Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

SINGULARITIES IN THE GEOMETRIC THEORY OF DIFFERENTIAL EQUATIONS

Ulrike Kant and Werner M. Seiler

Institut für Mathematik, Universität Kassel, 34132 Kassel, Germany

Abstract. We briefly recall the basic ideas of the Vessiot theory, a geometric approach to differential equations based on vector fields. Then we show that it allows to extend naturally some results on singularities for ordinary differential equations to maximally overdetermined partial differential equations.

1. **Geometric Theory of Differential Equations.** The geometric modelling of differential equations is based on the jet bundle formalism [7, 9, 11]. We formulate it here for sections of a fibred manifold $\pi : \mathcal{E} \to \mathcal{X}$. For simplicity, we will work in local coordinates, although we will use a "global notation" in order to avoid the tedious introduction of many neighbourhoods. We call coordinates $\mathbf{x} = (x^1, \dots, x^n)$ on the base space $\mathcal{X}$ *independent variables* and fibre coordinates $\mathbf{u} = (u^1, \dots, u^m)$ in the total space $\mathcal{E}$ *dependent variables*. Derivatives are written in the form $u^\alpha_\mu = \partial^{|\mu|}u^\alpha/\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$ with a multi index $\mu = [\mu_1, \dots, \mu_n]$. Adding all derivatives u^α_μ up to order q (collectively denoted by $\mathbf{u}^{(q)}$) defines a coordinate system for the q-th order jet bundle $J_q\pi$ which may be considered as the space of Taylor polynomials of maximal degree q. The hierarchy of jet bundles $(J_q\pi)_{q\geq 0}$ admits many natural fibrations. Important for us are $\pi^q_{q-1} : J_q\pi \to J_{q-1}\pi$ and $\pi^q : J_q\pi \to \mathcal{X}$.

Sections $\sigma : \mathcal{X} \to \mathcal{E}$ of π correspond to functions $\mathbf{u} = \mathbf{s}(\mathbf{x})$, as locally they can always be written in the form $\sigma(\mathbf{x}) = \big(\mathbf{x}, \mathbf{s}(\mathbf{x})\big)$. To such a section σ, we associate its *prolongation* $j_q\sigma : \mathcal{X} \to J_q\pi$, a section of the fibration π^q given by $j_q\sigma(\mathbf{x}) = \big(\mathbf{x}, \mathbf{s}(\mathbf{x}), \partial_{\mathbf{x}}\mathbf{s}(\mathbf{x}), \partial_{\mathbf{xx}}\mathbf{s}(\mathbf{x}), \dots\big)$.

The geometry of $J_q\pi$ is to a large extent determined by its *contact structure* describing intrinsically the relationship between the different types of coordinates. One way to realise it consists of considering the smallest distribution[1] $\mathcal{C}_q \subset T(J_q\pi)$ that contains the tangent spaces $T(\operatorname{im} j_q\sigma)$ of all prolonged sections. $\mathcal{C}_q$ is called the *contact distribution* and any vector field in it a *contact vector field*. In local coordinates, $\mathcal{C}_q$ is generated by the following two families of fields[2]

$$C^{(q)}_i = \partial_i + u^\alpha_{\mu+1_i}\partial_{u^\alpha_\mu} , \qquad 1 \leq i \leq n , \tag{1a}$$

$$C^\mu_\alpha = \partial_{u^\alpha_\mu} , \qquad |\mu| = q . \tag{1b}$$

2000 *Mathematics Subject Classification.* Primary: 35A30 Secondary: 34A09, 35A20, 35D99.

Key words and phrases. Regular Singularities, irregular singularities, impasse points, singular integrals, Vessiot distribution, generalised solutions, Clairaut equations.

The work of the second author was partially supported by DFG grant Se 770/8-1.

[1] In this article, we use the term "distribution" exclusively in the geometric sense of a smooth assignment of a subspace of its tangent space to each point on a manifold.

[2] We use the convention that a summation over repeated indices is understood.

An alternative approach to the contact structure, the *contact map*, was introduced in [8]. It is the unique map $\Gamma_q : J_q\pi\times_{\mathcal{X}}T\mathcal{X} \to T(J_{q-1}\pi)$ such that

$$\begin{tikzcd} J_q\pi\times_{\mathcal{X}}T\mathcal{X} \arrow[rr, "\Gamma_q"] & & T(J_{q-1}\pi) \\ & T\mathcal{X} \arrow[ul, "((j_q\sigma)\circ\tau_{\mathcal{X}})\times\mathrm{id}_{T\mathcal{X}}"] \arrow[ur, "T(j_{q-1}\sigma)"'] & \end{tikzcd} \tag{2}$$

is a commutative diagram for any section $\sigma : \mathcal{X} \to \mathcal{E}$. Because of its linearity over π^q_{q-1}, we may also consider it as a map $\Gamma_q : J_q\pi \to T^*\mathcal{X}\otimes_{J_{q-1}\pi}T(J_{q-1}\pi)$ with the local coordinate form:

$$\Gamma_q : (\mathbf{x}, \mathbf{u}^{(q)}) \mapsto \big(\mathbf{x}, \mathbf{u}^{(q-1)}; dx^i \otimes (\partial_{x^i} + u^\alpha_{\mu+1_i}\partial_{u^\alpha_\mu})\big)\ . \tag{3}$$

In this form one sees that $\operatorname{im}\Gamma_q = \mathcal{C}_{q-1}$ and hence $\mathcal{C}_q = (T\pi^q_{q-1})^{-1}(\operatorname{im}\Gamma_q)$. The following result is classical and explains how the contact structures characterises those sections of the fibration π^q which are prolongations of sections of π.

