall x_1, x_2 \in [0,1] \ .$$

□

3.2.10 Order of convergence. To provide information about the speed of convergence of multivariate Bernstein polynomial, we need the following intermediate results.

Lemma 3.2.11.

There is a constant C independent of n_i such that for all $(x_1, \cdots, x_m) \in [0,1]^m, \ i = 1, \cdots, m,$

$$\sum_{\left|\frac{k_1}{n_1}-x_1\right| \geq n_1^{-1/4}} \cdots \sum_{\left|\frac{k_m}{n_m}-x_m\right| \geq n_m^{-1/4}} \prod_{i=1}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \leq \frac{C}{\prod_{i=1}^{m} n_i^{3/2}} \ .$$

Proof:

$$\begin{aligned} &\sum_{\left|\frac{k_1}{n_1}-x_1\right| \geq n_1^{-1/4}} \cdots \sum_{\left|\frac{k_m}{n_m}-x_m\right| \geq n_m^{-1/4}} \prod_{i=1}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \\ &= \prod_{i=1}^{m} \sum_{\left|\frac{k_i}{n_i}-x_i\right| \geq n_i^{-1/4}} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} \ . \end{aligned}$$

Using the lemma 2.3.12 we have

$$\sum_{\left|\frac{k_1}{n_1}-x_1\right|\geq n_1^{-1/4}} \cdots \sum_{\left|\frac{k_m}{n_m}-x_m\right|\geq n_m^{-1/4}} \prod_{i=1}^{m} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i}$$

$$\leq \prod_{i=1}^{m}\left(\frac{C_i}{n_i^{3/2}}\right) \quad \text{with } C_i \text{ constants}$$

$$= \frac{C}{\prod_{i=1}^{m} n_i^{3/2}} \quad \text{with } C \text{ constant } \ (C = C_1 \cdots C_m).$$

□

Remark 3.2.12. Lemma 2.3.12 is still valid if the summation is not on all the k_i, $1 \leq i \leq m$ but on some of them. In this case, the upper bound will contain only terms involved in the summation.

Theorem 3.2.13.

Let $f(x_1, \cdots, x_m)$ be bounded in $[0,1]^m]$ and let $(a_1, \cdots, a_m)$ be a point of $[0,1]^m$ at which $f^{(2)\cdots(2)}(a_1, \cdots, a_m)$ exists and is continuous. Then, for $n_1 = n_2 = \cdots = n_m = n$

$$\lim_{n\to\infty} n\left[B_{n,\cdots,n}(f; a_1, \cdots, a_m) - f(a_1, \cdots, a_m)\right] = \frac{1}{2}\sum_{i=1}^{m} a_i(1-a_i)\frac{\partial^2 f}{\partial x_i^2}(a_1, \cdots, a_m).$$

Proof: Since $f^{(2)\cdots(2)}(a_1, \cdots, a_m)$ exists and is continuous, then using Taylor expansion of order two

$$\begin{aligned} & f(x_1, \cdots, x_m) \\ =& f(a_1, \cdots, a_m) + \sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(a_1, \cdots, a_m)(x_i - a_i) + \frac{1}{2}\left[\sum_{i=1}^{m}\frac{\partial^2 f}{\partial x_i^2}(a_1, \cdots, a_m)(x_i-a_i)^2\right. \\ & \left. + \sum_{i,j=1, i\neq j}^{m}\frac{\partial^2 f}{\partial x_i \partial x_j}(a_1, \cdots, a_m)(x_i-a_i)(x_j-a_j)\right] + s(x_1, \cdots, x_m)\sum_{j=1}^{m}(x_j-a_j)^2 \end{aligned}$$

where $\lim_{(x_1,\cdots,x_m)\to(a_1,\cdots,a_m)} s(x_1, \cdots, x_m) = 0$. set $(x_1, \cdots, x_m) = \left(\frac{k_1}{n}, \cdots, \frac{k_m}{n}\right)$, multiplying both side by $\prod_{i=1}^{m}\binom{n}{k_i} a_i^{k_i}(1-a_i)^{n-k_i}$ and sum from $k_i = 0$ to $k_i = n$, for all $i = 1, \cdots, m$, we obtain

