AAECC manuscript No.
(will be inserted by the editor)

A Combinatorial Approach to Involution and δ-Regularity II: Structure Analysis of Polynomial Modules with Pommaret Bases

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Received: date / Revised version: date

Abstract Much of the existing literature on involutive bases concentrates on their efficient algorithmic construction. By contrast, we are here more concerned with their structural properties. Pommaret bases are particularly useful in this respect. We show how they may be applied for determining the Krull and the projective dimension, respectively, and the depth of a polynomial module. We use these results for simple proofs of Hironaka's criterion for Cohen-Macaulay modules and of the graded form of the Auslander-Buchsbaum formula, respectively.

Special emphasis is put on the syzygy theory of Pommaret bases and its use for the construction of a free resolution which is generically minimal for componentwise linear modules. In the monomial case, the arising complex always possesses the structure of a differential algebra and it is possible to derive an explicit formula for the differential. Here a minimal resolution is obtained, if and only if a stable module is treated. These observations generalise results by Eliahou and Kervaire.

Using our resolution, we show that the degree of the Pommaret basis with respect to the degree reverse lexicographic term order is always the Castelnuovo-Mumford regularity. This approach leads to new proofs for a number of characterisations of this invariant proposed in the literature. This includes in particular the criteria of Bayer/Stillman and Eisenbud/Goto, respectively. We also relate Pommaret bases to the recent work of Bermejo/Gimenez and Trung on computing the Castelnuovo-Mumford regularity via saturations.

It is well-known that Pommaret bases do not always exist but only in so-called δ-regular coordinates. We show that several classical results in commutative algebra, holding only generically, are true for these special coordinates. In particular, they are related to regular sequences, independent sets of variables, saturations and Noether normalisations. Many properties of the generic initial ideal hold also for the leading ideal of the Pommaret basis with respect to the degree reverse lexicographic term order, although the latter one is in general not Borel-fixed. We present

a deterministic approach for the effective construction of δ-regular coordinates that is more efficient than all methods proposed in the literature so far.

1 Introduction

Rees [59] introduced a combinatorial decomposition of finitely generated polynomial modules and related it for graded modules to the Hilbert series. Later, more general decompositions of $\Bbbk$-algebras were studied by Stanley and several other authors (see e. g. [7, 13, 68, 69]), especially in the context of Cohen-Macaulay complexes but also for other applications like invariant theory or the theory of normal forms of vector fields with nilpotent linear part. Sturmfels and White [74] presented algorithms to compute various combinatorial decompositions.

Apparently all these authors have been unaware that similar decompositions are implicitly contained in the Janet-Riquier theory of differential equations [48]. In fact, they represent the fundamental idea underlying this theory. Involutive bases combine this idea with concepts from Gröbner bases. As we have seen in Part I, one may consider (strong) involutive bases as those Gröbner bases which automatically induce a combinatorial decomposition of the ideal they generate.

The main goal of this second part is to show that Pommaret bases possess a number of special properties not shared by other involutive bases which makes them particularly useful for the kind of structure analysis of polynomial modules typically needed in algebraic geometry. A number of important invariants can be directly read off a Pommaret basis without any further computations. One reason for this phenomenon is that Pommaret bases induce the special type of decomposition introduced by Rees [59] and which now carries his name.

In their classical works on singularities, Hironaka [44–46] and Grauert [32] developed a concept of standard bases for ideals in rings of power series. A closer analysis of their definitions shows that their bases correspond not to arbitrary Gröbner bases but to Pommaret bases. Later, Amasaki [3, 4] followed up these ideas and explicitly introduced Pommaret bases for polynomial ideals under the name Weierstraß bases because of their connection to the Weierstraß Preparation Theorem. In his study of their properties, Amasaki obtained to some extent results which are similar to the ones presented here, however in a different way.

This second part is organised as follows. Section 2 discusses the problem of δ-regularity and thus of the existence of Pommaret bases. It develops an effective method for the construction of δ-regular coordinates for any ideal without destroying too much sparsity. This method is based on a comparison of the Janet and Pommaret multiplicative variables for a given basis. As a first application, we determine the depth of a polynomial ideal $\mathcal{I}$ and a simple maximal $\mathcal{I}$-regular sequence.

