Fachbereich Mathematik **WiSe 2012/2013**
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Numerik steifer Probleme

Aufgabenblatt 12

Aufgabe 1

Suppose, we want to solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ by the multigrid method. Let $\mathbf{G}_l$ denote the iteration matrix of the smoother $S_l(\mathbf{x}_l, \mathbf{b}_l)$. Consider the following properties.

- *Smoothing property:* There exists a function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\eta(\nu) \to 0$ for $\nu \to \infty$ and
$$\|\boldsymbol{A}_l \boldsymbol{G}_l^{\nu}\| \leq \eta(\nu)\|\boldsymbol{A}_l\| \quad \forall l \geq 0.$$
- *Approximation property:* For some constant $C_A > 0$, independent of l, it holds
$$\|\boldsymbol{A}_l^{-1} - \boldsymbol{P}_{l,l-1}\boldsymbol{A}_{l-1}^{-1}\boldsymbol{R}_{l-1,l}\| \leq C_A\|\boldsymbol{A}_l\|^{-1} \quad \forall l \geq 1.$$

Prove that the iterative method $S_l(\mathbf{x}_l, \mathbf{b}_l)$ converges, if the smoothing property holds.

Furthermore, assume both the smoothing and approximation property. Prove that under these assumptions there exists $\bar{\nu} \in \mathbb{N}$ such that the two-grid method with $\nu \geq \bar{\nu}$ presmoothings and no postsmoothing converges. (4 P)

Aufgabe 2

Consider the possibly nonlinear system of equations $\mathbf{f}(\mathbf{u}) = \mathbf{s}$ to be solved by FAS-MG.

a) To which scheme does FAS-MG reduce if $\mathbf{f}$ is a linear operator?

b) Consider the following reduced two-grid variant of FAS-MG

   `Function ReducedFAS` $(\mathbf{u}_l, \mathbf{s}_l, l)$

   - $\mathbf{r}_l = \mathbf{s}_l - \mathbf{f}_l(\mathbf{u}_l)$
   - $\tilde{\mathbf{u}}_{l-1} = \mathbf{R}_{l-1,l}\mathbf{u}_l$
   - $\mathbf{s}_{l-1} = \mathbf{f}_{l-1}(\tilde{\mathbf{u}}_{l-1}) + \mathbf{R}_{l-1,l}\mathbf{r}_l$
   - Solve $\mathbf{f}_{l-1}(\mathbf{u}_{l-1}) = \mathbf{s}_{l-1}$ exactly
   - $\mathbf{u}_l = \mathbf{u}_l + \mathbf{P}_{l,l-1}(\mathbf{u}_{l-1} - \tilde{\mathbf{u}}_{l-1})$

   This method reduces to the coarse grid correction in case of a linear operator $\mathbf{f}$. Furthermore, assume that $\mathbf{P}_{l,l-1}$ has maximum rank $l-1$.

   Show that $\mathbf{u}_l$ is a fixed point of the above scheme if and only if $\mathbf{u}_l$ is an exact solution to the restricted fine grid problem, i.e. $\mathbf{R}_{l-1,l}\mathbf{f}_l(\mathbf{u}_l) = \mathbf{R}_{l-1,l}\mathbf{s}_l$.

(4 P)

Aufgabe 3
Consider Burgers' equation

$$u_t + f(u)_x = 0, \quad f(u) = \frac{1}{2}u^2,$$

for a function $u : \mathbb{R}_+ \times [0,2] \to \mathbb{R}$ depending on the variables $t \in \mathbb{R}_+$ and $x \in [0,2]$. We furthermore provide initial conditions $u_0(x) = u(0,x) = \sin(\pi x)$ and periodic boundary conditions. Discretize this equation in space on an equidistant grid by the FV method using the Godunov flux

$$\mathbf{H}(u^-, u^+) = \begin{cases} \min\limits_{u^- \le u \le u^+} f(u) & \text{if } u^- \le u^+, \\ \max\limits_{u^+ \le u \le u^-} f(u) & \text{otherwise.} \end{cases}$$

For time discretization, use the implicit Euler scheme and compute just one time step. To solve the resulting nonlinear systems now use FAS-MG. As smoothing methods, use 2-stage RK smoothers and play a bit with the parameters Δt^* and α_1. Also investigate the influence of the number of pre- and postsmoothings. Compare the parameter choices with respect to the number of iterations needed to fulfill a stopping criterion for the FAS-MG iteration of

$$\frac{\|\mathbf{f}_l(\mathbf{u}_l) - \mathbf{s}_l\|}{\|\mathbf{s}_l\|} < 10^{-9}.$$

(4 P)

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