$$\begin{aligned} & B_{n,\cdots,n}(f; a_1, \cdots, a_m) \\ =& f(a_1, \cdots, a_m) + \sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(a_1, \cdots, a_m)\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\left(\frac{k_i}{n}-a_i\right)\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i} \\ & + \frac{1}{2}\sum_{i=1}^{m}\frac{\partial^2 f}{\partial x_i^2}(a_1, \cdots, a_m)\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\left(\frac{k_i}{n}-a_i\right)^2\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i,j=1,i\neq j}^{m}\frac{\partial^2 f}{\partial x_i\partial x_j}(a_1,\cdots,a_m)\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\left(\frac{k_i}{n}-a_i\right)\left(\frac{k_j}{n}-a_j\right)\prod_{i=1}^{m}\binom{n}{k_i}\\
&\cdot a_i^{k_i}(1-a_i)^{n-k_i}+\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}s\left(\frac{k_1}{n},\cdots,\frac{k_m}{n}\right)\sum_{j=1}^{m}\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n}{k_i}\\
&\qquad\cdot a_i^{k_i}(1-a_i)^{n-k_i}\\
=&f(a_1,\cdots,a_m)+\sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(a_1,\cdots,a_m)\sum_{k_i=0}^{n}\left(\frac{k_i}{n}-a_i\right)\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\\
&+\frac{1}{2}\sum_{i=1}^{m}\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m)\sum_{k_i=0}^{n}\left(\frac{k_i}{n}-a_i\right)^2\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\\
&+\frac{1}{2}\sum_{i,j=1,i\neq j}^{m}\frac{\partial^2 f}{\partial x_i\partial x_j}(a_1,\cdots,a_m)\sum_{k_i=0}^{n}\left(\frac{k_i}{n}-a_i\right)\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\\
&\qquad\cdot\sum_{k_j=0}^{n}\left(\frac{k_j}{n}-a_j\right)\binom{n}{k_j}a_j^{k_j}(1-a_j)^{n-k_j}\\
&+\sum_{j=1}^{m}\left[\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}s\left(\frac{k_1}{n},\cdots,\frac{k_m}{n}\right)\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right].
\end{aligned}
$$

Hence, using the equations (2.2.2)

$$
\begin{aligned}
&B_{n,\cdots,n}(f;a_1,\cdots,a_m)\\
=&f(a_1,\cdots,a_m)+\sum_{i=1}^{m}\frac{a_i(1-a_i)}{2n}\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m)\\
&+\sum_{j=1}^{m}\left[\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}s\left(\frac{k_1}{n},\cdots,\frac{k_m}{n}\right)\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right].
\end{aligned}
$$

Let

$$
S=\sum_{j=1}^{m}\left[\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}s\left(\frac{k_1}{n},\cdots,\frac{k_m}{n}\right)\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right]
$$

and $\varepsilon>0$. We can find $n,\ i=1,\cdots,m$ sufficiently large that $|x_i-a_i|<\dfrac{1}{n^{1/4}}$ implies $|s(x_1,\cdots,x_m)|\leq\varepsilon$ $\left(\text{since } \lim_{(x_1,\cdots,x_m)\to(a_1,\cdots,a_m)}s(x_1,\cdots,x_m)=0\right)$.

Let $\Omega_i=\left\{k_i\in\{0,\cdots,n_i\}:\ \left|\dfrac{k_i}{n}-a_i\right|<\dfrac{1}{n^{1/4}}\right\}$ for $i=1,\cdots,m$ and $F=\{0,\cdots,n\}\times\cdots\times\{0,\cdots,n\}\setminus\{\Omega_1\times\cdots\times\Omega_m\}$.

$$
\begin{aligned}
|S| \leq & \sum_{j=1}^{m}\left[\sum_{\Omega_1}\cdots\sum_{\Omega_m}\left|s\left(\frac{k_1}{n},\cdots,\frac{k_m}{n}\right)\right|\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n_i}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right.\\
&\left.+\sum_{F}\left|s\left(\frac{k_1}{n},\cdots,\frac{k_m}{n}\right)\right|\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right]\\
\leq & \ \varepsilon\sum_{j=1}^{m}\left[\sum_{k_1=0}^{n}\cdots\sum_{k_m=0}^{n}\left(\frac{k_j}{n}-a_j\right)^2\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right]\\
&+M\sum_{F}\left[\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right]\\
&\text{where } \ M=\sup_{(x_1,\cdots,x_m)\in[0,1]^m}|s(x_1,\cdots,x_m)|\sum_{j=1}^{m}(x_j-a_j)^2\\
\leq & \ \varepsilon\sum_{j=1}^{m}\left[\sum_{k_j=0}^{n}\left(\frac{k_j}{n}-a_j\right)^2\binom{n}{k_j}a_j^{k_j}(1-a_j)^{n-k_j}\right]+M\sum_{F}\left[\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right]\\
\leq & \ \varepsilon\sum_{i=1}^{m}\left(\frac{a_i(1-a_i)}{n}\right)+M\sum_{F}\left[\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i}\right] \quad (\text{ using equations (2.2.2)}).
\end{aligned}
$$

