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Differential Algebra

4. Exercise Sheet

Exercise 1

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A map $f : X \to Y$ is *continuous*, if for an open set $O_Y \in \mathcal{O}_Y$ its preimage $f^{-1}(O_Y) = \{x \in X \mid f(x) \in O_Y\}$ is an open set, too (i. e. if $f^{-1}(O_Y) \in \mathcal{O}_X$).

(i) Prove that for $X = Y = \mathbb{R}$ with its standard topology this definition of continuity is equivalent to the δ-ϵ one.

(ii) Provide a concrete example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $O \subseteq \mathbb{R}$ such that $f(O)$ is *not* an open set.

(iii) Show that if $\mathcal{O}_X$ is the discrete topology *or* $\mathcal{O}_Y$ the indiscrete topology, then every map $f : X \to Y$ is continuous. If conversely $\mathcal{O}_X$ is the indiscrete topology *and* $\mathcal{O}_Y$ the discrete topology, then no map $f : X \to Y$ is continuous.

(iv) Prove that a map $f : X \to Y$ is continuous, if and only if the preimage of any closed set $C_Y \subseteq Y$ is a closed set in X.

(v) Consider for some differential field $(\mathbb{K}, \Delta)$ the spaces $\mathbb{K}^n$, $\mathbb{K}^m$ equipped with the Kolchin topology. Let $\mathbf{f} : \mathbb{K}^n \to \mathbb{K}^m$ be a differential polynomial map, i. e. $\mathbf{f}(\mathbf{u}) = \bigl(f_1(\mathbf{u}), \dots, f_m(\mathbf{u})\bigr)$ with some differential polynomials $f_1, \dots, f_m \in \mathbb{K}\{u_1, \dots, u_n\}$. Show that $\mathbf{f}$ is continuous.

(vi) A bijective map $f : X \to Y$ is a *homeomorphism*, if both f and f^{-1} are continuous.

  (a) Take $X = [0, 2\pi) \subset \mathbb{R}$ and $Y = \{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ both with the trace topology. Is the map $f : X \to Y$ with $f(t) = (\cos t, \sin t)$ a homeomorphism?

  (b) Show that a bijective map $f : X \to Y$ is a homeomorphism, if and only if f is continuous and closed (i. e. for any closed set $C_X \subseteq X$ we have that $f(C_X)$ is again a closed set). Does this statement remain true, if we everywhere replace *closed* by *open*?