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Differential Algebra

8. Exercise Sheet

Exercise 1

(i) Consider in the polynomial ring $R = K[x, y]$ the monomial ideal $I = \langle x^2, xy\rangle$ and give two different minimal primary decompositions of I.

(ii) Let I be an ideal in a noetherian ring R. Show that to $r \in R$ there exists an $s \in R \setminus I$ such that $rs \in I$, if and only if there is an associated prime ideal $P \in \operatorname{Ass}(I)$ with $r \in P$. Interpret the definition of a regular chain with the help of this result.

Exercise 2

We work in $\mathbb{Q}[x, y]$ with $x < y$.

(i) Consider the triangular set $A_1 = x^2 - 1 \vartriangle (x + 1)y - 1$. Show first that A_1 is a chain, but not a regular one. Verify then that $\mathcal{I}(A_1) = \langle x^2 - 1, 2y - 1\rangle$. Finally compare minimal primary decompositions of $\mathcal{I}(A_1)$ and $\mathcal{I}(A_1)_{<y}$, respectively. Discuss the implications of your results for solving the given polynomial system.

(ii) Consider the triangular set $A_2 = x \vartriangle y^2 - 1$. Show that it defines a regular chain. As in (i) compare minimal primary decompositions of $\mathcal{I}(A_2)$ and $\mathcal{I}(A_2)_{<y}$, respectively. What is different?

Please turn the page!

Exercise 3

Let $A = a_1 \triangle \cdots \triangle a_p$ be a triangular set in the polynomial ring $K[X]$. We write $K[X] = R_0[L_A]$ where $R_0 = K[X \setminus L_A]$. With $v_i = \mathrm{lead}(a_i)$ for $1 \leq i \leq p$, we define inductively $R_i = R_0[v_1, \ldots, v_i]$ (i. e. $R_p = K[X]$) and $S_i = K_0[v_1, \ldots, v_i]$ where $K_0 = \mathrm{Quot}(R_0)$. Finally, we define $V_A = K_p/\langle A \rangle$ which is both a ring and a vector space over K_0 and write π_A for the canonical projection $K_p \to V_A$.

(i) Prove that the following statements are equivalent for an arbitrary polynomial $f \in K[X]$:

(a) The homomorphism $\mu_f : V_A \to V_A\,,\ \pi_A(g) \mapsto \pi_A(fg)$ is surjective.

(b) $\pi_A(f)$ is a unit in V_A.

(c) $\exists\, 0 \neq h \in R_0,\ g \in R_p : fg \equiv h \mod \langle A \rangle_{R_p}$.

(d) $\langle A, f \rangle_{R_p} \cap R_0 \neq \{0\}$.

If any of these equivalent conditions is satisfied, we say that f is *invertible with respect to* A *and* $v_1, \ldots, v_p$. We also say that A *has invertible initials*, if $\mathrm{init}(a_i) \in R_{i-1}$ is invertible with respect to $a_1 \triangle \cdots \triangle a_{i-1}$ and $v_1, \ldots, v_{i-1}$ for all i.

(ii) We call $f \in K[X]$ *regular with respect to* A *and* $v_1, \ldots, v_p$, if $\left\langle \langle A \rangle_{S_p} : I_A^\infty, f \right\rangle_{S_p} \cap R_0 \neq \{0\}$ and say that A *has regular initials*, if $\mathrm{init}(a_i) \in R_{i-1}$ is regular with respect to $a_1 \triangle \cdots \triangle a_{i-1}$ and $v_1, \ldots, v_{i-1}$ for all i. Now assume that A has invertible initials (as defined in (i)) and show that the following statements are equivalent for an arbitrary polynomial $f \in K[X]$:

(a) f is regular with respect to A and $v_1, \ldots, v_p$.

(b) f is invertible with respect to A and $v_1, \ldots, v_p$.

(c) $\pi_A(f) \neq 0$ and $\pi_A(f)$ is not a zero divisor in V_A.

(d) The map μ_f is injective.

(iii) Prove that A has regular initials, if and only if A is a regular chain.