Using the same procedure as we have used to show the theorem 3.2.5, but with the lemma 3.2.11 instead of the lemma 3.2.3, we bounded the summation on F as follow:

$$
\begin{aligned}
\sum_{F}\prod_{i=1}^{m}\binom{n}{k_i}a_i^{k_i}(1-a_i)^{n-k_i} \leq & \sum_{i=1}^{m}\max_{1\leq i\leq m}\left\{\frac{C_i}{n^{3/2}}\right\}+\sum_{p=2}^{m-1}\sum_{1\leq i_1<\cdots<i_p\leq m}\max_{1\leq j\leq p}\left\{\frac{C_j}{n^{3/2}}\right\}\\
\leq & \ m\left(\max_{1\leq i\leq m}\left\{\frac{C_i}{n^{3/2}}\right\}\right)+\sum_{i=2}^{m-1}\binom{m}{i}\left(\max_{1\leq i\leq m}\left\{\frac{C_i}{n^{3/2}}\right\}\right)\\
= & \ (2^m-1)\left(\max_{1\leq i\leq m}\left\{\frac{C_i}{n^{3/2}}\right\}\right).
\end{aligned}
$$

Therefore,

$$
|S| \leq \ \varepsilon\sum_{i=1}^{m}\left(\frac{a_i(1-a_i)}{n}\right)+M\left[(2^m-1)\left(\max_{1\leq i\leq m}\left\{\frac{C_i}{n^{3/2}}\right\}\right)\right]
$$

and we have

$$
\begin{aligned}
&\left|n\left[B_{n,\cdots,n}(f;a_1,\cdots,a_m)-f(a_1,\cdots,a_m)\right]-\sum_{i=1}^{m}\frac{a_i(1-a_i)}{2}\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m)\right|\\
&=|nS| \leq \ \varepsilon\sum_{i=1}^{m}a_i(1-a_i)+M\left[(2^m-1)\left(\max_{1\leq i\leq m}\left\{\frac{C_i}{n^{1/2}}\right\}\right)\right].
\end{aligned}
$$

By taken n so large that $M\left[(2^m-1)\left(\max_{1\le i\le m}\left\{\frac{C_i}{n^{1/2}}\right\}\right)\right]<\varepsilon$ we obtain

$$\left|n\left[B_{n,\cdots,n}(f;a_1,\cdots,a_m)-f(a_1,\cdots,a_m)\right]-\sum_{i=1}^{m}\frac{a_i(1-a_i)}{2}\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m)\right|$$
$$=|nS|\le\ \varepsilon\left[1+\sum_{i=1}^{m}a_i(1-a_i)\right]\ .$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty}n\left[B_{n,\cdots,n}(f;a_1,\cdots,a_m)-f(a_1,\cdots,a_m)\right]=\sum_{i=1}^{m}\frac{a_i(1-a_i)}{2}\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m)\ .$$

□

Conclusion : The order of convergence of Bernstein polynomials with several variables for a bounded function $f(x_1,\cdots,x_m)$, $(x_1,\cdots,x_m)\in[0,1]^m$, in some neighborhood of $(a_1,\cdots,a_m)\in[0,1]^m$ containing in $[0,1]^m$, where $f^{(2)}(a_1,\cdots,a_m)$ exists, is least than Taylor's one. This convergence is very slow.

4. Conclusion

In this work, we have studied the properties of Bernstein polynomials with one variable, for a function $f(x)$ defined on $[0,1]$, given by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \ ,$$

and show how to generalized those properties to several variables by defining the polynomials

$$B_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_m=0}^{n_m} f\left(\frac{k_1}{n_1},\cdots,\frac{k_m}{n_m}\right)\prod_{i=1}^{m}\binom{n_i}{k_i} x_i^k(1-x_i)^{n_i-k_i}$$

called the $(n_1,\cdots,n_m)^{\text{th}}$ Bernstein polynomials for $f(x_1,\cdots,x_m)$ and which also keep the same properties. Amount those properties, the major's ones are:

1)* If $f\in\mathcal{C}[0,1]^m$, the $B_{n_1,\cdots,n_m}(f;\cdot)$ converges uniformly to f,

2)* If $f\in\mathcal{C}^{p_1\cdots p_m}[0,1]^m$, then $B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f;\cdot)$ converges uniformly to $f^{(p_1)\cdots,(p_m)}$, where $f\in\mathcal{C}^{p_1,\cdots,p_m}[0,1]^m$ means f is continuous on $[0,1]^m$ and $\dfrac{\partial^{p_i}f}{\partial x^{p_i}}$ is continuous on $[0,1]^m$, $0\le p_i\le n_i,\ i=1,\cdots,m,$

