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Linear Systems Theory

Exercise Sheet 2

Exercise 1

Let R be a commutative ring, $S = R[t]$ the ring of univariate polynomials with coefficients in R and $m, n \in \mathbb{N}$ fixed integers.

(i) Let $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$ be given matrices over R. Show that if the matrix A is non-singular, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

(The matrix $L = D - CA^{-1}B$ is sometimes called the *Schur complement* of the block A in the block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and has many applications. In particular, if L is also non-singular, then the inversion of M can be reduced to inverting A and L which is computationally much cheaper.)

(ii) Let $P_0, P_1, \dots, P_n \in S^{n\times n}$ be square polynomial matrices. Introduce the $mn \times mn$ matrices

$$K = \begin{pmatrix} I_n & & & 0 \\ & \ddots & & \\ & & I_n & \\ 0 & & & P_n \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & I_n & & 0 \\ \vdots & & \ddots & \\ 0 & & & I_n \\ -P_0 & -P_1 & \cdots & -P_{n-1} \end{pmatrix}.$$

Show that

$$\det(K - tL) = \det(P_0 t^n + P_1 t^{n-1} + \cdots + P_{n-1}t + P_n).$$

(Note that here the matrices P_i also depend on t and thus the determinant lives in the polynomial ring S.) Conclude that in the special case that the matrices P_i are constant

$$\det(sK - L) = \det(P_0 + P_1 s + \cdots + P_{n-1}s^{n-1} + P_n s^n)$$

for any value $s \in R$.

please turn over

Exercise 2

We consider the inhomogeneous linear differential equation $\dot{x} = ax + b$ where we assume that the coefficient functions $a \in \mathcal{C}^0(\mathbb{R})$ and $b \in L^1_{\text{loc}}(\mathbb{R})$ are continuous and locally integrable, respectively.

(i) Show that the equation has no *classical solution* $x \in \mathcal{C}^1(\mathbb{R})$, if b is not continuous.

(ii) Show that if $x \in L^1_{\text{loc}}(\mathbb{R})$, then also $ax + b \in L^1_{\text{loc}}(\mathbb{R})$.

(iii) $x \in L^1_{\text{loc}}(\mathbb{R})$ is called a *weak solution* of the differential equation, if for all test functions $\phi \in \mathcal{D}(\mathbb{R})$

$$\int_{-\infty}^{\infty} \big[x(t)\dot{\phi}(t) + a(t)x(t)\phi(t) + b(t)\phi(t)\big]\,dt = 0\,.$$

Explain in what sense one may consider such an x as a "solution" and show that x is a weak solution, if and only if the regular distribution T_x satisfies the equation $\dot{T}_x = T_{ax+b}$. (If $a \in \mathcal{C}^\infty$, then we may even write the right hand side as $aT_x + T_b$ and consider this as a distributional version of our given differential equation).

(iv) Let the function $a = c$ be a constant $c \in \mathbb{R}$ and $b = h$ the Heaviside function. Solve the differential equation $\dot{x} = cx + h$ first on the intervals $(-\infty, 0)$ and $(0, \infty)$ and combine the results into a globally defined function $x \in \mathcal{C}^0(\mathbb{R})$. Prove that this x is a weak solution.

Exercise 3

(i) Let $M \in \mathbb{R}^{n\times n}$ be a square matrix. Show that M is nilpotent, if and only if 0 is its only eigenvalue.

(ii) Compute the Kronecker-Weierstraß normal form for the matrix pencil defined by

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Preferably with the help of some computer algebra system!)