Proposition 1.1. *A section $\gamma : \mathcal{X} \to J_q\pi$ is of the form $\gamma = j_q\sigma$ for a section $\sigma : \mathcal{X} \to \mathcal{E}$, if and only if $\operatorname{im}\Gamma_q\big(\gamma(x)\big) = T_{\gamma(x)}\pi^q_{q-1}\big(T_{\gamma(x)}(\operatorname{im}\gamma)\big)$ for all points $x \in \mathcal{X}$ where γ is defined.*

A *differential equation* of order q is a fibred submanifold $\mathcal{R}_q \subseteq J_q\pi$ locally described as the zero set of some smooth functions on $J_q\pi$:

$$\mathcal{R}_q\ :\ \Big\{\ \Phi^\tau(\mathbf{x}, \mathbf{u}^{(q)}) = 0\ , \quad \tau = 1, \dots, t\ . \tag{4}$$

Note that this definition does not distinguish between scalar equations and systems. Indeed, when we speak of a differential equation in the sequel, we will always mean a system, if not explicitly stated otherwise.

Differentiating every equation in (4) yields the *prolonged equation* $\mathcal{R}_{q+1} \subseteq J_{q+1}\pi$ defined by all equations $\Phi^\tau = 0$ and $D_i\Phi^\tau = 0$ with the formal derivative

$$D_i\Phi^\tau = C^{(q)}_i(\Phi^\tau) + u^\alpha_{\mu+1_i}C^\mu_\alpha(\Phi^\tau)\ . \tag{5}$$

Iteration of this process gives the higher prolongations $\mathcal{R}_{q+r} \subseteq J_{q+r}\pi$. A subsequent projection leads to the differential equation $\mathcal{R}^{(1)}_q = \pi^{q+1}_q(\mathcal{R}_{q+1}) \subseteq \mathcal{R}_q$ which will be a proper submanifold, if integrability conditions are hidden. $\mathcal{R}_q$ is *formally integrable*, if at any prolongation order $r > 0$ the equality $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ holds. An *involutive* differential equation satisfies in addition some further algebraic conditions which we do not specify here (see [11] for more details). By the Cartan-Kuranishi theorem, every consistent differential equation can be completed to an equivalent involutive one by a finite number of prolongations and projections (extracting hidden integrability conditions). Therefore, without loss of generality, we will always assume in the sequel that we are dealing with an involutive equation.

Definition 1.2. A *classical solution* of the differential equation $\mathcal{R}_q \subseteq J_q\pi$ is a (local) section $\sigma : \mathcal{X} \to \mathcal{E}$ such that its prolongation satisfies $\operatorname{im} j_q\sigma \subseteq \mathcal{R}_q$.

In local coordinates, this definition obviously coincides with the usual notion of a solution. For involutive equations, it is straightforward to construct order by order formal power series solutions; if the equation is furthermore analytic, then one can show that the series converges and one obtains the Cartan-Kähler theorem as a generalisation of the familiar Cauchy-Kovalevskaya theorem.

2. **The Vessiot Distribution.** A key insight of Cartan was to introduce *infinitesimal solutions* or *integral elements* at a point $\rho \in \mathcal{R}_q$ as subspaces $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_q$ which are potentially part of the tangent space of a prolonged solution. We follow an approach pioneered by Vessiot [13] which is based on vector fields and dual to the more familiar Cartan-Kähler theory. Therefore our definition of an integral element is not the standard one, but one can easily prove the equivalence [11].

Definition 2.1. Let $\mathcal{R}_q \subseteq J_q\pi$ be a differential equation of order q. A linear subspace $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_q$ is an *integral element* at the point $\rho \in \mathcal{R}_q$, if a point $\hat{\rho} \in \mathcal{R}_{q+1}$ exists such that $\pi_q^{q+1}(\hat{\rho}) = \rho$ and $\mathcal{U}_\rho \subseteq \operatorname{im}\Gamma_{q+1}(\hat{\rho})$.

According to Proposition 1.1, the tangent spaces $T_\rho(\operatorname{im} j_q\sigma)$ of prolonged solutions of $\mathcal{R}_q$ are always integral elements. However, the converse is not true: integral elements are only *candidates* for such tangent spaces. In particular, one should note that integral elements are defined at single points $\rho \in \mathcal{R}_q$, hence the construction of a classical solution defined in some neighbourhood $\mathcal{U} \subseteq \mathcal{X}$ of a point $x \in \mathcal{X}$ requires integral elements over all points $y \in \mathcal{U}$. Thus for obtaining classical solutions from infinitesimal ones distributions of integral elements must be considered.

By Proposition 1.1, the tangent spaces $T_\rho(\operatorname{im} j_q\sigma)$ of prolonged sections at points $\rho \in J_q\pi$ are always subspaces of the contact distribution $\mathcal{C}_q|_\rho$. If the section σ is a solution of $\mathcal{R}_q$, it furthermore satisfies by definition $\operatorname{im} j_q\sigma \subseteq \mathcal{R}_q$ and hence $T(\operatorname{im} j_q\sigma) \subseteq T\mathcal{R}_q$. These considerations motivate the following construction.