The following section studies combinatorial decompositions of general polynomial modules using involutive bases. A trivial application, already noticed by Janet [48] and Stanley [68], is the determination of the Hilbert series and thus of the Krull dimension. For Pommaret bases an alternative characterisation of the dimension can be given which is related to Gröbner's approach via maximal independent sets of variables [35, 50]. Extending our previous results on the depth from submodules to arbitrary polynomial modules, we obtain as a simple corollary Hironaka's criterion for Cohen-Macaulay modules.

Section 4 discusses the relation between δ-regularity and Noether normalisation. It turns out that searching δ-regular coordinates for an ideal $\mathcal{I}$ is equivalent to putting simultaneously $\mathcal{I}$ and all primary components of $\mathrm{lt}_{\prec}\mathcal{I}$ into Noether position. As a by-product we provide a number of equivalent characterisations for monomial ideals possessing a Pommaret basis and show how an irredundant primary decomposition of such ideals can be easily obtained. These results are heavily based on recent work by Bermejo and Gimenez [12].

Section 5 develops the syzygy theory of involutive bases. We show that the involutive standard representations of the non-multiplicative multiples of the generators induce a Gröbner basis (for an appropriately chosen term order) of the first syzygy module. Essentially, this involutive form of Schreyer's theorem follows from the ideas behind Buchberger's second criterion for redundant S-polynomials. For Janet and Pommaret bases the situation is even better, as the arising Gröbner basis is then again a Janet and Pommaret basis, respectively.

In the next three sections we construct by iteration of this result free resolutions of minimal length. We first outline the construction for arbitrary polynomial modules with a Pommaret basis. Then we specialise to monomial modules where one can always endow the resolution with the structure of a differential algebra and provide an explicit formula for the differential. Most of these results are inspired by and generalisations of the work of Eliahou and Kervaire [27]. Finally, we study those modules for which the obtained resolution is even minimal. It turns out that minimality is obtained only for componentwise linear modules. As a by-product, we develop an effective method for deciding whether or not a module is componentwise linear.

In Section 9 we show that the degree of a Pommaret basis with respect to the degree reverse lexicographic order equals the Castelnuovo-Mumford regularity of the ideal. Together with our method for the construction of δ-regular coordinates, this result leads to a simple effective method for the computation of this important invariant. As corollaries we recover characterisations of the Castelnuovo-Mumford regularity previously proposed by Bayer/Stillman [10] and Eisenbud/Goto [25]. In the following section we discuss the relation between regularity and saturation from the point of view of Pommaret bases. Here we make contact with recent works of Trung [78] and Bermejo/Gimenez [12].

Finally, we apply the previously developed syzygy theory to the construction of involutive bases in iterated polynomial algebras of solvable type. A rather technical appendix clarifies the relation between Pommaret bases and the Sturmfels-White approach [74] to the construction of Rees decompositions.

2 Pommaret Bases and δ-Regularity

We saw in Part I (Example 2.12) that not every monoid ideal in $\mathbb{N}_0^n$ possesses a finite Pommaret basis: the Pommaret division is not Noetherian. Obviously, this also implies that there are polynomial ideals $\mathcal{I} \subseteq \mathcal{P} = \Bbbk[x_1, \dots, x_n]$ without a finite Pommaret basis for a given term order. However, we will show that at the level of *polynomial* ideals this problem may be considered as solely a question of choosing "good" variables $\mathbf{x}$. For this purpose, we take in the sequel the following point of view: term orders are defined for exponent vectors, i. e. on the monoid $\mathbb{N}_0^n$; performing a linear change of variables $\tilde{\mathbf{x}} = A\mathbf{x}$ leads to new exponent vectors in each polynomial which are then sorted according to the same term order as before.