3)* If $f\in\mathcal{C}^{p_1\cdots p_m}[0,1]^m,\ 0\le p_i\le n_i,\ i=1,\cdots,m,$ and $A\le f^{(p_1)\cdots(p_m)}(x_1,\cdots,x_m)\le B,\quad (x_1,\cdots,x_m)\in[0,1]^m$, then

$$A\le\prod_{i=1}^{m}\frac{n_i^{p_i}}{n_i(n_i-1)\cdots(n_i-p_i+1)}B^{p_1,\cdots,p_m}_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m)\le B,\ (x_1,\cdots,x_m)\in[0,1]^m,$$

4)* If $f(x_1,\cdots,x_m)$ is convex in $[0,1]^m$, then, for $n_i=2,3,\cdots,\quad i=1,\cdots,m$, we have

$$B_{n_1-1,\cdots,n_m-1}(f;x_1,\cdots,x_m)\ge B_{n_1,\cdots,n_m}(f;x_1,\cdots,x_m),\quad (x_1,\cdots,x_m)\in[0,1]^m,$$

5)* If $f(x_1,\cdots,x_m)$ is bounded in $[0,1]^m]$ and let $(a_1,\cdots,a_m)$ is a point of $[0,1]^m$ at which $f^{(2)\cdots(2)}(a_1,\cdots,a_m)$ exists and is continuous, then, for $n_1=n_2=\cdots=n_m=n$

$$\lim_{n\to\infty} n\left[B_{n,\cdots,n}(f;a_1,\cdots,a_m)-f(a_1,\cdots,a_m)\right]=\frac{1}{2}\sum_{i=1}^{m}a_i(1-a_i)\frac{\partial^2 f}{\partial x_i^2}(a_1,\cdots,a_m).$$

Using Bernstein polynomials with one and several variables we have a proof and the generalization of Weierstrass theorem. More better, we also have an explicit representation of polynomials for uniform approximation.

Currently, it is shown that the convergence order of the discussed polynomials is very slow. Therefore, we recommend in our future work to improve this speed of convergence.

The proofs we have provided for multivariate Bernstein polynomial are our own contribution. We are in process of cheeking if there are references in relation to this but we believe that some of these proofs might be new results.

Appendix A.

A.1 Proof of some lemmas and theorems from chapter 2

For a well understanding about the proof in Chapter 3, let us recall some proofs of chapter 2. All those proofs have been done in [Dav75], but here we give them with some explanations which are necessary to understand them very well.

A.1.1 Proof of some lemmas from chapter 2.

Lemma 2.3.2

Proof: $\left|\frac{k}{n}-x\right| \geq \delta \Longrightarrow \frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2 \geq 1$. Hence

$$\begin{aligned}\sum_{\left|\frac{k}{n}-x\right|\geq\delta} \binom{n}{k} x^k(1-x)^{n-k} &\leq \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \frac{1}{\delta^2}(\frac{k}{n}-x)^2 \binom{n}{k} x^k(1-x)^{n-k} \\ &\leq \frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 \binom{n}{k} x^k(1-x)^{n-k} \\ &\leq \frac{1}{\delta^2}\frac{x(1-x)}{n} \quad \text{(using one of equations (2.2.2))} \\ &\leq \frac{1}{4n\delta^2} \quad \text{since} \quad \forall x\in[0,1], \quad x(1-x)\leq\frac{1}{4}\ .\end{aligned}$$

□

Lemma 2.3.5

Proof: We write

$$B_{n+p}(f;x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n+p}\right)\binom{n+p}{k} x^k(1-x)^{n+p-k}$$

and differentiate p times, giving

$$B^{(p)}_{n+p}(f;x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n+p}\right)\binom{n+p}{k} P(x)$$

$$\text{where} \quad P(x) = \frac{d^p}{dx^p} x^k(1-x)^{n+p-k}\ .$$

We now use the Leibniz rule to differentiate the product x^k and $(1-x)^{n+p-k}$. We find that

$$\frac{d^j}{dx^j}x^k = \begin{cases} \frac{k!}{(k-j)!}x^{k-j}\ , & k-j\geq 0 \\ 0\ , & k-j<0 \end{cases}$$

and

$$\frac{d^{p-j}}{dx^{p-j}}(1-x)^{n+p-k} = \begin{cases} (-1)^{p-j}\frac{(n+p-k)!}{(n+j-k)!}(1-x)^{n+j-k}\ , & k-j\leq n \\ 0\ , & k-j>n. \end{cases}$$

