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Differential Algebra

2. Exercise Sheet

Exercise 1

In this exercise we want to correct the treatment of Example 3.3 in the lecture where the second step was not completely sound. We consider in $\mathbb{K}\{u\}$, where $(\mathbb{K}, \delta)$ is a differential field containing the rational numbers $\mathbb{Q}$, the ascending chain of differential ideals $\mathcal{I}_n = \left[u^2, (u')^2, \dots, (u^{(n-1)})^2\right]$. Our goal is to show that we always have a proper inclusion $\mathcal{I}_n \subset \mathcal{I}_{n+1}$. For this purpose, we introduce the following linear subspaces of $\mathbb{K}\{u\}$: $\mathcal{V}_n = \left\langle u^{(i)}u^{(j)} \mid i + j = n \right\rangle_{\mathbb{K}}$.

(i) Show that $\dim_{\mathbb{K}} \mathcal{V}_{2n} = \dim_{\mathbb{K}} \mathcal{V}_{2n+1} = n + 1$.

(ii) Show that $\mathcal{V}_{2n} = \left\langle \delta^2(\mathcal{V}_{2n-2}), (u^{(n)})^2 \right\rangle_{\mathbb{K}}$.
Hint: Express any element $\delta^2\left(u^{(i)}u^{(j)}\right) \in \delta^2(\mathcal{V}_{2n-2})$ and $(u^{(n)})^2$ as a linear combination of the generators of $\mathcal{V}_{2n}$. Then show that these relations define an invertible change of bases.

(iii) Show that $\mathcal{I}_n \cap \mathcal{V}_{2n} = \left\langle \delta^2(\mathcal{V}_{2n-2}) \right\rangle_{\mathbb{K}}$ and conclude that $\mathcal{I}_n \subset \mathcal{I}_{n+1}$.

(iv) Does this proof also work, if $\mathbb{K}$ is the finite field $\mathbb{F}_2$?

Exercise 2

Let $f_1, \dots, f_r \in \mathbb{K}\{U\}$ be *linear* differential polynomials (i. e. $\deg f_i = 1$ for all i) and consider the differential ideal $\mathcal{P} = [f_1, \dots, f_r]$ generated by them. Show that $\mathcal{P}$ is either the whole ring or a prime differential ideal.

Exercise 3

Let $F, H \subset \mathbb{K}\{U\}$ be finite subsets, $v \in \Theta U$ a differential variable and $\prec$ a ranking. Assume that $f \in \langle \Theta F_{\prec v} \rangle : H^\infty$. Show that for any derivative operator $\theta \in \Theta$ we have $\theta f \in \langle \Theta F_{\prec \theta v} \rangle : H^\infty$.

Exercise 4

(i) (Pseudo)-divide the polynomials $f = 5x^3y^2 - 10xy^3$ and $g = 2x^2y + x^2 + xy^3$ once over the rational numbers $\mathbb{Q}$ and once over the integers $\mathbb{Z}$ using the lexicographic order with $x > y$.

(ii) Let R be an integral domain, $f, g \in R[x] \setminus R$ and $c = \mathrm{lc}(g)$. A pseudo-division algorithm yields $d \in \{c\}^\infty$, $q, r \in R[x]$ such that $df = qg + r$ with $\deg r < \deg g$. Assume that a different algorithm yields $\hat{d} \in \{c\}^\infty$, $\hat{q}, \hat{r} \in R[x]$ such that $\hat{d}f = \hat{q}g + \hat{r}$ with $\deg \hat{r} < \deg g$. Discuss the relation between d, q, r and $\hat{d}, \hat{q}, \hat{r}$.