Ex. 2:

(i) $P = \int_0^\varepsilon e^{At} BB^T e^{A^T t}\, dt, \quad Q = \int_0^\varepsilon e^{A^T t} C^T C e^{At}\, dt$

$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T^{-1}AT, T^{-1}B, CT, D)$

recall $e^{\tilde{A}t} = e^{T^{-1}ATt} = T^{-1}e^{At}T$

$\Rightarrow \tilde{P} = \int_0^\varepsilon T^{-1}e^{At}\cancel{T T^{-1}} BB^T \cancel{T^{-T}T^T} e^{A^T t}T^{-T}\, dt = T^{-1}PT^{-T}$

analogously $\tilde{Q} = T^T Q T$

$\Rightarrow \tilde{P}\tilde{Q} = T^{-1}PT^{-T}T^T Q T = T^{-1}PQT$

(ii) • $P > 0 \Rightarrow \exists\ V$ orthogonal, $\delta_1, \dots, \delta_n \in \mathbb{R}_+ : P = V \underbrace{\Delta}_{= \operatorname{diag}(\delta_1, \dots, \delta_n)} V^T$

$\rightsquigarrow$ set $R = V\Delta^{1/2} \Rightarrow R^T = \Delta^{1/2}V^T \Rightarrow P = RR^T$

("numerical" proof $\rightsquigarrow$ Cholesky decomposition)

note R obviously nonsingular

• set $S = R^T Q R \Rightarrow (Q > 0)\ S > 0$

$\Rightarrow \operatorname{Spec}(S) \subset \mathbb{R}_+$

claim: $\operatorname{Spec}(PQ) = \operatorname{Spec}(S)$

proof:

- aside: X, Y nonsingular real matrices $\Rightarrow \operatorname{Spec}(XY) = \operatorname{Spec}(YX)$

  proof: $\lambda \in \operatorname{Spec}(XY) \Rightarrow \exists\, v \neq 0 : XYv = \lambda v$

  $\Rightarrow (YX)(Yv) = \lambda Yv$ with $Yv \neq 0 \Rightarrow \lambda \in \operatorname{Spec}(YX)$

  converse analogously

- $\operatorname{Spec}(PQ) = \operatorname{Spec}(RR^TQ) \underset{\text{(aside)}}{=} \operatorname{Spec}(R^TQR) = \operatorname{Spec}(S)$ ∎

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(iii) • $S > 0 \Rightarrow \exists\, U$ orthogonal, $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+ : S = U^T \Delta U$, $\Delta = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

note $\{\lambda_1, \dots, \lambda_n\} = \mathrm{Spec}(PQ)$

so $T = R U \Delta^{-1/4}$ non-singular

• $\tilde{P} = T^{-1} P T^{-T} (\Delta^{1/4} U^T R^{-1}) R R^T (R^{-T} U \Delta^{1/4}) = \Delta^{1/2}$

$\tilde{Q} = T^T Q T = (\Delta^{-1/4} U^T R^T)(R^{-T} S R^{-1})(R U \Delta^{-1/4}) = \Delta^{1/2}$

$(U^T S U = \Delta)$

$$\Rightarrow \tilde{P} = \tilde{Q} = \Delta^{1/2} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$$
$$= \mathrm{diag}(\sigma_1, \dots, \sigma_n)$$

• in the new representation

$\vec{e}_i^T \tilde{P}^{-1} \vec{e}_i = \frac{1}{\sigma_i}$, $\vec{e}_i^T \tilde{Q} \vec{e}_i = \sigma_i$

$\Rightarrow$ the "larger" σ_i, the "better" $\vec{e}_i$ for both controllability and observability

note: by previous exercise $\vec{x}_0^T Q x_0 = \| L \vec{x}_0 \|_Y^2 = \| \vec{y}(\cdot) \|_Y^2$

thus it makes sense to consider this as output energy