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Linear Systems Theory

Exercise Sheet 1

Exercise 1

(i) Let δ be the Dirac distribution and $f \in \mathcal{C}^\infty(\mathbb{R})$ a smooth function. Show that $f\delta = f(0)\delta$.

(ii) Let $T \in \mathcal{D}'(\mathbb{R})$ be a distribution and $f \in \mathcal{C}^\infty(\mathbb{R})$ a smooth function. Show that $(fT)' = f'T + fT'$, i. e. the usual Leibniz rule holds also for distributional derivatives.

(iii) Let h be the Heaviside function and consider the smooth function $f(x) = \sin(x)h(x)$. Compute the second derivative f'' of f in the distributional sense.

Exercise 2

We consider a (mathematical) pendulum with a control. Its equations of motion are given by (for simplicity, all physical constants are set to 1):

$$\dot{x}_1(t) = x_2(t)\,, \quad \dot{x}_2(t) = \sin\bigl(x_1(t)\bigr) + u(t)$$

where x_1 represents the angle of the pendulum and x_2 its angular velocity. Our goal is to determine an input u that steers this system from a given initial position $\vec{x}(0) = \vec{x}_0 \in \mathbb{R}^2$ to the origin $\vec{x}(\tau) = 0$ within a prescribed time $\tau > 0$. This can be achieved with a simple, yet effective trick.

(i) Introduce the new "control" $v = \sin(x_1) + u$ and compute the explicit solution of the initial value problem $\dot{x}_1 = x_2,\ \dot{x}_2 = v,\ \vec{x}(0) = \vec{x}_0$ in dependence of v.

(ii) Make the linear ansatz $v(t) = \alpha + \beta t$ with constants $\alpha, \beta \in \mathbb{R}$ and use the terminal condition $\vec{x}(\tau) = 0$ to express α, β in terms of the parameters $\vec{x}_0$ and τ.

(iii) Determine the arising input u and verify that it really solves our problem.

(iv) *(Optional)* Visualise your results numerically for the values $\tau = 1$, $\vec{x}_0 = \binom{0.5}{0}$ – using for example MAPLE or MATLAB. It is instructive to note that naive discretisations like the explicit Euler method will fail here due to instabilities. One needs either a good solver (like the built-in ones in the above systems) or should use at least the implicit Euler method.

please turn over

Exercise 3

(i) Show that both solutions of the quadratic equation $\lambda^2 + p\lambda + q = 0$ with real coefficients $p, q \in \mathbb{R}$ possess a negative real part, if and only if $p > 0$ and $q > 0$.

(ii) Linearisation of the pendulum equations from Exercise 2 about the upright position $\vec{x} = 0$ yields the linear differential system

$$\dot{x}_1 = x_2 \,, \quad \dot{x}_2 = x_1 + u \,.$$

This time our goal consists of finding a choice of the control u such that the upright position $\vec{x} = 0$ becomes asymptotically stable (i. e. all eigenvalues of the matrix of the linear system should have negative real parts).

Recall that x_1 denotes the angular position and x_2 the angular velocity of the pendulum. A natural approach is then the following one. If $x_1 > 0$, then the pendulum is to the right of its desired position and thus one should apply a torque $u < 0$ acting counter-clockwise and similarly one should choose $u > 0$ for $x_1 < 0$. Thus one is lead to the simple ansatz $u = \alpha x_1$ for some $\alpha < 0$. However, this ansatz will *not* solve our problem. The same holds true for the ansatz $u = \beta x_2$. Derive conditions on $\alpha, \beta \in \mathbb{R}$ such that the combined ansatz $u = \alpha x_1 + \beta x_2$ renders the upright position asymptotically stable and explain why the simpler approaches cannot work.