Prof. Werner M. Seiler, Ph.D.
FB 10 Mathematik und Naturwissenschaften
Institut für Mathematik

UNI KASSEL VERSITÄT

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Linear Systems Theory

Exercise Sheet 6

Exercise 1

The *energy* required to control a system $\dot{x} = Ax + Bu$ from 0 to some state $\bar{x}$ in time ϵ using the input $u(t)$ is defined as the integral $E(u) = \int_0^\epsilon \|u(t)\|^2 dt$. In this exercise, we will show that the minimal energy required for this process is given by

$$\min\left\{E(u) \mid \phi(\epsilon, 0, 0, u) = \bar{x}\right\} = \bar{x}^T W(\epsilon)^{-1} \bar{x}$$

where W denotes the controllability Gramian and that this minimum is achieved with the special input constructed in the lecture.

(i) Show that $W(t)$ satisfies the differential equation $\dot{W} = AW + WA^T + BB^T$.

(ii) Let $x(t)$ be the solution of the system for some input u and set $V(t) = x(t)^T W(t)^{-1} x(t)$. Show that its derivative is given by

$$\dot{V}(t) = -\|B^T W(t)^{-1} x(t)\|^2 + 2\langle u(t), B^T W(t)^{-1} x(t)\rangle\,.$$

(iii) Use the result from (ii) to show that any input u with $\phi(\epsilon, 0, 0, u) = \bar{x}$ satisfies the estimate $E(u) \geq \bar{x}^T W(\epsilon)^{-1} \bar{x}$. Finally show that equality holds for the input

$$u(t) = B^T e^{A^T(\epsilon - t)} W(\epsilon)^{-1} \bar{x}\,.$$

please turn over

Exercise 2

In this exercise, we study two concrete physical systems. However, no specific physical knowledge is required for the solution.

(i) Let V be the voltage applied to an electric water kettle and x_1, x_2 the temperature of the heater coil and of the water, respectively. The change of x_1 is proportional to the electric power fed into the system minus the heat loss to the water. The electric power is proportional to V^2, the heat loss to the temperature difference $x_1 - x_2$. The change of x_2 is proportional to the heat loss of the coil. Thus we obtain the mathematical model

$$\dot{x}_1 = aV^2 - b(x_1 - x_2)\,, \qquad \dot{x}_2 = c(x_1 - x_2)$$

with real constants $a, b, c \in \mathbb{R}$. With $u = V^2$ as input, it represents a state space system. Discuss its stability and controllability in dependence of the parameters a, b, c.

(ii) The "bipendulum" consists of a horizontal rod with a pendulum attached to each end. If the rod is moved horizontally, the pendula begin to swing. After some idealisations, this system is described by the following equations

$$\ddot{y}_1 + \omega_1^2 y_1 = u\,, \qquad \ddot{y}_2 + \omega_2^2 y_2 = u$$

where y_i is the angle between the ith pendulum and the vertical direction, $\omega_i > 0$ the characteristic frequency of the ith pendulum (determined by its length and the gravitational acceleration) and the input u is proportional to the acceleration of the rod. Considering y_1, y_2 as the output of the system, transform it into a four-dimensional state space form by rewriting it as a first-order system. Now discuss its stability and controllability.