Definition 2.1 *The variables* $\mathbf{x}$ *are* δ-regular *for the ideal* $\mathcal{I} \subseteq \mathcal{P}$ *and the term order* $\prec$*, if* $\mathcal{I}$ *possesses a finite Pommaret basis for* $\prec$*.*

Given our definition of an involutive basis, it is obvious that δ-regularity concerns the existence of a Pommaret basis for the monoid ideal $\mathrm{le}_\prec \mathcal{I}$. A coordinate transformation generally yields a new leading ideal which may possess a Pommaret basis. In fact, we will show in this section that for every polynomial ideal $\mathcal{I} \subseteq \mathcal{P}$ variables $\mathbf{x}$ exist such that $\mathcal{I}$ has a finite Pommaret basis provided that the chosen term order $\prec$ is class respecting.[1]

Besides showing the mere existence of δ-regular variables, we will develop in this section an effective approach to recognising δ-singular coordinates and transforming them into δ-regular ones. It is inspired by the work of Gerdt [29] on the relation between Pommaret and Janet bases and the key ideas have already been used in the context of the combined algebraic-geometric completion to involution of linear differential equations [39]. However, the approach presented in [39] contains gaps[2] and we develop here a modified version avoiding the problems of [39]. We begin by proving two useful technical lemmata. The number $\max_{h\in\mathcal{H}} \deg h$ is called the *degree* of the finite set $\mathcal{H} \subset \mathcal{P}$ and denoted by $\deg \mathcal{H}$.

Lemma 2.2 *Let the set* $\mathcal{H}$ *be a homogeneous Pommaret basis of the homogeneous ideal* $\mathcal{I} \subseteq \mathcal{P}$*. Then for any degree* $q \geq \deg \mathcal{H}$ *a Pommaret basis of the truncated ideal* $\mathcal{I}_{\geq q} = \bigoplus_{p\geq q} \mathcal{I}_p$ *is given by*

$$\mathcal{H}_q = \left\{ x^\mu h \mid h \in \mathcal{H},\ |\mu| + \deg h = q,\ \forall j > \operatorname{cls} h : \mu_j = 0 \right\} . \tag{1}$$

Conversely, if $\mathcal{I}_{\geq q}$ *possesses a finite Pommaret basis, then so does* $\mathcal{I}$*.*

Proof According to the conditions in (1), each polynomial $h \in \mathcal{H}$ is multiplied by terms x^μ containing only variables which are multiplicative for it. Thus trivially $\operatorname{cls}(x^\mu h) = \operatorname{cls}\mu$. Furthermore, $\mathcal{H}_q$ is involutively head autoreduced, as $\mathcal{H}$ is. Now let $f \in \mathcal{I}_{\geq q}$ be an arbitrary homogeneous polynomial. As $\mathcal{H}$ is a Pommaret basis of $\mathcal{I}$, it has a standard representation $f = \sum_{h\in\mathcal{H}} P_h h$ with polynomials $P_h \in \Bbbk[x_1,\dots,x_{\operatorname{cls} h}]$. Hence f can be written as a linear combination of polynomials $x^\nu h$ where $|\nu| = \deg f - \deg h \geq q - \deg h$ and where x^ν contains only multiplicative variables. We decompose $\nu = \mu + \rho$ with $|\mu| = q - \deg h$ and $\rho_j = 0$ for all $j > \operatorname{cls}\mu$. Thus $x^\nu h = x^\rho (x^\mu h)$ with $x^\mu h \in \mathcal{H}_q$ and x^ρ contains

[1] Recall from the appendix of Part I that any class respecting term order coincides on terms of the same degree with the reverse lexicographic order.

[2] I am indebted to Daniel Robertz and an anonymous referee for pointing out these gaps.

only variables multiplicative for it. But this trivially implies the existence of a standard representation $f = \sum_{h' \in \mathcal{H}_q} P_{h'} h'$ with $P_{h'} \in \Bbbk[x_1, \dots, x_{\operatorname{cls} h'}]$ and thus $\mathcal{H}_q$ is a Pommaret basis of $\mathcal{I}_{\geq q}$.

