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Linear Systems Theory

Exercise Sheet 9

Exercise 1

Consider two linear systems

$$\dot{\mathbf{x}}_i = A_i\mathbf{x}_i + B_i\mathbf{u}_i\,, \qquad \mathbf{y}_i = C_i\mathbf{x}_i + D_i\mathbf{u}_i$$

with $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $C_i \in \mathbb{R}^{p_i\times n_i}$, $D_i \in \mathbb{R}^{p_i\times m_i}$ for $i = 1, 2$. In analogy to electrical circuits, we define the *series* and the *parallel connection* of the systems. For the series connection we must assume $p_1 = m_2$ and then simply set $\mathbf{u}_2 = \mathbf{y}_1$, i. e. the output of the first system becomes the input of the second one. The parallel connection requires $p_1 = p_2$ and $m_1 = m_2$. It is defined by giving both systems the same input $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$ and adding their outputs $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$.

(i) Derive the state space representations of the systems obtained by the two types of interconnection setting $\mathbf{x} = \binom{\mathbf{x}_1}{\mathbf{x}_2}$.

(ii) Show that if the series connection is observable, then both original systems are observable, too. Provide a concrete counter example for the converse statement.

(iii) Assume that (A_1, C_1) and (A_2, C_2) are observable matrix pairs and that in addition the matrix

$$\begin{pmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{pmatrix}$$

has full column rank for all eigenvalues λ of A_1. Prove that these assumptions are sufficient for the observability of the series connection.

(iv) Show that if the parallel connection is observable, then both original systems are observable, too.

(v) Prove that the converse of (iv) is true, if we assume additionally that A_1 and A_2 have no common eigenvalue.

Remark: Analogous statements hold for the *controllability* of the series and the parallel connection. Only in (iii) the additional condition must then be replaced by requiring that the matrix

$$\begin{pmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{pmatrix}$$

has full row rank for all $\lambda \in \operatorname{spec} A_2$.

please turn over

Exercise 2

Let $\mathcal{D}$ be a commutative ring with 1 and $R_i \in \mathcal{D}^{g_i \times q}$ for $i = 1, 2$ two matrices over $\mathcal{D}$ with the same number of columns. A matrix $R \in \mathcal{D}^{g \times q}$ is a *common left multiple* of R_1 and R_2, if there exist two further matrices T_1, T_2 over $\mathcal{D}$ such that $R = T_1 R_1 = T_2 R_2$. It is even a *least common left multiple*, if for any other common left multiple $\tilde{R}$ a matrix X exists such that $\tilde{R} = XR$.

(i) Show that for a Noetherian ring $\mathcal{D}$ any two matrices with the same number q of columns possess a least common left multiple.

Hint: consider the left kernel of the matrix

$$S = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \\ I_q & I_q \end{pmatrix}$$

(which is finitely generated in the case of a Noetherian ring $\mathcal{D}$) and interpret its generators as the rows of a matrix $(A\ B\ \ -C)$.

(ii) Prove that R is a least common left multiple of R_1 and R_2, if and only if

$$\mathcal{D}^{1 \times g_1} R_1 \cap \mathcal{D}^{1 \times g_2} R_2 = \mathcal{D}^{1 \times g} R\,.$$

What does this characterisation imply for the uniqueness of least common left multiples?

Now we choose $\mathcal{D} = \mathbb{R}[s]$ and introduce the behaviours $\mathcal{B}_i = \left\{ \mathbf{w} \in \mathcal{A}^q \mid R(\frac{d}{dt})\mathbf{w} = 0 \right\}$ for some signal space $\mathcal{A}$ in which the fundamental principle holds. Furthermore, we write $\mathcal{Q} = \mathbb{R}(s)$ for the quotient field of $\mathcal{D}$. The *rank* of a matrix R over the ring $\mathcal{D}$ is defined as its rank over the field $\mathcal{Q}$.

(iii) Show that any least common left multiple of R_1 and R_2 represents the behaviour $\mathcal{B}_1 + \mathcal{B}_2$.

Hint: use first a representation of $\mathcal{B}_1 + \mathcal{B}_2$ with the matrix S defined above and then apply the fundamental principle.

(iv) Show that the following statements are equivalent:

1. $\operatorname{rank} R_1 + \operatorname{rank} R_2 = \operatorname{rank} \binom{R_1}{R_2}$.
2. $\mathcal{Q}^{1 \times g_1} R_1 \cap \mathcal{Q}^{1 \times g_2} R_2 = 0$.
3. $\mathcal{D}^{1 \times g_1} R_1 \cap \mathcal{D}^{1 \times g_2} R_2 = 0$.
4. Any common left multiple of R_1 and R_2 is zero.
5. $\mathcal{B}_1 + \mathcal{B}_2 = \mathcal{A}^q$.

Hint: for proving the implication "5 $\Rightarrow$?" use the concept of an input dimension.