Definition 2.2. The *Vessiot distribution* of a differential equation $\mathcal{R}_q \subseteq J_q\pi$ is the distribution $\mathcal{V}[\mathcal{R}_q] \subseteq T\mathcal{R}_q$ defined by

$$\mathcal{V}[\mathcal{R}_q] = T\mathcal{R}_q \cap \mathcal{C}_q|_{\mathcal{R}_q} . \tag{6}$$

Computing the Vessiot distribution is straightforward and requires only some linear algebra. It follows from Definition 2.2 that any vector field X contained in $\mathcal{V}[\mathcal{R}_q]$ is a contact field and thus can be written as a linear combination of the basic fields (1): $X = a^i C_i^{(q)} + b_\mu^\alpha C_\alpha^\mu$. On the other hand, X must be tangent to the differential equation $\mathcal{R}_q$. Hence, if $\mathcal{R}_q$ is locally described by the system (4), then X must satisfy the equations $d\Phi^\tau(X) = X(\Phi^\tau) = 0$. Evaluation of this condition yields the following linear system of equations for the coefficients a^i, b_μ^α:

$$C_i^{(q)}(\Phi^\tau)a^i + C_\alpha^\mu(\Phi^\tau)b_\mu^\alpha = 0 , \quad \tau = 1, \dots, t . \tag{7}$$

Note that X is vertical with respect to π^q, if and only if all coefficients a^i vanish.

Determining the Vessiot distribution of $\mathcal{R}_q$ requires essentially the same computations as prolonging it. Indeed, the prolongation $\mathcal{R}_{q+1}$ is locally described by the original equations $\Phi^\tau = 0$ together with the prolonged equations

$$C_\alpha^\mu(\Phi^\tau)u_{\mu+1_i}^\alpha + C_i^{(q)}(\Phi^\tau) = 0 , \quad \tau = 1, \dots, t , \ i = 1, \dots, n . \tag{8}$$

For an ordinary differential equation, where $n = 1$, these coincide with (7), if we set $a^1 = 1$ and $b_\mu^\alpha = u_{\mu+1_1}^\alpha$, i. e. if we consider only transversal solutions of (7).

Remark 2.3. If a formally integrable differential equation $\mathcal{R}_q \subset J_q\pi$ contains equations of different orders, then only the equations of highest order are relevant for the determination of the Vessiot distribution. Assume that locally the differential equation $\mathcal{R}_q$ is described by the system

$$\Phi^\tau(\mathbf{x}, \mathbf{u}^{(q)}) = 0 , \qquad 1 \leq \tau \leq t , \tag{9a}$$

$$\Psi^\sigma(\mathbf{x}, \mathbf{u}^{(q-1)}) = 0 , \qquad 1 \leq \sigma \leq s . \tag{9b}$$

Here we assume that the Jacobian of Φ^τ with respect to all derivatives of order q has maximal rank, i. e. that no lower-order equations can be extracted from (9a). If we follow the above described ansatz for determining $\mathcal{V}[\mathcal{R}_q]$, then the first subsystem (9a) contributes the conditions (7) and the second subsystem (9b) yields $C_i^{(q)}(\Psi^\sigma)a^i = 0$. Since Ψ^σ does not depend on any derivatives of order q, (5) implies that $D_i\Psi^\sigma = C_i^{(q)}(\Psi^\sigma)$. By assumption, $\mathcal{R}_q$ is formally integrable and hence $D_i\Psi^\sigma$ vanishes on $\mathcal{R}_q$, as otherwise an integrability condition would arise. Thus (9b) does not provide additional conditions on the Vessiot distribution.

Example 2.4. *Assume that we are given a first-order* ordinary *differential equation $\mathcal{R}_1 \subset J_1\pi$ of the standard form $\mathbf{u}_x = \mathbf{f}(x,\mathbf{u})$. Then a straightforward computation yields that the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ is everywhere one-dimensional and generated by the vector field*

$$X = \partial_x + f^\alpha\partial_{u^\alpha} + \left(\frac{\partial f^\alpha}{\partial x} + f^\beta\frac{\partial f^\alpha}{\partial u^\beta}\right)\partial_{u^\alpha_x} \tag{10}$$

(here we exploited that on $\mathcal{R}_1$ one can substitute u^α_x by f^α). One recognises the expression in the parentheses as the value of u^α_{xx} obtained by prolonging the equation.

X is projectable from $\mathcal{R}_1$ to $\mathcal{E}$ and the result $T\pi^1_0(X) = \partial_x + f^\alpha\partial_{u^\alpha}$ is the evolution vector field associated with the given differential equation. Hence we may consider $\mathcal{V}[\mathcal{R}_1]$ as a "lift" of this evolution field to $\mathcal{R}_1$. Here this lift is not particularly interesting, but it is useful for analysing singularities of implicit equations.

