Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNIKASSEL VERSITÄT

7 Januar 2019

Differential Algebra

6. Exercise Sheet

Exercise 1

We consider in the differential ring $\mathbb{Q}(x,y)\{u\}$ with the obvious derivations the linear system A comprising $f_1 = u_{zz} - yu_{xx}$ and $f_2 = u_{yy}$ (a very classical example due to Janet).

(i) Complete A to a differentially coherent set using differential pseudo-S-polynomials (as we are dealing with a linear system this is analogous to a Gröbner basis computation and there is no need to use a pseudo-division as all initials are one).

(ii) Describe the set of all parametric derivatives of the system and construct an initial value problem leading to a unique power series solution around the point $(0,0,0)$. What is the size of the solution space?

(iii) Show that all solutions of this system are polynomial and derive the general solution of the system.

Exercise 2

Describe the set of all parametric derivatives of the "monomial" differential system $\{u_{zzz}, u_{yyz}, u_{xyz}\} \subset \mathbb{Q}(x,y,z)\{u\}$ and construct an initial value problem leading to a unique power series solution around the point $(0,0,0)$. Express the solution of this initial value problem in terms of the initial data.

Exercise 3

(i) We consider the differential polynomial $f_a = (u')^2 - u^2 + a(x^2 - x) \in \mathbb{R}(x)\{u\}$ depending on a parameter a together with the initial conditions $u(\bar{x}) = b_0$ and $u'(\bar{x}) = b_1$. Discuss the existence and uniqueness of formal power series solutions depending on the parameters $a, b_0, b_1, \bar{x} \in \mathbb{R}$ (do not try to make a complete classification – stop after order 3). What is different for $b_1 = 0$?

(ii) We consider the differential polynomial $f_2 = xu'' - uu' \in \mathbb{Q}(x)\{u\}$ together with the initial conditions $u(0) = b_0$ and $u'(0) = 0$. Show first that differentiating f_2 yields the following differential polynomials f_q of order $q \geq 3$:

$$f_q = xu^{(q)} - (u - q + 2)u^{(q-1)} - \sum_{j=1}^{\lfloor (q-1)/2 \rfloor} a_{q,j} u^{(j)} u^{(q-j-1)}$$

where the coefficients $a_{q,j}$ are defined recursively by

$$a_{q,1} = \begin{cases} 1 & \text{for } q = 3, \\ q-1 & \text{for } q > 3, \end{cases} \qquad a_{q,j} = a_{q-1,j-1} + a_{q-1,j} \quad \text{for } 1 < j < \lfloor (q-1)/2 \rfloor$$

$$a_{\lfloor (q-1)/2 \rfloor} = \begin{cases} a_{q-1,(q-3)/2} & \text{if } q \text{ odd}, \\ a_{q-1,(q-4)/2} + 2a_{q-1,(q-2)/2} & \text{if } q \text{ even}. \end{cases}$$

Then use this result to determine all power series solutions of the given initial value problem.