The converse is also very simple. Let $\mathcal{H}_q$ be a finite Pommaret basis of the truncated ideal $\mathcal{I}_{\geq q}$ and $\mathcal{H}_p$ head autoreduced $\Bbbk$-linear bases of the components $\mathcal{I}_p$ for $0 \leq p < q$. If we set $\mathcal{H} = \bigcup_{p=0}^{q} \mathcal{H}_p$, then $\operatorname{le}_\prec \mathcal{H}$ is obviously a weak Pommaret basis of the full monoid ideal $\operatorname{le}_\prec \mathcal{I}$ and by Proposition 5.7 of Part I an involutive head autoreduction yields a strong basis. □

Lemma 2.3 *With the same notations as in Lemma 2.2, let $\mathcal{N} = \operatorname{le}_\prec \mathcal{H}_q$. If $\nu \in \mathcal{N}$ with $\operatorname{cls} \nu = k$, then*[3] *$\nu - 1_k + 1_j \in \mathcal{N}$ for all $k < j \leq n$. Conversely, let $\mathcal{N} \subseteq (\mathbb{N}_0^n)_q$ be a set of multi indices of degree q. If for each $\nu \in \mathcal{N}$ with $\operatorname{cls} \nu = k$ and each $k < j \leq n$ the multi index $\nu - 1_k + 1_j$ is also contained in $\mathcal{N}$, then the set $\mathcal{N}$ is involutive for the Pommaret division.*

Proof j is non-multiplicative for ν. As $\mathcal{N}$ is an involutive basis of $\operatorname{le}_\prec \mathcal{I}_{\geq q}$, it must contain a multi index μ with $\mu \,|_P\, \nu + 1_j$. Obviously, $\operatorname{cls}(\nu + 1_j) = k$ and thus $\operatorname{cls} \mu \geq k$. Because of $|\mu| = |\nu|$, the only possibility is $\mu = \nu + 1_j - 1_k$. The converse is trivial, as each non-multiplicative multiple of $\nu \in \mathcal{N}$ is of the form $\nu + 1_j$ with $j > k = \operatorname{cls} \nu$ and hence has $\nu - 1_k + 1_j$ as an involutive divisor. □

As in concrete computations one always represents an ideal $\mathcal{I} \subseteq \mathcal{P}$ by some finite generating set $\mathcal{F} \subset \mathcal{I}$, we also introduce a notion of regularity for such sets. Assume that the given set $\mathcal{F}$ is involutively head autoreduced with respect to an involutive division L and a term order $\prec$. In general, $\mathcal{F}$ is not an involutive basis of $\mathcal{I}$, but its involutive span $\langle \mathcal{F} \rangle_{L,\prec}$ is only a proper subset of $\mathcal{I}$.

We consider now a linear change of coordinates $\tilde{\mathbf{x}} = A\mathbf{x}$ defined by a regular matrix $A \in \Bbbk^{n \times n}$. It transforms each $f \in \mathcal{P}$ into a polynomial $\tilde{f} \in \tilde{\mathcal{P}} = \Bbbk[\tilde{x}_1, \dots, \tilde{x}_n]$ of the same degree. Thus a given set $\mathcal{F} \subset \mathcal{P}$ is transformed into a set $\tilde{\mathcal{F}} \subset \tilde{\mathcal{P}}$ which generally is no longer involutively head autoreduced. Performing an involutive head autoreduction yields a set $\tilde{\mathcal{F}}^\triangle$. Again $\tilde{\mathcal{F}}^\triangle$ will in general not be an involutive basis of the transformed ideal $\tilde{\mathcal{I}} \subseteq \tilde{\mathcal{P}}$.

Since we are dealing with homogeneous polynomials, we can use Hilbert functions to measure the size not only of ideals but also of involutive spans. Recall that the Hilbert function of the ideal $\mathcal{I}$ is given by $h_\mathcal{I}(r) = \dim \mathcal{I}_r$ for all integers $r \geq 0$. For an involutively head autoreduced set $\mathcal{F}$ we define similarly $h_{\mathcal{F},L,\prec}(r) = \dim(\langle \mathcal{F} \rangle_{L,\prec})_r$. Obviously, $h_{\mathcal{F},L,\prec}(r) \leq h_\mathcal{I}(r)$ for all $r \geq 0$ with equality holding only, if $\mathcal{F}$ is an involutive basis. The same inequality is true for the Hilbert function $h_{\tilde{\mathcal{F}}^\triangle,L,\prec}$ defined by the transformed basis $\tilde{\mathcal{F}}^\triangle$.

