Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNIKASSEL VERSITÄT

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Differential Algebra

5. Exercise Sheet

Exercise 1

(i) Prove Lemma 6.4. *Hint:* Use first that for any differential polynomials $a, b \in \mathbb{K}\{U\}$ and any derivative operator $\theta \in \Theta$ we have $\theta(ab) \equiv a\theta(b) \bmod \langle \hat{\theta}(b) : \hat{\theta} \mid \theta \wedge \hat{\theta} \neq \theta \rangle$ and then apply Exercise 3 from Sheet 2.

(ii) Show with a concrete counter example that the condition for differential coherence provided by Corollary 6.5 is not necessary.

Exercise 2

Given sets $F, H \subset \mathbb{K}\{U\}$ and a subset $G \subset \Theta(F)$, we say that G is *differentially coherent relative to F away from H*, if for all pairs $g_1, g_2 \in G$ such that $v_1 = \operatorname{lead}(g_1)$ and $v_2 = \operatorname{lead}(g_2)$ have a common derivative $v = \theta_1 v_1 = \theta_2 v_2$ we have $S_{\Delta,v}(g_1, g_2) \in \langle \Theta F_{<v} \rangle : H^\infty$.

(i) Assume that $G_1 \subset G_2 \subset \Theta(F)$ and that G_2 is differentially coherent relative to F away from H. Show that then also G_1 is differentially coherent relative to F away from H.

(ii) Show that G is differentially coherent relative to F away from H, if and only if for every pair $g_1, g_2 \in G$ the subset $\{g_1, g_2\}$ is differentially coherent relative to F away from H.

(iii) Assume that $G_1 \subset G_2 \subset \Theta G_1$. Show that if G_1 is differentially coherent relative to F away from H, then G_2 has the same property. Show furthermore that if G_1 is differentially coherent (relative to itself away from H_{G_1}), then the same is true for G_2 (relative to itself away from H_{G_2}).

(iv) Show that G is differentially coherent relative to F away from H, if and only if ΘG is coherent relative to ΘF away from H (in the algebraic sense!). In particular, F is differentially coherent, if and only if ΘF is coherent.

(v) Let $(\mathbb{K}, \Delta)$ be a differential field with $\Delta = \{\delta_1, \delta_2\}$ and consider the ring of differential polynomials $\mathbb{K}\{u, v, w\}$ with an orderly ranking. Verify for $G = \{\delta_1 w - u, \delta_2 w - v\}$ the following statements. G is an autoreduced set. The differential polynomial $f = \delta_1 v - \delta_2 u$ is partially reduced with respect to G. Furthermore, we have $f \in [G] : H_G^\infty$, but $f \notin \langle G \rangle : H_G^\infty$. The set G is not differentially coherent, but the set $F = G \cup \{f\}$ is. G is differentially coherent relative to F away from H_F.