Fachbereich Mathematik **WiSe 2012/2013**
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Numerik steifer Probleme

Aufgabenblatt 9

Aufgabe 1

a) Let $\mathbf{A}, \mathbf{P} \in \mathbb{R}^{n\times n}$ be regular matrices. Show that $\sigma(\mathbf{PA}) = \sigma(\mathbf{AP})$.

This means that the spectra corresponding to left and right preconditioning are equal.

b) Note that any SPD matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ may be decomposed as $\mathbf{B} = \mathbf{Q}^T\mathbf{DQ}$, where the orthonormal matrix $\mathbf{Q}$ contains the eigenvectors of $\mathbf{B}$ and the diagonal matrix $\mathbf{D}$ the eigenvalues. For $\alpha \in \mathbb{R}$, we define the SPD matrix $\mathbf{B}^\alpha = \mathbf{Q}^T\mathbf{D}^\alpha\mathbf{Q}$.

Let now $\mathbf{A}, \mathbf{P} \in \mathbb{R}^{n\times n}$ be SPD matrices. Show that $\sigma(\mathbf{PA}) = \sigma(\mathbf{P}^{1/2}\mathbf{AP}^{1/2})$. Suppose, we want to solve the system $\mathbf{Ax} = \mathbf{b}$. Is the CG algorithm directly applicable to the new preconditioned system $\mathbf{P}^{1/2}\mathbf{AP}^{1/2}\mathbf{u} = \mathbf{P}^{1/2}\mathbf{b}$?

Aufgabe 2

Consider the initial value problem with $\mathbf{y}(t_0) = \mathbf{y}_0$ and

$$\dot{\mathbf{y}} = \begin{pmatrix} -100 & 0 \\ y_1 & -10 \end{pmatrix} \mathbf{y}.$$

Suppose, you solve the system using the implicit Euler scheme and a Jacobian-Free Newton-GMRES method. How does the approximation to the matrix vector product look like? How would this change, if this would be a stage inside a DIRK method for the same equation? (4 P)

Aufgabe 3

Consider the nonlinear convection diffusion equation

$$\partial_t u = \beta \cdot \nabla u + \nabla \cdot (u^3 \nabla u),$$

for $u(x)$, $x \in (0,1)^2$, $t \in [0, 0.002]$ with

$$\beta = -200 \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix},$$

$\gamma = 0.35\pi$. As initial data, choose $u_0 = 2$ in $[0.2, 0.3]^2$ and $u_0 = 1$ everywhere else. At the boundary we set $u = 1$.

Discretize this in space using upwind for the convective term and a five-point-stencil with central differences for the second term with $\Delta x = 1/100$. Use boundary conditions appropriate for upwind and the time adaptive method of Cash in time.

Implement a Jacobian-Free Newton-GMRES method and use that to solve the resulting nonlinear equation systems. For the approximation of the matrix vector product $\mathbf{Aq}$, choose $\epsilon = \sqrt{\epsilon_{\text{machine}}/\|\mathbf{q}\|_2}$, where $\epsilon_{\text{machine}}$ is machine precision. Plot the solution.

To check your result, use the explicit Euler method with a tiny time step to obtain a reference solution and compare your error to that. (4 P)

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