According to Lemma 5.12 of Part I, the set $\mathcal{F}$ defines a direct sum decomposition of the involutive span $\langle \mathcal{F} \rangle_{L,\prec}$. This observation allows us to provide a simple explicit formula for the Hilbert function

$$h_{\mathcal{F},L,\prec}(r) = \sum_{f \in \mathcal{F}} \binom{r - q_f + k_f - 1}{r - q_f} \tag{2}$$

[3] Recall from Part I that ℓ_i denotes for any number $\ell \in \mathbb{N}$ the multi index where all entries except the ith one vanish and the ith one is given by ℓ.

where $q_f = \deg f$ and k_f denotes the number of multiplicative variables of f (for $r < q_f$ we understand that the binomial coefficient is zero). Indeed, the binomial coefficient in (2) is easily seen to give the number of multiplicative multiples of f of degree r and thus the contribution of f to the involutive span at this degree.

Obviously, a linear change of coordinates does not affect the Hilbert function of an ideal and thus we find $h_{\mathcal{I}} = h_{\tilde{\mathcal{I}}}$. However, this is not true for the Hilbert functions of the involutive spans $\langle\mathcal{F}\rangle_{L,\prec}$ and $\langle\tilde{\mathcal{F}}^{\triangle}\rangle_{L,\prec}$, respectively. We may now measure the effect of the made coordinate transformation by comparing the asymptotic behaviour of $h_{\mathcal{F},L,\prec}$ and $h_{\tilde{\mathcal{F}}^{\triangle},L,\prec}$.

Definition 2.4 *Let the finite set $\mathcal{F} \subset \mathcal{P}$ be involutively head autoreduced with respect to the Pommaret division and a term order $\prec$. The coordinates $\mathbf{x}$ are* asymptotically regular *for $\mathcal{F}$ and $\prec$, if after any linear change of coordinates $\tilde{\mathbf{x}} = A\mathbf{x}$ the inequality $h_{\mathcal{F},P,\prec}(r) \geq h_{\tilde{\mathcal{F}}^{\triangle},P,\prec}(r)$ holds for all sufficiently large values $r \gg 0$.*

Example 2.5 Let us reconsider Example 2.12 of Part I. It corresponds to the set $\mathcal{F} = \{xy\} \subset \Bbbk[x, y]$ with the degree reverse lexicographic order. Independent of how we order the variables, the class of xy is 1. Hence we have $h_{\mathcal{F},P,\prec}(r) = 1$ for all $r > 1$. After the change of coordinates $x = \tilde{x} + \tilde{y}$ and $y = \tilde{y}$, we obtain the set $\tilde{\mathcal{F}} = \{\tilde{y}^2 + \tilde{x}\tilde{y}\} \subset \Bbbk[\tilde{x}, \tilde{y}]$. Its leading term is $\tilde{y}^2$ which is of class 2 implying that $h_{\tilde{\mathcal{F}},P,\prec}(r) = \binom{r-1}{r-2} = r - 1$ for $r > 1$. Thus the original coordinates are not asymptotically regular for $\mathcal{F}$ and we know from Part I that they are also not δ-regular for the ideal $\mathcal{I} = \langle\mathcal{F}\rangle$. $\triangleleft$

Note that, given variables $\mathbf{x}$, generally asymptotic regularity for a finite set $\mathcal{F}$ according to Definition 2.4 and δ-regularity for the ideal $\mathcal{I} = \langle\mathcal{F}\rangle$ according to Definition 2.1 are independent properties. For a concrete instance where the two concepts differ see Example 2.7 below. The main point is that δ-regularity for the ideal $\mathcal{I}$ is concerned with the monoid ideal $\operatorname{le}_{\prec}\mathcal{I}$ whereas asymptotic regularity for the set $\mathcal{F}$ depends on the ideal $\langle\operatorname{le}_{\prec}\mathcal{F}\rangle \subseteq \operatorname{le}_{\prec}\mathcal{I}$. However, in some cases the two notions are related. A simple instance is given by the following result.

