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Differential Algebra

1. Exercise Sheet

Exercise 1

(i) Prove that the following rules are valid for any derivation δ on a ring R:

- $\delta(r^m) = mr^{m-1}\delta(r)$;
- $\delta^m(rs) = \sum_{i=0}^{m} \binom{m}{i} \delta^i(r)\delta^{m-i}(s)$;
- $\delta(r/s) = \frac{\delta(r)s - r\delta(s)}{s^2}$ (assuming that s is a unit in R).

(ii) Prove that the only derivation on the ring $\mathbb{Z}$ and on the field $\mathbb{Q}$, respectively, is the zero map mapping every element to zero.

(iii) Prove that the constants form a subring in a differential ring (R, δ) and a subfield in a differential field (K, δ).

(iv) Let (K, δ) be a differential field and $K \subset L$ an algebraic field extension of K (i.e. for every element $a \in L$ there exists a polynomial $f \in K[x]$ such that $f(a) = 0$). Show that we can extend the derivation δ from K to L and that this extension is unique.

Exercise 2

Given a ring R, the *dual numbers over* R are defined as the ring $R[\epsilon]/\langle\epsilon^2\rangle \cong R \oplus R\epsilon$. Elements of this ring can be represented in the form $r + s\epsilon$. Their addition and multiplication is done as for complex numbers, except that we have the rule $\epsilon^2 = 0$. Alternatively, one can represent elements as matrices of the form $\left(\begin{smallmatrix} r & s \\ 0 & r \end{smallmatrix}\right)$ and apply the usual matrix operations.

(i) Show that a map $\delta : R \to R$ is a derivation, if and only if the map $\psi : R \to R \oplus R\epsilon$ defined by $\psi(r) = r + \delta(r)\epsilon$ is a ring homomorphism.

(ii) Assume that (R, δ) is a differential ring which is also an integral domain so that we can define the quotient field $Q = \operatorname{Quot}(R)$. Use (i) to show that δ can be extended to Q so that Q becomes a differential field. (If you are familiar with localisations, consider more generally a multiplicatively closed subset $S \subset R$ and show that δ can be extended to $S^{-1}R$.)

Exercise 3

(i) Let (R, δ) be a differential ring and $I \lhd R$ a differential ideal. Show that δ induces a derivation on the factor ring R/I.

(ii) Let (R, δ) and (S, ∂) be two differential rings. A *differential ring homomorphism* is a ring homomorphism $\phi : R \to S$ such that $\partial(\phi(r)) = \phi(\delta(r))$. Show that the kernel of a differential ring homomorphism is a differential ideal and that its image is a differential subring.

(iii) Let (R, δ) be a differential ring and consider the finitely generated (algebraic) ideal $I = \langle r_1, \ldots, r_n\rangle$. Show that I is a differential ideal, if and only if $\delta(r_i) \in I$ for all i.