Determination of the Kronecker-Weierstraß normal form of a regular matrix pencil

We simplify the use of linear algebra commands by preloading the LinearAlgebra package.

```
> with(LinearAlgebra) :
```

We first define the two given matrices.

```
> K := <<0|0|0|0>, <0|0|0|0>, <0|0|1|0>, <1|0|0|0>>;
  L := <<0|1|1|1>, <2|-1|0|0>, <1|0|0|0>, <0|0|0|1>>;
```

$$K := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$L := \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(1)}$$

Then we must guess a value of s such that sK-L is non-singular. As it is not difficult to see that L is already non-singular, we simply choose s=0. This also has the advantage that det(sK-L)=1 which simplifies subsequent matrices, as no fractions appear. Different choices of s will lead to different results, as one can verify experimentally. Not permitted is e.g. s=-1, as then a singular matrix is obtained.

```
> s := 0 :
  KL := s·K - L :
  det(KL);
```

$$1 \quad \textbf{(2)}$$

Next we define some auxiliary matrices and compute a Jordan normal form J plus the corresponding transformation matrix T.

```
> KLI := KL^(-1) :
  KK := KLI.K :
  LL := KLI.L :
> JJ := JordanForm(KK, output = ['J','Q']) :
  J := JJ[1]; T := JJ[2];
```

$$J := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T := \begin{bmatrix} 0 & -1 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 1 & -2 & 1 \end{bmatrix} \quad \textbf{(3)}$$

A rather annoying "feature" of Maple is here that the Jordan blocks are not properly sorted. Thus the above matrix J is not in the right form to continue with our algorithm. We must first perform a permutation of J and T. We subsequently check that we did this correctly.

> $P := \langle\langle 0|0|1|0\rangle, \langle 1|0|0|0\rangle, \langle 0|1|0|0\rangle, \langle 0|0|0|1\rangle\rangle; \mathrm{PI} := P^{(-1)}$:
$J := \mathrm{PI}.J.P,\ T := T.P;$
$T^{(-1)}.\ KK.\ T - J;$

$$P := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T := \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 2 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(4)}$$

Finally, we extract the relevant submatrices from the properly sorted Jordan normal form and assemble the transformation matrices U and V according to our algorithm.

> $E1 := SubMatrix(J, 1..2, 1..2);$
$E2 := SubMatrix(J, 3..4, 3..4);$

$$E1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$E2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(5)}$$

> $E1I := E1^{(-1)} : E2I := (s \cdot E2 - Matrix(2, shape = identity))^{(-1)} :$

> $U := \langle\langle E1I[1, 1] | E1I[1, 2] | 0 | 0\rangle, \langle E1I[2, 1] | E1I[2, 2] | 0 | 0\rangle, \langle 0 | 0 | E2I[1, 1] | E2I[1, 2]\rangle, \langle 0 | 0 | E2I[2, 1] | E2I[2, 2]\rangle\rangle . T^{(-1)} . KLI;$
$V := T;$

$$U := \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$V := \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 2 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & 3 & 1 \end{bmatrix} \quad \textbf{(6)}$$

Now we can admire the constructed Kronecker-Weierstraß normal form. The nilpotent submatrix N is in this case just the zero matrix (and thus automatically in Jordan normal form). The submatrix A is already in Jordan normal form. This last fact is due to the "clever" choice of s=0. For other values of s, A will look differently (as the matrices U and V).

> $U.K.V;\ U.L.V;$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(7)}$$

>

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