Ex. 2:

(i) • let $(A \;\; B \;\; -C)$ be a matrix such that its rows generate the kernel of S $\Rightarrow$ $AR_1 = C = BR_2$

$\Rightarrow$ C is common left multiple

• let $\tilde{C} = \tilde{A} R_1 = \tilde{B} R_2$ be another common left multiple

$\Rightarrow$ $(\tilde{A} \;\; \tilde{B} \;\; -\tilde{C}) S = 0$

$\Rightarrow$ (definition of $(A \;\; B \;\; -C)$) $\exists X: \; (\tilde{A} \;\; \tilde{B} \;\; -\tilde{C}) = X (A \;\; B \;\; -C)$

i.e. the rows of $(\tilde{A} \;\; \tilde{B} \;\; -\tilde{C})$ are linear combinations of the rows of $(A \;\; B \;\; -C)$

$\Rightarrow$ $\tilde{C} = XC$ $\Rightarrow$ C least common left multiple

(ii) "$\Rightarrow$": • let R be a common left multiple of R_1, R_2

$\Rightarrow$ $\exists T_1, T_2: \; R = T_1 R_1 = T_2 R_2$

$\Rightarrow$ $D^{1 \times g} R \subseteq D^{1 \times g_1} R_1 \cap D^{1 \times g_2} R_2$

• R even a <u>least</u> common left multiple

$\Rightarrow$ every row vector in $D^{1 \times g_1} R_1 \cap D^{1 \times g_2} R_2$ must be expressible as linear combination of the rows of R

$\Rightarrow$ $D^{1 \times g_1} R_1 \cap D^{1 \times g_2} R_2 \subseteq D^{1 \times g} R$

"$\Leftarrow$": the inclusion "$\supseteq$" implies R is common left multiple

—— " —— "$\subseteq$" yields R is even a least c.l.m.

let $R, \tilde{R}$ be two least common multiples

$\Rightarrow$ $\exists X, Y: \; R = X\tilde{R} \wedge \tilde{R} = YR$

$\Rightarrow$ the rows of R and of $\tilde{R}$, resp., span the same module

(iii) $\mathcal{B}_1 + \mathcal{B}_2 = \left\{ w \in \mathcal{A}^q \;\middle|\; \exists\, w_1, w_2 \in \mathcal{A}^q : \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \\ I & I \end{pmatrix} \left(\tfrac{d}{dt}\right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} w \right\}$

$\rightsquigarrow$ latent variables description of $\mathcal{B}_1 + \mathcal{B}_2$

let $(A \;\; B \;\; -C)$ be a matrix generating the kernel of S

$\Rightarrow$ (fundamental principle)

$$\mathcal{B}_1 + \mathcal{B}_2 = \left\{ w \in \mathcal{A}^q \;\middle|\; C\left(\tfrac{d}{dt}\right) w = 0 \right\}$$

(i) $\Rightarrow$ C least common multiple of R_1, R_2

(iv) "1. $\Leftrightarrow$ 2.":

$$\dim\Big(\underbrace{Q^{1\times g_1} R_1 + Q^{1\times g_2} R_2}_{=\, Q^{1\times(g_1+g_2)} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}}\Big) = \dim\left(Q^{1\times g_1} R_1\right) + \dim\left(Q^{1\times g_2} R_2\right) - \dim\left(Q^{1\times g_1} R_1 \cap Q^{1\times g_2} R_2\right)$$

in other words: $\operatorname{rank}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \operatorname{rank} R_1 + \operatorname{rank} R_2 - \dim\left(Q^{1\times g_1} R_1 \cap Q^{1\times g_2} R_2\right)$

"2. $\Rightarrow$ 3.": obvious since $\mathcal{D} \subseteq Q$

"3. $\Rightarrow$ 2.": assume $\exists\, a_1 \in Q^{1\times g_1}, a_2 \in Q^{1\times g_2}: a_1 R_1 = a_2 R_2$

$\Rightarrow \exists\, \tilde{a}_1 \in \mathcal{D}^{1\times g_1}, \tilde{a}_2 \in \mathcal{D}^{1\times g_2}, d_1, d_2 \in \mathcal{D}\setminus\{0\}: a_1 = \frac{\tilde{a}_1}{d_1}, a_2 = \frac{\tilde{a}_2}{d_2}$

$\Rightarrow d_2 \tilde{a}_1 R_1 = d_1 \tilde{a}_2 R_2$

$\Rightarrow$ (3.) $d_2 \tilde{a}_1 R_1 = 0 \Rightarrow (d_2 \neq 0)\; \tilde{a}_1 R_1 = 0 \Rightarrow a_1 R_1 = 0$

"3. $\Rightarrow$ 4.": by (ii) 3. $\Rightarrow$ least common multiple of R_1, R_2 is 0

any c.l.m. is a multiple of a least one $\Rightarrow$ 4.

"4. $\Rightarrow$ 3.": obvious

"3. $\Rightarrow$ 5.": (iii) $\Rightarrow$ $\mathcal{B}_1 + \mathcal{B}_2$ represented by l.c.l.m. of R_1, R_2

3. $\Rightarrow$ l.c.l.m. of R_1, R_2 zero $\Rightarrow$ $\mathcal{B}_1 + \mathcal{B}_2 = \mathcal{A}^q$

"5. $\Rightarrow$ 3.": $\mathcal{B}_1 + \mathcal{B}_2 = \mathcal{A}^q \Leftrightarrow \operatorname{idim}(\mathcal{B}_1 + \mathcal{B}_2) = q$

let R be a representation matrix of $\mathcal{B}_1 + \mathcal{B}_2$, i.e. a l.c.l.m of R_1, R_2 $\Rightarrow$ $\operatorname{idim}(\mathcal{B}_1 + \mathcal{B}_2) = q - \operatorname{rank} R$ $\Rightarrow$ $\operatorname{rank} R = 0$

$\Rightarrow R = 0$