Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNIKASSEL VERSITÄT

17 Januar 2019

Differential Algebra

7. Exercise Sheet

Exercise 1

Let R be a commutative ring and $I \lhd R$ an ideal. We denote the set of all prime ideals in R containing I by $\mathrm{Var}(I) = \{P \lhd R \mid P \text{ prime} \wedge I \subseteq P\}$.

(i) Show that $\mathrm{Var}(I)$ is not empty and contains at least one minimal element with respect to inclusion (thus every ideal has minimal associated primes).

(ii) Show that for any $P \in \mathrm{Var}(I)$ there exists a minimal associated prime $P' \in \mathrm{minAss}(I)$ such that $I \subseteq P' \subseteq P$.

Exercise 2

Let R be a commutative ring and $I \lhd R$ an ideal. Prove the following statements:

(i) If R is noetherian, then there exists an $\ell \in \mathbb{N}$ such $\sqrt{I}^{\ell} \subseteq I$.

(ii) I is primary, if and only if every zero divisor in the factor ring R/I is nilpotent.

(iii) Let $P, Q \lhd R$ be ideals which contain I. Then Q is P-primary, if and only if Q/I is P/I-primary in the factor ring R/I.

(iv) Let $\phi : S \to R$ be a homomorphism of commutative rings and assume that I is primary. Then $\phi^{-1}(I) \lhd S$ is also primary.

Exercise 3

Let $P \lhd R$ be a prime.

(i) Let $Q_1, \dots, Q_n$ be P-primary ideals. Then $\bigcap_{i=1}^{n} Q_i$ is P-primary, too.

(ii) Let Q be a P-primary ideal. Show that for any $r \in R \setminus Q$ the ideal $Q : r$ is again P-primary and for any $r \in R \setminus P$ we have $Q : r = Q$.