Thus the p^{th} derivative of $x^k(1-x)^{n+p-k}$ is

$$P(x) = \sum_{j=0,(0\le k-j\le n)}^{p} (-1)^{p-j}\binom{p}{j}\frac{k!}{(k-j)!}\frac{(n+p-k)!}{(n+j-k)!}x^{k-j}(1-x)^{n+j-k}$$

We also note that

$$\binom{n+p}{k}\frac{k!}{(k-j)!}\frac{(n+p-k)!}{(n+j-k)!} = \frac{(n+p)!}{n!}\binom{n}{k-j}\ , \qquad \text{(A.1.1)}$$

hence

$$\begin{aligned}
&B_{n+p}^{(p)}(f;x)\\
&= \sum_{k=0}^{n+p}\sum_{j=0,(0\le k-j\le n)}^{p} f\left(\frac{k}{n+p}\right)\binom{p}{j}\binom{n+p}{k}\frac{k!}{(k-j)!}\frac{(n+p-k)!}{(n+j-k)!}(-1)^{p-j}x^{k-j}(1-x)^{n+j-k}\\
&= \sum_{k=0}^{n+p}\sum_{j=0,(0\le k-j\le n)}^{p} f\left(\frac{k}{n+p}\right)\binom{p}{j}\frac{(n+p)!}{n!}\binom{n}{k-j}(-1)^{p-j}x^{k-j}(1-x)^{n+j-k} \quad \text{using equation (A.1.1)}\\
&= \frac{(n+p)!}{n!}\sum_{t=0}^{n}\sum_{j=0}^{p} f\left(\frac{t+j}{n+p}\right)\binom{p}{j}\binom{n}{t}(-1)^{p-j}x^t(1-x)^{n-t}\\
&\qquad \text{with } t=k-j \ \ i.e. \ \begin{cases} 0\le t\le n\\ 0\le j\le p\end{cases}\\
&= \frac{(n+p)!}{n!}\sum_{t=0}^{n}\left[\sum_{j=0}^{p} f\left(\frac{t+j}{n+p}\right)\binom{p}{j}(-1)^{p-j}\right]\binom{n}{t}x^t(1-x)^{n-t}\\
&= \frac{(n+p)!}{n!}\sum_{t=0}^{n}\triangle^p f\left(\frac{t}{n+p}\right)\binom{n}{t}x^t(1-x)^{n-t}
\end{aligned}$$

where $\triangle$ is applied with step size $h = \dfrac{1}{n+p}$. $\square$

Lemma 2.3.12

Proof: Let $n\ge 1,\ m\ge 0$ be integers, and consider for $x\in[0,1]$ the sum

$$S_{m,n}(x) = \sum_{k=0}^{n}(k-nx)^m\binom{n}{k}x^k(1-x)^{n-k} \qquad \text{(A.1.2)}$$

which satisfy the relation

$$\begin{cases} S_{m+1,n}(x) = x(1-x)\left[S'_{m,n}(x) + mnS_{m-1,n}(x)\right]\\ S_{0,n}(x)=1 \qquad S_{1,n}(x)=0 \qquad S_{2,n}(x)=nx(1-x).\end{cases}$$

We may conclude from this recurrence that each sum $S_{m,n}(x)$ is a polynomial in x. Using this recurrence relation, we have for some constant C, $|S_{6,n}(x)|\le Cn^3$ for $x\in[0,1]$.

In as much as $\left|\frac{k}{n} - x\right| \geq n^{-1/4}$ implies $\frac{(k-nx)^6}{n^{9/2}} \geq 1$

$$\begin{aligned}
\sum_{\left|\frac{k}{n}-x\right|\geq n^{-1/4}} \binom{n}{k} x^k(1-x)^{n-k} &\leq \sum_{\left|\frac{k}{n}-x\right|\geq n^{-1/4}} \frac{1}{n^{9/2}}(k-nx)^6 \binom{n}{k} x^k(1-x)^{n-k} \\
&\leq \frac{1}{n^{9/2}} \sum_{k=0}^{n} (k-nx)^6 \binom{n}{k} x^k(1-x)^{n-k} \\
&= \frac{1}{n^{9/2}} S_{6,n}(x) \quad \text{(using equation (2.2.2) with } m=6) \\
&\leq \frac{Cn^3}{n^{9/2}} = \frac{C}{n^{3/2}} \ .
\end{aligned}$$

□

A.1.2 Proof of some theorems from chapter 2.