Geometrically, an ordinary differential equation in standard form defines an Ehresmann connection on the underlying fibred manifold $\pi : \mathcal{E} \to \mathcal{X}$. If the base space $\mathcal{X}$ is higher-dimensional, then connections are defined by first-order partial differential equations locally described by systems of the form

$$\mathcal{R}_1 \,:\, \left\{\; u^\alpha_i = \phi^\alpha_i(\mathbf{x},\mathbf{u})\;, \quad \alpha = 1,\dots,m\;,\; i = 1,\dots,n\;. \right. \tag{11}$$

Indeed, such a system induces a horizontal bundle $\mathcal{H} \subset T\mathcal{E}$ generated by the vector fields $Y_i = \partial_{x^i} + \phi^\alpha_i\partial_{u^\alpha}$. The connection is flat, i. e. the distribution $\mathcal{H}$ involutive in the sense that it is closed under Lie brackets, if and only if $\mathcal{R}_1$ is formally integrable. The Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ is here n-dimensional and generated by the fields

$$X_i = \partial_{x^i} + \phi^\alpha_i\partial_{u^\alpha} + \left(\frac{\partial \phi^\alpha_j}{\partial x^i} + \phi^\beta_i\frac{\partial \phi^\alpha_j}{\partial u^\beta}\right)\partial_{u^\alpha_j}\;, \quad 1 \le i \le n\;. \tag{12}$$

Again $T\pi^1_0(X_i) = Y_i$, so that we may consider $\mathcal{V}[\mathcal{R}_1]$ as a "lift" of the horizontal bundle $\mathcal{H}$, and in the parentheses one finds the value of the derivative u^α_{ij} given by the prolonged equation.

It should be stressed that we allow that the rank of a distribution varies from point to point. In fact, this will be important for certain types of singularities. The following, fairly elementary result is the basis of Vessiot's approach to the existence theory of solutions for the differential equation $\mathcal{R}_q$. It relates solutions of $\mathcal{R}_q$ with certain subdistributions of the Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$.

Lemma 2.5. *If the section $\sigma : \mathcal{X} \to \mathcal{E}$ is a solution of the equation $\mathcal{R}_q$, then the tangent bundle $T(\operatorname{im} j_q\sigma)$ is an n-dimensional involutive subdistribution of $\mathcal{V}[\mathcal{R}_q]|_{\operatorname{im} j_q\sigma}$ transversal to the fibration π^q. Conversely, if $\mathcal{U} \subseteq \mathcal{V}[\mathcal{R}_q]$ is an n-dimensional transversal involutive subdistribution, then any integral manifold of $\mathcal{U}$ has locally the form $\operatorname{im} j_q\sigma$ for a solution σ of $\mathcal{R}_q$.*

Because of this simple observation, Vessiot proposed to search for n-dimensional, transversal involutive subdistributions of $\mathcal{V}[\mathcal{R}_q]$ and presented a concrete procedure consisting of an algebraic and a differential step for the actual construction of all such subdistributions. Fesser [2] provided recently a rigorous analysis of this procedure and showed that it succeeds, if and only if the differential equation $\mathcal{R}_q$ is involutive (see also [3] or [11, Sects 9.5/6]). Here we only mention briefly how (distributions of) integral elements can be characterised algebraically.

Proposition 2.6 ([2, 3])**.** *Let $\mathcal{U} \subseteq \mathcal{V}[\mathcal{R}_q]$ be a subdistribution of the Vessiot distribution of constant rank k not containing vertical vectors. Then the spaces $\mathcal{U}_\rho$ are k-dimensional integral elements for all points $\rho \in \mathcal{R}_q$, if and only if $[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{V}[\mathcal{R}_q]$.*

Definition 2.7. A *generalised solution* of the differential equation $\mathcal{R}_q \subseteq J_q\pi$ is an n-dimensional integral manifold $\mathcal{N} \subseteq \mathcal{R}_q$ of the Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$.

Note that a generalised solution lives in the jet bundle $J_q\pi$ and not in $\mathcal{E}$. If $\sigma : \mathcal{X} \to \mathcal{E}$ is a classical solution, then the image of its prolongation $j_q\sigma : \mathcal{X} \to J_q\pi$ is a generalised solution. However, not every generalised solution $\mathcal{N}$ projects on a classical one: this will be the case, if and only if the tangent bundle $T\mathcal{N}$ is everywhere transversal to the fibration $\pi^q : J_q\pi \to \mathcal{X}$. It follows from the discussion in Example 2.4 that a partial differential equation of the form considered there possesses generalised solutions, if and only if it is formally integrable. Generalised solutions permit in particular the treatment of multivalued solutions, as they often appear in wave equations [6, 14].

3. **Maximally Overdetermined Systems.** We call a first-order partial differential equation $\mathcal{R}_1 \subset J_1\pi$ in n independent and m dependent variables locally described by the system $\Phi^\tau(\mathbf{x}, \mathbf{u}^{(1)}) = 0$ *maximally overdetermined* (or *holonomic*), if $\dim \mathcal{R}_1 \leq n + m$. Obviously, this requires that the system comprises at least mn equations. The partial differential equations considered in Example 2.4 obviously belong to this class, as for them $\dim \mathcal{R}_1 = n + m$.

