Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNIKASSEL VERSITÄT

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Linear Systems Theory

Exercise Sheet 3

Exercise 1

Prove the following generalisation of the fundamental principle presented in the lecture to non-square matrices. Let $R \in \mathbb{R}[s]^{p\times q}$ be a polynomial matrix with full row rank, i. e. $\operatorname{rank} R = p$, and $\mathcal{A} = \mathcal{D}'(\mathbb{R})$ or $\mathcal{C}^\infty(\mathbb{R})$. Then the inhomogeneous linear system $R(\frac{d}{dt})u = v$ possesses for arbitrary right hand sides $v \in \mathcal{A}^p$ a solution $w \in \mathcal{A}^q$.

Exercise 2

Consider the simple electric circuit from the first lecture. By eliminating the latent variables (i. e. the currents and voltages at the individual elements of the circuit), derive a differential equation relating the total voltage U and the total current I. Do this first by "clever" manipulations of the equations and then systematically using the technique presented in the lecture. *Note:* only for such small examples one is able to do this elimination by hand; for larger systems only the systematic approach is feasible (but may require the use of a computer algebra system).

Exercise 3

Let $\lambda \in \mathbb{C}$ be a complex number and $m \in \mathbb{N}$.

(i) Show that for all exponents $k \in \mathbb{N}$

$$\frac{d^m}{dt^m}(t^k e^{\lambda t}) = e^{\lambda t}\Big(\frac{d}{dt} + \lambda\Big)^m (t^k)$$

and conclude that for all polynomials $P \in \mathbb{C}[s]$ and $a \in \mathbb{C}[t]$

$$P\Big(\frac{d}{dt}\Big)\big(a(t)e^{\lambda t}\big) = e^{\lambda t} P\Big(\frac{d}{dt} + \lambda\Big)a(t)\,.$$

(ii) Choose $P = (s-\lambda)^m$. Show that for a polynomial $a \in \mathbb{C}[t]$ the function $y(t) = a(t)e^{\lambda t}$ is a solution of $P(\frac{d}{dt})y = 0$, if and only if $\deg a < m$.

(iii) Let $\lambda_1 \neq \lambda_2$ be complex numbers and $m_1, m_2 \in \mathbb{N}$ integers. We set $P_i = (s-\lambda_1)^{m_2} \in \mathbb{C}[s]$ for $i = 1, 2$. Show that $P_1P_2y = 0$, if and only if the function y can be decomposed $y = y_1 + y_2$ where y_i is a solution of $P_iy_i = 0$.

(iv) Assume that $\lambda \notin \mathbb{R}$ and set $P = (s-\lambda)^m(s-\bar{\lambda})^m$. Show that the function $y(t) = a(t)e^{\lambda t} + b(t)e^{\bar{\lambda}t}$ is a real-valued solution of $Py = 0$, if and only if $\deg a < m$ and $b = \bar{a}$.