Theorem 2.3.3

Proof: The function $f(x)$ is assumed bounded in $[0,1]$. Hence, for some $M > 0$, $|f(x)| \leq M$ and for any two values $\alpha, \beta \in [0,1]$,

$$|f(\alpha_1, \cdots, \alpha_m) - f(\beta_1, \cdots, \beta_m)| \leq 2M \ .$$

Let $x \in [0,1]$ be a point of continuity of f. Given $\varepsilon > 0$, we can find δ depending on x and ε such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$, whenever $|x-y| < \delta$. Since $B_n(1;x) = 1$ we have

$$f(x) = f(x)B_n(1;x) = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k(1-x)^{n-k} \ .$$

$$\begin{aligned}
&|f(x) - B_n(f;x)| \\
&\leq \sum_{k=0}^{n} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k(1-x)^{n-k} \\
&\leq \sum_{\left|\frac{k}{n}-x\right|<\delta} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k(1-x)^{n-k} + \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k(1-x)^{n-k} \\
&\leq \frac{\varepsilon}{2} \sum_{\left|\frac{k}{n}-x\right|<\delta} \binom{n}{k} x^k(1-x)^{n-k} + 2M \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \binom{n}{k} x^k(1-x)^{n-k} \\
&\leq \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} + 2M\left(\frac{1}{4n\delta^2}\right) \quad \text{(using lemma 2.3.2)} \\
&\leq \frac{\varepsilon}{2} + \frac{M}{2n\delta^2} \quad \text{(using one of the equations 2.2.1)} \ .
\end{aligned}$$

For n sufficiently large, we have $\frac{M}{2n\delta^2} < \frac{\varepsilon}{2}$. Then $|f(x) - B_n(f;x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose now that $f \in \mathcal{C}[0,1]$, then f is uniformly continuous on $[0,1]$ i.e. given $\varepsilon > 0$, we can find δ depending just on ε such that $|f(x) - f(y)| < \dfrac{\varepsilon}{2}$ for all $x, y \in [0,1]$ satisfying $|x - y| < \delta$. Hence the inequality

$$|f(x) - B_n(f;x)| < \varepsilon$$

holds independently of the x selected and the convergence to $f(x)$ is uniform in $[0,1]$. □

Theorem 2.3.8

Proof: We have already show that the the above result holds for $p = 0$. We have to see the case when p are not all zero.

We begin with the expression for $B^p_{n+p}(f;x)$ given in (3.2.2) and write (using (2.1.1) with $h = \dfrac{1}{n+p}$)

$$\triangle^p f\left(\frac{t}{n+p}\right) = \frac{f^{(p)}(\varepsilon_t)}{(n+p)^p}$$

where $\dfrac{t}{n+p} < \varepsilon_t < \dfrac{t+p}{n+p}$.

$$B^p_{n+p}(f;x) = \frac{(n+p)!}{n!(n+p)^p} \sum_{t=0}^{n} f^{(p)}(\varepsilon_t) \binom{n}{t} x^t (1-x)^{n-t} .$$

We then approximate $f^{(p)}(\varepsilon_t)$ writing

$$f^{(p)}(\varepsilon_t) = f^{(p)}\left(\frac{t}{n}\right) + \left[f^{(p)}(\varepsilon_{t_1}) - f^{(p)}\left(\frac{t}{n}\right)\right] .$$

We thus obtain

$$\frac{n!(n+p)^p}{(n+p)!} B^p_{n+p}(f;x) = T_1(x) + T_2(x) \quad \text{where}$$

$$\begin{cases} T_1(x) = \displaystyle\sum_{t=0}^{n} f^{(p)}\left(\frac{t}{n}\right)\binom{n}{t} x^t(1-x)^{n-t} \\ T_2(x) = \displaystyle\sum_{t=0}^{n} \left[f^{(p)}(\varepsilon_t) - f^{(p)}\left(\frac{t}{n}\right)\right]\binom{n}{t} x^t(1-x)^{n-t}. \end{cases}$$

Since $\dfrac{t}{n+p} < \dfrac{t}{n} < \dfrac{t+p}{n+p}$, $t = 0, \cdots, n$ it follows from the bounds on ε_t that

$$\left|\varepsilon_t - \frac{t}{n}\right| < \frac{t+p}{n+p} - \frac{t}{n+p} = \frac{p}{n+p} .$$

From the uniform continuity of $f^{(p)}(x)$, given $\varepsilon > 0$, we can find n_0 such that for all $n \geq n_0$ and all t,

$$\left|f^{(p)}(\varepsilon_t) - f^{(p)}\left(\frac{t}{n}\right)\right| < \varepsilon.$$

Thus $T_2(x)$ converges uniformly to zero on $[0,1]$. We have

$$\frac{n!(n+p)^p}{(n+p)!} \longrightarrow 1 \quad \text{as} \quad n \to \infty ,$$

and we see from theorem (3.2.5) with $f^{(p)}$ in place of f that $T_1(x)$ converges uniformly to $f^{(p)}(x)$. This completes the proof. □