Definition 3.1. Let $\mathcal{R}_1$ be a maximally overdetermined, involutive differential equation in n independent variables locally described by the system $\Phi^\tau(\mathbf{x}, \mathbf{u}^{(1)}) = 0$. A point $\rho \in \mathcal{R}_1$ is called

- **(i):** *regular*, if $\mathcal{V}_\rho[\mathcal{R}_1]$ is n-dimensional and transversal to the fibration π^1,
- **(ii):** *regular singular*, if $\mathcal{V}_\rho[\mathcal{R}_1]$ is n-dimensional, but not transversal to π^1,
- **(iii):** *irregular singular* or *s-singular*, if $\dim \mathcal{V}_\rho[\mathcal{R}_1] = n + s$ with $s > 0$.

Regular singularities are also called *impasse points*.

This definition extends a terminology introduced by Arnold [1] for ordinary differential equations. Rabier [10] objects that it may be confused with similar notions in the Fuchs–Frobenius theory of linear ordinary differential equations in the complex plane. While this is certainly true, it is equally true that from a geometric point of view this terminology appears very natural, as the classification is based on whether the Vessiot distribution behaves regularly or singularly at ρ in the sense that its dimension jumps.

Proposition 3.2. *Let the maximally overdetermined, involutive differential equation $\mathcal{R}_1 \subset J_1\pi$ in n independent and m dependent variables be locally described by the system $\Phi^\tau(\mathbf{x}, \mathbf{u}^{(1)}) = 0$. The point $\rho = (\bar{\mathbf{x}}, \bar{\mathbf{u}}^{(1)}) \in \mathcal{R}_1$ is regular, if and only if*

$$\operatorname{rank} \left(\frac{\partial \Phi^\tau}{\partial u_i^\alpha} \right)_\rho = mn \, . \tag{13}$$

The point ρ is regular singular, if and only if it is not regular and

$$\operatorname{rank}\left(\frac{\partial\Phi^\tau}{\partial u_i^\alpha}\;\middle|\;\frac{\partial\Phi^\tau}{\partial x^j}+u_j^\beta\frac{\partial\Phi^\tau}{\partial u^\beta}\right)_\rho = mn\,. \tag{14}$$

Proof. This claim is a simple consequence of the local computation of the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$. The usual ansatz $X = a^i C_i^{(1)} + b_i^\alpha C_\alpha^i$ yields a linear system of $t \geq mn$ equations for the $(m+1)n$ coefficients a^i, b_i^α:

$$C_\alpha^i(\Phi^\tau)b_i^\alpha + C_i^{(1)}(\Phi^\tau)a^i = 0\,, \qquad 1 \leq \tau \leq t\,. \tag{15}$$

Obviously, we obtain an n-dimensional distribution, if and only if (14) is satisfied. $\mathcal{V}[\mathcal{R}_1]$ is in addition transversal to π^1, if and only if no solution exists for which all coefficients a^i vanish, which is the case if already (13) is satisfied. □

Example 3.3. *We consider the maximally overdetermined, involutive differential equation $\mathcal{R}_1$ locally described by the fully nonlinear system*

$$u_x^2 + u^2 + (x^2-1)e^{2y} = 0\,, \tag{16a}$$

$$u_y - u = 0\,. \tag{16b}$$

As usual, we make the ansatz $X = a^1C_1^{(1)} + a^2C_2^{(1)} + b_1C_1^1 + b_2C_1^2$ for a general vector field in the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$. Then we obtain modulo (16) for the coefficients the following linear system:

$$(xe^{2y} + uu_x)a^1 + u_x^2a^2 + u_xb_1 = 0\,, \tag{17a}$$

$$-u_xa^1 - ua^2 + b_2 = 0\,. \tag{17b}$$

Thus $\mathcal{V}[\mathcal{R}_1]$ is generically spanned by the two vector fields

$$X_1 = u_xC_1^{(1)} - (xe^{2y} + uu_x)C_1^1 + u_x^2C_1^2\,, \tag{18a}$$

$$X_2 = C_2^{(1)} - u_xC_1^1 + uC_1^2\,. \tag{18b}$$

Obviously, for $u_x = 0$ the field X_1 becomes vertical. If in addition $x = 0$, then the first equation in (17) vanishes identically and $\mathcal{V}[\mathcal{R}_1]$ becomes three-dimensional. Hence all points $\rho \in \mathcal{R}_1$ with $u_x \neq 0$ are regular, those with $u_x = 0$ and $x \neq 0$ are regular singular and those with $u_x = x = 0$ are irregular singular.

Away from the irregular singularities, the Vessiot distribution is involutive and through every point a unique generalised solution exists as a leaf of the corresponding foliation. For $u_x \neq 0$ this generalised solution projects onto a classical one. At the irregular singularities the leaves of the foliation intersect and thus infinitely many solutions go through these points (which may be considered as a generalised form of the folded foci *discussed in [1] for ordinary differential equations). This can be seen as follows. The singular behaviour concerns only the x-dependency of the solution and thus can be studied by analysing the integral curves of the vector field $\hat{X}_1 = u_x(\partial_x + u_x\partial_u) - (cx + uu_x)\partial_{u_x}$ for a non-negative constant c on the hypersurface $u_x^2 + u^2 + c(x^2-1) = 0$ in the three-dimensional space with coordinates (x, u, u_x). This field has a focus at the points with $u_x = x = 0$ and thus infinitely many integral curves meet there.*

Another way to understand the properties of $\mathcal{R}_1$ consists of noting that the second equation (16b) implies that any solution is of the form $u(x, y) = v(x)e^y$. Entering this ansatz into (16a) yields that $v(x)$ must satisfy the ordinary differential equation $v_x^2 + v^2 + x^2 = 1$ which will be studied in more details in the next section.

