Fachbereich Mathematik
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Numerik steifer Probleme

Aufgabenblatt 3

Aufgabe 1
Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 20 & & & & \\ & -2 & 20 & & & \\ & & -3 & 20 & & \\ & & & \ddots & \ddots & \\ & & & & -19 & 20 \\ & & & & & -20 \end{pmatrix}.$$

a) Compute the eigenvectors of the matrix $\mathbf{A}$ by solving $(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{v}_i = \mathbf{0}$.

b) Compute numerically the inverse of $\mathbf{T} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ and determine its largest element. (Observe that $\mathbf{T}$ is thus very badly conditioned.)

c) Compute numerically the solutions of

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \qquad y_i(0) = 1,\; i = 1, \dots, 20.$$

Plot the components $y_i(t)$ for $t \in [0, 8]$ in one diagram (use logarithmic scale for the y-axis). What do you observe?

(4 P)

Aufgabe 2
Prove that the stability function of a Runge-Kutta method is given by

$$R(z) = \frac{\det(\mathbf{I} - z\mathbf{A} + z\mathbf{e}\mathbf{b}^T)}{\det(\mathbf{I} - z\mathbf{A})},$$

where $\mathbf{I}$ is the identity matrix of size s and $\mathbf{e}$ is the s-vector of all ones. Hint: Use Cramer's Rule. (4 P)

Aufgabe 3
Consider the equation

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t),$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{A}$ is normal (this means that $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$) and for all eigenvalues of $\mathbf{A}$, the real part is nonpositive. Prove that a Runge-Kutta method is A-stable if and only if it is unconditionally contractive in $\|\cdot\|_2$ (Definition 6) on the class of problems just described.

Hint: Use the fact that $\mathbf{A}$ is normal if and only if it is diagonalizable by a unitary matrix, i.e. $\mathbf{A} = \mathbf{U}^*\mathbf{D}\mathbf{U}$, where $\mathbf{U} \in \mathbb{C}^{n \times n}$ is unitary and $\mathbf{D} \in \mathbb{C}^{n \times n}$ is a diagonal matrix. (4 P)

Abgabe: **Montag, 12.11.2012, in der Vorlesung**