Proposition 2.6 *Let the coordinates $\mathbf{x}$ be δ-regular for the ideal $\mathcal{I} \subseteq \mathcal{P}$ and the term order $\prec$. If the set $\mathcal{H}$ is a Pommaret basis of $\mathcal{I}$ for $\prec$, then the coordinates $\mathbf{x}$ are asymptotically regular for $\mathcal{H}$ and $\prec$.*

Proof If the set $\mathcal{H}$ is a Pommaret basis of $\mathcal{I}$, then the two Hilbert functions $h_{\mathcal{I}}$ and $h_{\mathcal{H},P,\prec}$ coincide. As obviously for any generating set $\mathcal{F}$ of the ideal $\mathcal{I}$ in any coordinate system the inequality $h_{\mathcal{F},P,\prec}(r) \leq h_{\mathcal{I}}(r)$ holds for all $r \geq 0$, our coordinates are indeed asymptotically regular. $\square$

δ-Regularity of the used coordinates $\mathbf{x}$ represents a trivial necessary condition for the existence of Pommaret bases for an ideal $\mathcal{I} \subseteq \mathcal{P}$. From an algorithmic point of view, their asymptotic regularity for the current basis $\mathcal{H}$ is equally important for the effective construction of a Pommaret basis by the completion Algorithm 3 of Part I. Even if the used coordinates $\mathbf{x}$ are δ-regular for the ideal $\mathcal{I}$, it may still happen that the algorithm will not terminate, as it tries to construct an infinite Pommaret basis for the monoid ideal $\langle\operatorname{le}_{\prec}\mathcal{H}\rangle$.

Example 2.7 One of the simplest instance where this termination problem occurs is not for an ideal but for a submodule of the free $\mathbb{k}[x,y]$-module with basis $\{\mathbf{e}_1,\mathbf{e}_2\}$. Consider the set $\mathcal{F}=\{y^2\mathbf{e}_1, xy\mathbf{e}_1+\mathbf{e}_2, x\mathbf{e}_2\}$ and any term order for which $xy\mathbf{e}_1 \succ \mathbf{e}_2$. The used coordinates are not asymptotically regular for $\mathcal{F}$, as any transformation of the form $x=\bar{x}+a\bar{y}$ with $a\neq 0$ will increase the Hilbert function. Nevertheless, the used coordinates are δ-regular for the submodule $\langle\mathcal{F}\rangle$: adding the generator $y\mathbf{e}_2$ (the S-"polynomial" of the first two generators) makes $\mathcal{F}$ to a reduced Gröbner basis which is simultaneously a minimal Pommaret basis.

Note that the termination of the involutive completion algorithm depends here on the precise form of the used term order. If we have $xy^k\mathbf{e}_2 \prec xy^2\mathbf{e}_1$ for all exponents $k\in\mathbb{N}$, then the algorithm will not terminate, as in the kth iteration it will add the generator $xy^k\mathbf{e}_2$. Otherwise, it will treat at some stage the non-multiplicative product $y\cdot(xy\mathbf{e}_1+\mathbf{e}_2)$ and thus find the decisive generator $y\mathbf{e}_2$. This is in particular the case for any degree compatible order.

Another simple example is provided by the set $\mathcal{F}=\{z^2-1, yz-1, x\}\subset\mathbb{k}[x,y,z]$ together with the lexicographic term order $\prec_{\text{lex}}$. The involutive completion algorithm will iterate infinitely adding all monomials of the form xy^k with $k\geq 1$. Nevertheless, a finite Pommaret basis of $\langle\mathcal{F}\rangle$ for $\prec_{\text{lex}}$ exists and is given by $\mathcal{H}=\{z-y, y^2-1, xy, x\}$. $\lhd$

Remark 2.8 For Definition 2.4 of asymptotic regularity for a finite set $\mathcal{F}$, the behaviour at lower degrees is irrelevant and it suffices to consider the involutive span of $\mathcal{F}$ for degrees beyond $q=\deg\mathcal{F}$. Thus we can proceed as in Lemma 2.2 and replace $\mathcal{F}$ by the set $\mathcal{F}_q$ defined in analogy to (1), i.e. we consider all multiplicative multiples of degree q of elements of $\mathcal{F}$. If we perform a coordinate transformation and a subsequent involutive head autoreduction, then we obtain a set $\tilde{\mathcal{F}}_q^{\triangle}$ where again all elements are of degree q.