Another type of singular behaviour is the classical notion of a *singular integral* which occurs if a submanifold of irregular singularities can be considered as a differential equation of its own. A famous example of an ordinary differential equation possessing a singular integral is the Clairaut equation $u = xu_x + f(u_x)$ with a function f such that $f'' \neq 0$. Izumiya [4] defined a square system of n first-order partial differential equations $\Phi^\tau(\mathbf{x}, u^{(1)}) = 0$ for a single dependent variable u to be of *Clairaut type*, if smooth functions $B_{ij}(\mathbf{x}, u^{(1)})$, $D^t_{i\sigma}(\mathbf{x}, u^{(1)})$ exist such that

$$C_i^{(1)}(\Phi^\tau) = B_{ij}C_1^j(\Phi^\tau) + D^\tau_{i\sigma}\Phi^\sigma \tag{19}$$

and where the coefficients B_{ij} satisfy in addition

$$B_{ij} = B_{ji} \ , \tag{20a}$$

$$C_i^{(1)}(B_{jk}) + B_{\ell i}C_1^\ell(B_{jk}) = C_k^{(1)}(B_{ji}) + B_{\ell k}C_1^\ell(B_{ji}) \ . \tag{20b}$$

Izumiya and Kurokawa [5] provided a classification of such systems.

Proposition 3.4. *Let $\mathcal{R}_1$ be a maximally overdetermined equation of Clairaut type. On the submanifold of all regular points the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ is spanned by the n vector fields*

$$X_i = C_i^{(1)} - B_{ij}C_1^j \ , \quad 1 \leq i \leq n \ , \tag{21}$$

and it is involutive. All other points are irregular singular.

Proof. Making our usual ansatz $X = a^iC_i^{(1)} + b_jC_1^j$ for a field in $\mathcal{V}[\mathcal{R}_1]$ and exploiting the relation (19) defining equations of Clairaut type, we obtain the system

$$(a^iB_{ij} + b_j)C_1^j(\Phi^\tau) = 0 \ , \qquad 1 \leq \tau \leq n \ . \tag{22}$$

At regular points the square matrix $\big(C_1^j(\Phi^\tau)\big)$ is invertible and we obtain as solution (21). If this matrix is singular, the solution space is of higher dimension and thus we are at an irregular singularity. One easily verifies by direct computation that the fields (21) form an involutive distribution, if and only if (20) is satisfied. □

Thus we conclude that the terminology "Clairaut type" is a bit misleading for this class of equations, as the defining conditions only ensure that we are dealing with an involutive differential equation. A truly Clairaut-like behaviour emerges only, if almost all points on $\mathcal{R}_1$ are regular and the irregularity condition $\det\big(C_1^j(\Phi^\tau)\big) = 0$ is compatible with $\mathcal{R}_1$. In this case the differential equation $\hat{\mathcal{R}}_1 \subset \mathcal{R}_1$ obtained by adding the irregularity condition is also involutive. Since $\dim \hat{\mathcal{R}}_1 = n$, this augmented equation has exactly one solution—the singular integral.

Example 3.5. *Consider the following maximally overdetermined equation in two independent variables x, y*

$$\mathcal{R}_1 : \left\{ \ u_x^2 - u = 0 \ , \quad u_y = 0 \ . \right. \tag{23}$$

One easily verifies that it is of Clairaut type with the only non-vanishing coefficients $B_{11} = -1/2$ and $D^1_{22} = -1$. Points with $u_x \neq 0$ are regular and, according to Proposition 3.4, the Vessiot distribution is there spanned by the two vector fields

$$X_1 = C_1^{(1)} + \frac{1}{2}C_1^1 \ , \qquad X_2 = C_2^{(1)} \ . \tag{24}$$

The general solution is here given by $u(x, y) = \frac{1}{4}(x+a)^2$ and defines a one-parameter family of parabolas.

The singular integral is trivially given by $u(x,y) = 0$, as adding the irregularity condition $u_x = 0$ to (23) leads to the augmented equation $\hat{\mathcal{R}}_1$ defined by the system $u = u_x = u_y = 0$. It defines an envelope of the general solution (which is typical for Clairaut equations). Compared with Example 3.3, we find here a very different behaviour at an irregular singular point ρ, although in both cases $\mathcal{V}_\rho\mathcal{R}_1$ becomes three-dimensional. In Example 3.3 the singular points are folded foci where a generalised solution exists for every two-dimensional subspace of $\mathcal{V}_\rho[\mathcal{R}_1]$. Here two distinguished two-dimensional subspaces exists: one is defined by continuing the fields (24) to ρ and corresponds to the unique member of the general solution going through ρ; the other one is $\mathcal{V}_\rho[\hat{\mathcal{R}}_1]$ (generated by the vectors $C_1^{(1)}|_\rho$ and $C_2^{(1)}|_\rho$) and corresponds to the singular integral.

4. **Ordinary Differential Equations.** The Vessiot distribution has been used for the analysis of singularities of ordinary differential equations for a long time (though not the term "Vessiot distribution"). Much of the theory is devoted to scalar equations of first order [1]. We will extend some classical results to arbitrary formally integrable systems. The following generalised existence and uniqueness theorem for ordinary differential equations explains the terminology "impasse point."