Theorem 2.3.10

Proof: Set $t = \dfrac{x}{1-x}$. Then

$$\left\{\begin{array}{l} x^k(1-x)^{-k} = t^k \\ 1+t = (1-x)^{-1} \end{array}\right. \qquad i.e. \qquad \left\{\begin{array}{l} x^k(1-x)^{-k} = t^k \\ x^k(1-x)^{-k-1} = t^k(1+t) \end{array}\right. \tag{A.1.3}$$

$$\begin{aligned}
& (1-x)^{-n}\left[B_{n-1}(f;x) - B_n(f;x)\right] \\
= & \sum_{k}^{n} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} x^k(1-x)^{-k-1} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{-k} \\
= & \sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^k(1+t) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k \quad \text{(using (A.1.3))} \\
= & \sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^k(1+t) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k \\
= & \sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^k(1+t) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k \\
= & \sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^k + \sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^{k+1} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k.
\end{aligned}$$

By using the change of variable, replacing $k+1$ by k in the second term of the left of our previous equality, we have

$$\begin{aligned}
& (1-x)^{-n}\left[B_{n-1}(f;x) - B_n(f;x)\right] \\
= & \sum_{k=0}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^k + \sum_{k=1}^{n} f\left(\frac{k-1}{n-1}\right)\binom{n-1}{k-1} t^k - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k \\
= & \sum_{k=1}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k} t^k + \sum_{k=1}^{n-1} f\left(\frac{k-1}{n-1}\right)\binom{n-1}{k-1} t^k - \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\binom{n}{k} t^k \\
& \qquad + f(0) + f(1) - f(0) - f(1) \\
= & \sum_{k=1}^{n-1}\left[f\left(\frac{k}{n-1}\right)\binom{n-1}{k} + f\left(\frac{k-1}{n-1}\right)\binom{n-1}{k-1} - f\left(\frac{k}{n}\right)\binom{n}{k} t^k\right.
\end{aligned}$$

It is obvious to show that $\dbinom{n-1}{k} = \dbinom{n}{k}\dfrac{n-k}{n}$ and $\dbinom{n-1}{k-1} = \dbinom{n}{k}\dfrac{k}{n}$. Then

$$\begin{aligned}
& (1-x)^{-n}\left[B_{n-1}(f;x) - B_n(f;x)\right] \\
= & \sum_{k=1}^{n-1}\binom{n}{k}\left[\frac{n-k}{n} f\left(\frac{k}{n-1}\right) + \frac{k}{n} f\left(\frac{k-1}{n-1}\right) - f\left(\frac{k}{n}\right)\right] t^k \\
= & \sum_{k=1}^{n-1}\binom{n}{k} d_k t^k \quad \text{where } d_k = \frac{n-k}{n} f\left(\frac{k}{n-1}\right) + \frac{k}{n} f\left(\frac{k-1}{n-1}\right) - f\left(\frac{k}{n}\right).
\end{aligned}$$

Let us write

$$\lambda = \frac{n-k}{n}, \quad x_1 = \frac{k}{n-1}, \quad x_2 = \frac{k-1}{n-1}, \quad x_3 = \frac{k}{n} \ .$$

It follows that $1-\lambda = \dfrac{k}{n}$ and $\lambda x_1 + (1-\lambda)x_2 = \dfrac{k}{n} = x_3$. Hence

$$\begin{aligned} d_k =& \lambda f(x_1) + (1-\lambda) f(x_2) - f(x_3) \\ =& \lambda f(x_1) + (1-\lambda) f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \\ \geq& 0 \ \text{ since } f \text{ is convex.} \end{aligned}$$

So $(1-x)^{-n}\left[B_{n-1}(f;x) - B_n(f;x)\right] \geq 0,$ and we deduce that $B_{n-1}(f;x) \geq B_n(f;x), \quad x \in [0,1]$.

Let $f \in \mathcal{C}[0,1]$ and f is linear in each intervals $\left[\dfrac{j-1}{n-1}, \dfrac{j}{n-1}\right]$, $j = 1, \cdots, n-1$; then for $k = 1, \cdots, n-1$

$$\begin{aligned} d_k =& \frac{n-k}{n} f\left(\frac{k}{n-1}\right) + \frac{k}{n} f\left(\frac{k-1}{n-1}\right) - f\left(\frac{k}{n}\right) \\ =& f\left(\frac{n-k}{n}\frac{k}{n-1} + \frac{k}{n}\frac{k-1}{n-1} - \frac{k}{n}\right) \\ =& f(0) \\ =& 0 \quad \forall 0 \leq k \leq n-1. \end{aligned}$$

