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Linear Systems Theory

Exercise Sheet 8

Exercise 1

We consider an observable system $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{y} = C\mathbf{x}$ with $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ and a fixed time $\epsilon > 0$. If we introduce the output space $\mathcal{Y} = L^2\big([0,\epsilon], \mathbb{R}^p\big)$ of square-integrable functions $\mathbf{f} : [0,\epsilon] \rightarrow \mathbb{R}^p$, then we can define the *observability operator*

$$L : \begin{cases} \mathbb{R}^n & \longrightarrow & \mathcal{Y} \\ \mathbf{x}_0 & \longmapsto & (t \mapsto Ce^{At}\mathbf{x}_0) \end{cases}$$

mapping an initial condition $\mathbf{x}(0) = \mathbf{x}_0$ to the corresponding output function of the system.

(i) Show that L is a bounded linear operator and that the observability of the system is equivalent to the injectivity of L.

(ii) Construct the adjoint operator to L, i. e. find a linear operator $L^* : \mathcal{Y} \rightarrow \mathbb{R}^n$ such that the equation

$$\big\langle L\mathbf{x}_0, \mathbf{y}(\cdot)\big\rangle_{\mathcal{Y}} = \big\langle \mathbf{x}_0, L^*\mathbf{y}(\cdot)\big\rangle_{\mathbb{R}^n}$$

holds for all $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{y}(\cdot) \in \mathcal{Y}$. Show that the observability Gramian $W(\epsilon)$ is then given by $W(\epsilon) = L^*L$. Conclude that $L\mathbf{x}_0 = \mathbf{y}(\cdot)$ can be inverted to $\mathbf{x}_0 = W(\epsilon)^{-1}L^*\mathbf{y}(\cdot)$.

(iii) Assume that due to perturbations we measure $\hat{\mathbf{y}}(\cdot)$ instead of the true output $\mathbf{y}(\cdot)$. Show that the corresponding vector $\hat{\mathbf{x}}_0 = W(\epsilon)^{-1}L^*\hat{\mathbf{y}}(\cdot)$ satisfies for all vectors $\mathbf{x}_0 \in \mathbb{R}^n$ the estimate

$$\|L\hat{\mathbf{x}}_0 - \hat{\mathbf{y}}(\cdot)\|_{\mathcal{Y}} \leq \|L\mathbf{x}_0 - \hat{\mathbf{y}}(\cdot)\|_{\mathcal{Y}}$$

with equality holding only for $\mathbf{x}_0 = \hat{\mathbf{x}}_0$. Thus the output function corresponding to $\hat{\mathbf{x}}_0$ is the best possible approximation of the measured output $\hat{\mathbf{y}}(\cdot)$.

please turn over

Exercise 2

We consider the controllable and observable system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$. Let $P = W_c(\epsilon)$ and $Q = W_o(\epsilon)$ be the controllability and the observability Gramian, respectively, for a fixed time $\epsilon > 0$. Recall that $\mathbf{x}_0^T P^{-1}\mathbf{x}$ is the input energy required to steer the system from $\mathbf{x}_0$ to the origin in time ϵ. We similarly define the output energy as $\mathbf{x}_0^T Q\mathbf{x}_0$. The state $\mathbf{x}_0 \in \mathbb{R}^n$ is considered to be "good" with respect to controllability, if $\mathbf{x}_0^T P^{-1}\mathbf{x}_0$ is "small", as then not much energy is needed to steer it to the origin. The state $\mathbf{x}_0$ is considered to be "bad" with respect to observability, if $\mathbf{x}_0^T Q\mathbf{x}_0$ is "small", as it is then hard to distinguish $\mathbf{x}_0$ from the origin.
It is desirable to find a system representation such that with respect to both criteria simultaneously any standard basis vector $\mathbf{e}_i \in \mathbb{R}^n$ is either "good" or "bad", so that we can decide about the "quality" of each state component. Thus we perform a coordinate transformation $\mathbf{x} = T\tilde{\mathbf{x}}$ with a non-singular matrix $T \in \mathbb{R}^{n\times n}$ to a new representation with matrices $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T^{-1}AT, T^{-1}B, CT, D)$. We achieve our goal, if the transformed Gramians $\tilde{P}$, $\tilde{Q}$ are equal and diagonal. This process is called *balancing*.

(i) Show that the original and the transformed Gramians are related by the congruences

$$\tilde{P} = T^{-1}PT^{-T}\,, \qquad \tilde{Q} = T^T QT\,.$$

Conclude that the matrices PQ and $\tilde{P}\tilde{Q}$ are similar.

(ii) Show that all eigenvalues of PQ are real and positive.

(iii) Prove the existence of a non-singular transformation matrix T such that

$$\tilde{P} = \tilde{Q} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$$

where the numbers $\sigma_i > 0$ are the square roots of the eigenvalues of PQ (they are called the *Hankel singular values* of the system at time ϵ). How would you use this result to define the "quality" of the state $\mathbf{e}_i$ in the new representation?

Hint for (ii,iii): Decompose $P = RR^T$ and set $S = R^T QR$; now $S > 0$ and $\operatorname{Spec}(S) = \operatorname{Spec}(PQ)$; transform $U^T SU = \Lambda$ with Λ diagonal and U orthogonal; finally set $T = RU\Lambda^{-1/4}$.
Remark: Balancing is an important technique for *model reduction*, i. e. for finding a smaller model with essentially the same properties. Here the reduction is performed by discarding all state components with a small Hankel singular value.