Theorem 4.1. *Let $\mathcal{R}_1$ be a formally integrable first-order ordinary differential equation such that everywhere $\dim \mathcal{V}[\mathcal{R}_1] = 1$ (i. e. $\mathcal{R}_1$ contains no irregular singular points). If $\rho \in \mathcal{R}_1$ is a regular point, then there exists a unique classical solution σ with $\rho \in \operatorname{im} j_1\sigma$. This solution can be extended until $\operatorname{im} j_1\sigma$ reaches either the boundary of $\mathcal{R}_1$ or a regular singular point. If $\rho \in \mathcal{R}_1$ is a regular singular point, then either two solutions σ_1, σ_2 with $\rho \in \operatorname{im} j_1\sigma_i$ exist or only one solution whose second derivative blows up.*

Proof. As a one-dimensional distribution, $\mathcal{V}[\mathcal{R}_1]$ is trivially involutive and the Frobenius theorem guarantees for each point $\rho \in \mathcal{R}_1$ the existence of a unique generalised solution $\mathcal{N}_\rho$ with $\rho \in \mathcal{N}_\rho$. This generalised solution is a smooth curve which can be extended until it reaches the boundary of $\mathcal{R}_1$ and around each regular point $\bar{\rho} \in \mathcal{N}_\rho$ it projects onto the graph of a classical solution σ.

Assume that in an open, simply connected neighbourhood of ρ the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ is generated by the vector field X. If ρ is an impassse point, then X_ρ is vertical to π^1, i. e. its ∂_x-component vanishes. The behaviour of the projection $\hat{\mathcal{N}}_\rho = \pi^1_0(\mathcal{N}_\rho)$ depends on whether the ∂_x-component changes its sign at ρ. If the sign changes, then $\hat{\mathcal{N}}_\rho$ has two branches corresponding to two classical solutions which both either end or begin at $\hat{\rho} = \pi^1_0(\rho)$ (see Figure 1 below for an example). Otherwise $\hat{\mathcal{N}}_\rho$ is still the graph of a classical solution, but it follows immediately from a comparison of (7) and (8) that the second derivative of this solution at $\hat{\rho}$ must be infinite. □

Our final result concerns irregular singularities of an involutive equation $\mathcal{R}_1$. If $\mathcal{R}_1$ is not underdetermined, then it follows from Proposition 3.2 that these form a submanifold of codimension at least 2. Thus, if $\rho \in \mathcal{R}_1$ is an irregular singularity, then we can find an open, simply connected submanifold $\mathcal{U} \subset \mathcal{R}_1$ such that $\rho \in \overline{\mathcal{U}}$ and that everywhere in $\mathcal{U}$ the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ is one-dimensional. Then $\mathcal{V}[\mathcal{R}_1]$ can be described within $\mathcal{U}$ as the span of a single vector field X.

Theorem 4.2. *In the situation described above, any smooth extension of the vector field X vanishes at the irregular singularity ρ.*

Proof. Following Remark 2.3, we may assume without loss of generality that $\mathcal{R}_1$ is described in $\mathcal{U}$ by a square system $\Phi^\tau(x, \mathbf{u}, \mathbf{u}') = 0$ with as many equations as unknown. We denote by A the Jacobian $\partial\Phi^\tau/\partial(u')^\alpha$ and by $\mathbf{v}$ the vector with the components $v^\tau = C_1^{(1)}(\Phi^\tau)$. For determining the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$, we make the usual ansatz $X = aC_1^{(1)} + b^\alpha C_\alpha^1$. If we collect the coefficients b^α in the vector $\mathbf{b}$, then the linear system $A\mathbf{b} + a\mathbf{v} = 0$ must be solved.

Let A^* be the adjunct matrix of A, i. e. $AA^* = A^*A = \det(A)\mathbb{1}$. Then at any point within $\mathcal{U}$ all solutions are multiples of $a = \det A$ and $\mathbf{b} = -A^*\mathbf{v}$ so that we may assume without loss of generality that

$$X = \det(A)C_1^{(1)} - (A^*\mathbf{v})^\alpha C_\alpha^1 \,. \tag{25}$$

At any singular point the determinant $\det A$ vanishes. By Proposition 3.2, we must distinguish two cases for the irregular singularity ρ. If $\operatorname{rank} A < n-1$ at ρ, then it follows immediately that $A^* = 0$, too, and thus we find $X_\rho = 0$, as claimed. If $\operatorname{rank} A = n-1$ at ρ, then we must also have $\operatorname{rank}(A \mid \mathbf{v}) = n-1$ there, as otherwise ρ was regular singular. Thus at ρ the vector $\mathbf{v}$ is linearly dependent of the columns of A. Since (by Cramer's rule) the components of $A^*\mathbf{v}$ can be expressed as determinants where one column of A is replaced by $\mathbf{v}$, we see that in this case $A^*\mathbf{v} = 0$ at ρ and thus also $X_\rho = 0$. □