That implies

$$(1-x)^{-n}\left[B_{n-1}(f;x) - B_n(f;x)\right] = 0 \ ,$$

and then we have

$$B_{n-1}(f;x) = B_n(f;x), \quad x \in [0,1].$$

Conversely, if $B_{n-1}(f;x) = B_n(f;x)$, for all $x \in [0,1]$ then

$$\sum_{k=1}^{n-1} \binom{n}{k} d_k t^k = 0$$

$$\Longrightarrow \binom{n}{k} d_k t^k = 0 \quad \forall k = 1, \cdots, n-1 \ \Longrightarrow d_k = 0 \quad \forall k = 1, \cdots, n-1$$

$$\text{since } \binom{n}{k} \neq 0 \ \forall k = 1, \cdots, n-1 \ .$$

Since $f \in \mathcal{C}[0,1]$ and f is convex, then f is linear in each interval $\left[\dfrac{j-1}{n-1}, \dfrac{j}{n-1}\right]$, $j = 1, \cdots, n-1,$.
□

Theorem 2.3.13

Proof: Since $f''(x_0)$ exists and is continuous, then using Taylor expansion of order two

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2} + s(x)(x-x_0)^2$$

where $\lim_{x\to x_0} s(x) = 0$. set $x = \frac{k}{n}$, multiplying both side by $\binom{n}{k} x_0^k(1-x_0)^{n-k}$ and sum from $k=0$ to $k=n$, we obtain

$$\begin{aligned}
B_n(f;x) =& f(x_0) + f'(x_0)\sum_{k=0}^{n}\left(\frac{k}{n}-x_0\right)\binom{n}{k} x_0^k(1-x_0)^{n-k} + \frac{f''(x_0)}{2}\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k}\\
&+\sum_{k=0}^{n} s\left(\frac{k}{n}\right)\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k}\\
=& f(x_0) + \frac{x_0(1-x_0)f''(x_0)}{2n} + \sum_{k=0}^{n} s\left(\frac{k}{n}\right)\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k}\\
&\text{(using the equations (2.2.2))}\ .
\end{aligned}$$

Let

$$S = \sum_{k=0}^{n} s\left(\frac{k}{n}\right)\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k}$$

and $\varepsilon > 0$. We can find n sufficiently large that $|x-x_0| < \frac{1}{n^{1/4}}$ implies $|s(x)| \le \varepsilon$ (since $\lim_{x\to x_0} s(x) = 0$).

$$\begin{aligned}
|S| \le & \sum_{\left|\frac{k}{n}-x_0\right|<n^{-1/4}} \left|s\left(\frac{k}{n}\right)\right|\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k}\\
&+\sum_{\left|\frac{k}{n}-x_0\right|\ge n^{-1/4}} \left|s\left(\frac{k}{n}\right)\right|\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k}\\
\le\ & \varepsilon\sum_{k_1=0}^{n}\left(\frac{k}{n}-x_0\right)^2\binom{n}{k} x_0^k(1-x_0)^{n-k} + M\sum_{\left|\frac{k}{n}-x_0\right|\ge n^{-1/4}}\binom{n}{k} x_0^k(1-x_0)^{n-k}\\
&\text{where } M = \sup_{0\le x\le 1}|s(x)|(x-x_0)^2\\
\le\ & \varepsilon\left(\frac{x_0(1-x_0)}{n}\right) + \frac{MC}{n^{3/2}} \quad \text{for some constant } C\\
&\text{(using lemma (2.3.12) and one of the equations (2.2.2))}
\end{aligned}$$

$$\begin{aligned}
&\left|B_n(f;x_0) - f(x_0) - \frac{x_0(1-x_0)}{2n}f''(x_0)\right| = |S| \le \frac{\varepsilon x_0(1-x_0)}{n} + \frac{MC}{n^{3/2}}\\
\Longrightarrow\ &\left|n\left[B_n(f;x_0) - f(x_0)\right] - \frac{x_0(1-x_0)}{2}f''(x_0)\right| = |nS| \le \varepsilon x_0(1-x_0) + \frac{MC}{n^{1/2}}\ .
\end{aligned}$$

By taken n so large that $\frac{MC}{n^{1/2}} < \varepsilon/2$ we have

$$\left|n\left[B_n(f;x_0) - f(x_0)\right] - \frac{x_0(1-x_0)}{2}f''(x_0)\right| = |nS| \le \varepsilon\left[x_0(1-x_0)+1\right]\ .$$

Since ε is arbitrary, then

$$\lim_{n\to\infty} n\left[B_n(f;x_0) - f(x_0)\right] = \frac{x_0(1-x_0)}{2} f''(x_0)\ .$$

□

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