Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNIKASSEL VERSITÄT

June 4th, 2013

Linear Systems Theory

Exercise Sheet 5

Exercise 1

For two arbitrary matrices $A \in \mathbb{R}^{k\times\ell}$ and $B \in \mathbb{R}^{m\times n}$, the *Kronecker product* (sometimes also called *tensor product*) is defined as the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1\ell}B \\ \vdots & \vdots & & \vdots \\ a_{k1}B & a_{k2}B & \cdots & a_{kl}B \end{pmatrix} \in \mathbb{R}^{km\times\ell n}\,.$$

For any matrix $A \in \mathbb{R}^{k\times\ell}$, we denote by $\operatorname{vec}(A) \in \mathbb{R}^{k\ell}$ the vector obtained by stacking all the columns of A above each other:

$$\operatorname{vec}(A) = (a_{11}, \dots, a_{k1}, a_{12}, \dots, a_{k2}, \dots, a_{1\ell}, \dots, a_{k\ell})^T\,.$$

(i) Show that we obtain for any matrices X, Y such that AX and BY are defined

$$(A \otimes B)(X \otimes Y) = (AX) \otimes (BY)\,.$$

(ii) Use (i) to show that for square matrices A, B the spectrum of $A \otimes B$ (i. e. the set of all eigenvalues of this matrix) is given by

$$\operatorname{spec}(A \otimes B) = \{\lambda\mu \mid \lambda \in \operatorname{spec}(A), \mu \in \operatorname{spec}(B)\}\,.$$

Hint: Show first that the matrix $A \otimes B$ is similar to the Kronecker product of the Jordan normal forms of A and B.

(iii) Show that for any three matrices A, B, X such that the product AXB is defined

$$\operatorname{vec}(AXB) = (B^T \otimes A)\operatorname{vec}(X)\,.$$

(iv) Show that the *Sylvester equation*

$$AX + XB = C$$

has for given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$ a unique solution $X \in \mathbb{R}^{n\times m}$, if and only if $\lambda + \mu \neq 0$ for all $\lambda \in \operatorname{spec}(A)$ and $\mu \in \operatorname{spec}(B)$.

please turn over

Exercise 2

For the computations required by this exercise, it is recommended to use some computer algebra system like MAPLE.

(i) Is the system given in Exercise 2 of last week asymptotically stable?

(ii) Compute the matrix exponential e^{At} for the matrix

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} .$$

Is the system $\dot{\mathbf{x}} = A\mathbf{x}$ (asymptotically) stable? Compute the unique solution for the initial data $\mathbf{x}(0) = \binom{1}{3}$.

(iii) Consider the state space system $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, $y = C\mathbf{x}$ with

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad C = \begin{pmatrix} 1 \\ -2 \end{pmatrix} .$$

Show that it is *not* asymptotically stable, but that the associated input/output system obtained after eliminating the state $\mathbf{x}$ is asymptotically stable. Do you have any explanation for this observation?