Example 4.3. *We consider the fully nonlinear ordinary differential equation $\mathcal{R}_1$ given by $u_x^2 + u^2 + x^2 = 1$ which obviously is the unit sphere in $J_1\pi$. One easily verifies with the help of Proposition 3.2 that all points off the equator $u_x = 0$ are regular and that on the equator all points are regular singular except for the two 1-singular points $(0, \pm 1, 0)$. Indeed, except for these two irregular singularities $\mathcal{V}[\mathcal{R}_1]$ is globally spanned by the vector field $X = u_x\partial_x + u_x^2\partial_u - (x + uu_x)\partial_{u_x}$. Figure 1 shows the direction defined by $\mathcal{V}[\mathcal{R}_1]$ at some points on $\mathcal{R}_1$. The singular points are marked red; they are zeros (nodes) for the field X, as predicted by Theorem 4.2.*

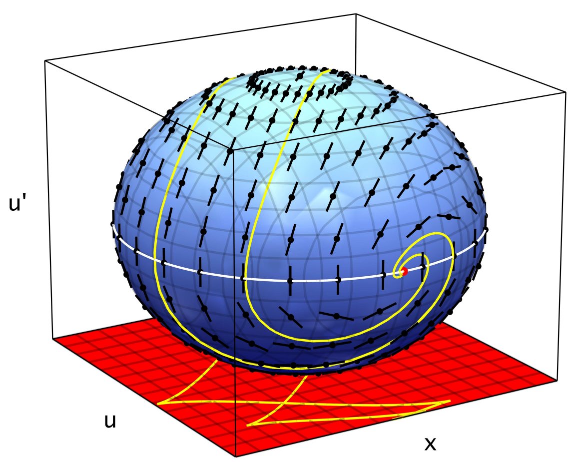

FIGURE 1. Two generalised solutions on $\mathcal{R}_1$

Figure 1 also shows two generalised solutions of $\mathcal{R}_1$. All generalised solutions are smooth curves connecting the two irregular singular points. They cross repeatedly *the equator and if one considers their projections to $\mathcal{E}$ (the red plane in the figure), then one sees that these form then a cusp. The projection defines a classical solution only in the segments between two cusps.*

5. **Conclusions.** We showed that the geometric theory of differential equations provides in form of the Vessiot distribution a very natural tool for singularity analysis. We exploited it for the generalisation of some results for scalar ordinary differential equations to maximally overdetermined systems of partial differential equations or to systems of differential algebraic equations. Since the Vessiot theory yields for any involutive partial differential equation a "covering" by an infinite family of maximally overdetermined systems [2, 3], our results can be seen as a step towards the singularity analysis of arbitrary partial differential equations.

The full power of the geometric approach becomes apparent in the ease with which we could prove Theorem 4.1. Rabier [10] presented a similar result with a considerably longer analytic proof. The length of his proof is essentially due to the fact that—in our geometric language—Rabier works in $\mathcal{E}$ and the simple geometry behind an impasse point becomes visible only in $J_1\pi$.

The Vessiot theory is also useful for the *numerical* integration around impasse points [12]. Away from irregular singularities, it is trivial to determine the Vessiot distribution numerically and then to integrate it with any standard method for ordinary differential equations. In fact, Figure 1 was produced that way.

REFERENCES

[1] V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, 2^{nd} ed., Grundlehren der mathematischen Wissenschaften 250, Springer-Verlag, New York, 1988.

[2] D. Fesser, *On Vessiot's theory of partial differential equations*, Ph.D. thesis, Fachbereich Mathematik, Universität Kassel, 2008.

[3] D. Fesser and W.M. Seiler, *Existence and construction of Vessiot connections*, SIGMA **5** (2009), 092.

[4] S. Izumiya, *Systems of Clairaut type*, Colloq. Math. **66** (1993), 219–226.

[5] S. Izumiya and Y. Kurokawa, *Holonomic systems of Clairaut type*, Diff. Geom. Appl. **5** (1995), 219–235.

[6] V.V. Lychagin, *Geometric theory of singularities of solutions of nonlinear differential equations*, J. Soviet Math. **51** (1990), 2735–2757.

[7] ———, *Homogeneous geometric structures and homogeneous differential equations*, The Interplay between Differential Geometry and Differential Equations (V.V. Lychagin, ed.), Amer. Math. Soc. Transl. 167, Amer. Math. Soc., Providence, 1995, pp. 143–164.

[8] M. Modugno, *Covariant quantum mechanics*, Unpublished manuscript, Dept. of Mathematics, University of Florence, 1999.

[9] J.F. Pommaret, *Systems of partial differential equations and Lie pseudogroups*, Gordon & Breach, London, 1978.

[10] P.J. Rabier, *Implicit differential equations near a singular point*, J. Math. Anal. Appl. **144** (1989), 425–449.

[11] W.M. Seiler, *Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra*, Algorithms and Computation in Mathematics 24, Springer-Verlag, Berlin, 2009.

[12] J. Tuomela, *On singular points of quasilinear differential and differential-algebraic equations*, BIT **37** (1997), 968–977.

[13] E. Vessiot, *Sur une théorie nouvelle des problèmes généraux d'intégration*, Bull. Soc. Math. Fr. **52** (1924), 336–395.

[14] A.M. Vinogradov, *Geometric singularities of solutions of nonlinear partial differential equations*, Differential Geometry and its Applications, Math. Appl. (East European Ser.) 27, Reidel, Dordrecht, 1987, pp. 35–379.

E-mail address: {kant,seiler}@mathematik.uni-kassel.de