It is now very easy to decide which Hilbert function becomes asymptotically larger. Let $\beta_q^{(k)}$ denote the number of generators in $\mathcal{F}_q$ which are of class k and similarly $\tilde{\beta}_q^{(k)}$ for the set $\tilde{\mathcal{F}}_q^{\triangle}$. In our special case, it follows immediately from (2) that both Hilbert functions are actually polynomials for $r\geq q$. Furthermore, an expansion of the binomial coefficients in (2) shows that if we write the Hilbert function in the form $h_{\mathcal{F},P,\prec}(q+r)=\sum_{k=0}^{n-1}h_i r^i$, then each coefficient h_i is determined by a linear combination of $\beta_q^{(i+1)},\dots,\beta_q^{(n)}$ with positive coefficients. Thus we must simply compare first $\beta_q^{(n)}$ and $\tilde{\beta}_q^{(n)}$, then $\beta_q^{(n-1)}$ and $\tilde{\beta}_q^{(n-1)}$, and so on, until for the first time one coordinate system leads to a larger value; the corresponding Hilbert function is asymptotically larger. $\lhd$

Choosing an arbitrary reference coordinate system $\hat{\mathbf{x}}$, we identify every system of coordinates $\mathbf{x}$ with the unique regular transformation matrix $A\in\mathbb{k}^{n\times n}$ for which $\mathbf{x}=A\hat{\mathbf{x}}$. The next result says that asymptotic regularity for a set $\mathcal{F}$ of polynomials is a generic property in the sense of the Zariski topology, i.e. almost all coordinates are asymptotically regular for $\mathcal{F}$.

Proposition 2.9 *The coordinate systems $\mathbf{x}$ which are asymptotically singular for a finite involutively head autoreduced set $\mathcal{F}\subset\mathcal{P}$ and a term order $\prec$ form a Zariski closed proper subset of $\mathbb{k}^{n\times n}$.*

Proof By the considerations in Remark 2.8, it suffices to consider the case that all elements of $\mathcal{F}$ possess the same degree. Let us now perform a coordinate transformation $\bar{\mathbf{x}} = A\mathbf{x}$ with an undetermined matrix A, i. e. we are treating its entries as parameters. This obviously leads to an asymptotically regular coordinate system, as each polynomial in $\tilde{\mathcal{F}}^{\triangle}$ will get its maximally possible class. Asymptotically singular coordinates are defined by the vanishing of certain (leading) coefficients. These coefficients are polynomials in the entries of A. Thus the set of all asymptotically singular coordinate systems can be described as the zero set of an ideal. □

Our goal is an effective criterion for recognising that coordinates are asymptotically singular for a given set $\mathcal{F}$ and a class respecting term order. The basic idea consists of comparing the multiplicative variables assigned by the Pommaret and the Janet division, respectively. The definitions of these two divisions are apparently quite different. Somewhat surprisingly, they nevertheless yield very similar sets of multiplicative indices, as shown by Gerdt and Blinkov [30, Prop. 3.10].

Proposition 2.10 *Let the finite set $\mathcal{N} \subset \mathbb{N}_0^n$ be involutively autoreduced with respect to the Pommaret division. Then $N_P(\nu) \subseteq N_{J,\mathcal{N}}(\nu)$ for all $\nu \in \mathcal{N}$.*

For later use, we mention the following simple corollaries which further study the relationship between the Janet and the Pommaret division. Recall that any set $\mathcal{N} \subset \mathbb{N}_0^n$ is involutively autoreduced with respect to the Janet division. We first note that by an involutive autoreduction of $\mathcal{N}$ with respect to the *Pommaret* division its *Janet* span can become only larger but not smaller.

