Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNIKASSEL VERSITÄT

21 November 2018

Differential Algebra

3. Exercise Sheet

Exercise 1

(i) Design an algorithm to compute for a given polynomial $f \in \mathbb{K}[X]$ and a given triangular set $A \subset \mathbb{K}[x]$ a reduction $\operatorname{red}(f, A)$. Discuss variants of your algorithm leading to different results.

(ii) Assume that $A = a_1 \vartriangle \cdots \vartriangle a_m$ and $v_i = \operatorname{lead}(a_i)$ for all i. Show that we may take

$$\operatorname{red}(f, A) = \operatorname{prem}\bigl(\cdots \operatorname{prem}(\operatorname{prem}(f, a_m, v_m), a_{m-1}, v_{m-1}), \dots, a_1, v_1\bigr)\,.$$

(iii) Prove the correctness and the termination of Algorithm 4.17 for differential reduction.

Exercise 2

Prove Proposition 4.20.

Exercise 3

We work in the ring $\mathbb{Q}[x, y, z]$ with the ranking $x < y < z$.

(i) Consider the triangular set $A = x^2 \vartriangle xy - 1 \vartriangle xz + y$. Show that $\operatorname{red}\bigl(\operatorname{init}(A_z), A_{<z}\bigr) \neq 0$, but that nevertheless $\operatorname{red}(A_z, A_{<z}) = 1$.

(ii) Consider the triangular set $A = x^2 - x \vartriangle xy - 1 \vartriangle (x - 1)z + xy$. Show that the question whether A is a chain (i. e. whether $\operatorname{rank}\bigl(\operatorname{red}(A_v, A_{<v})\bigr) = \operatorname{rank}(A_v)$ for all $v \in L_A$) depends on the used algorithm for pseudo-division. Find one algorithm for which A is a chain and another one for which it is not.

Exercise 4

Consider in the ring $\mathbb{Q}[x, y]$ with the ranking $x > y$ the polynomials $f = 5x^3y^2 - 10xy^3$, $g_1 = 2x^2y + x^2 + xy^3$ and $g_2 = 2x^2y + x^2 + y^3$. Compute a characteristic set for the ideals $\langle f, g_1\rangle$ and $\langle f, g_2\rangle$, respectively. *Hint:* Compute first an autoreduced set by repeated pseudo-division; then argue why it is of lowest rank and thus a characteristic set.