Corollary 2.11 *Let $\mathcal{N} \subset \mathbb{N}_0^n$ be an arbitrary finite set of multi indices and set $\mathcal{N}_P = \mathcal{N} \setminus \{\nu \in \mathcal{N} \mid \exists \mu \in \mathcal{N} : \mu \,|_P\, \nu\}$, i. e. we eliminate all multi indices possessing a Pommaret divisor in $\mathcal{N}$. Then $\langle \mathcal{N} \rangle_J \subseteq \langle \mathcal{N}_P \rangle_J$.*

Proof If $\mu^{(1)} \,|_P\, \mu^{(2)}$ and $\mu^{(2)} \,|_P\, \nu$, then trivially $\mu^{(1)} \,|_P\, \nu$. Thus for each eliminated multi index $\nu \in \mathcal{N} \setminus \mathcal{N}_P$ another multi index $\mu \in \mathcal{N}_P$ exists with $\mu \,|_P\, \nu$. Let $\operatorname{cls} \mu = k$. By the proposition above $\{1, \dots, k\} \subseteq N_{J,\mathcal{N}_P}(\mu)$. Assume that an index $j > k$ exists with $j \in N_{J,\mathcal{N}}(\nu)$. By definition of the Pommaret division, $\mu_i = \nu_i$ for all $i > k$. Thus $\mu \in (\nu_{j+1}, \dots, \nu_n)$ and $j \in N_{J,\mathcal{N}}(\mu)$. As by the second condition on an involutive division $N_{J,\mathcal{N}}(\mu) \subseteq N_{J,\mathcal{N}_P}(\mu)$ for all $\mu \in \mathcal{N}_P$, we conclude that $j \in N_{J,\mathcal{N}_P}(\mu)$ and $\mathcal{C}_{J,\mathcal{N}}(\nu) \subset \mathcal{C}_{J,\mathcal{N}_P}(\mu)$. But this immediately implies $\langle \mathcal{N} \rangle_J \subseteq \langle \mathcal{N}_P \rangle_J$. □

This observation implies that any Pommaret basis is simultaneously a Janet basis (a similar result is contained in [29, Thm. 17]). Thus if $\mathcal{H}$ is a Pommaret basis, then $X_{P,\prec}(h) = X_{J,\mathcal{H},\prec}(h)$ for all polynomials $h \in \mathcal{H}$.

Corollary 2.12 *Let the finite set $\mathcal{H} \subset \mathcal{P}$ be involutive with respect to the Pommaret division (and some term order). Then $\mathcal{H}$ is also involutive for the Janet division.*

Proof By the proposition above, it is obvious that the set $\mathcal{H}$ is at least weakly involutive with respect to the Janet division. For the Janet division any weakly involutive set is strongly involutive, if no two elements have the same leading terms. But as $\mathcal{H}$ is a Pommaret basis, this cannot happen. □

We show now that for an asymptotically regular coordinate system and a class respecting term order the inclusions in Proposition 2.10 must be equalities. In other words, if a variable x_ℓ exists which is multiplicative for an element of $\mathcal{F}$ with respect to the Janet division but non-multiplicative with respect to the Pommaret division, then the used coordinates are asymptotically singular for $\mathcal{F}$. Our proof is constructive in the sense that it shows us how to find coordinates leading to a larger Hilbert function.

Theorem 2.13 *Let the finite set $\mathcal{F} \subset \mathcal{P}$ be involutively head autoreduced for the Pommaret division and a class respecting term order $\prec$ and the field $\Bbbk$ be infinite. If the set $\mathcal{F}$ possesses more multiplicative variables for the Janet division than for the Pommaret division, then the coordinates $\mathbf{x}$ are asymptotically singular for it.*

Proof By the proposition above, we have $X_{P,\prec}(f) \subseteq X_{J,\mathcal{F},\prec}(f)$ for all $f \in \mathcal{F}$. Assume that for a polynomial $h \in \mathcal{F}$ the strict inclusion $X_{P,\prec}(h) \subset X_{J,\mathcal{F},\prec}(h)$ holds. Thus a variable $x_\ell \in X_{J,\mathcal{F},\prec}(h)$ with $\ell > k = \operatorname{cls} h$ exists. If we define the set $\mathcal{F}_q$ for $q = \deg \mathcal{F}$ as in Remark 2.8, then $\mathcal{F}_q$ contains in particular the generator $x_k^{q-\deg h} h$ which is still of class k. It is easy to see that the variable x_ℓ is also Janet multiplicative for this generator. Hence we may again assume without loss of generality that all elements of $\mathcal{F}$ are of the same degree q.

We perform now the following linear change of variables: $x_i = \tilde{x}_i$ for $i \neq k$ and $x_k = \tilde{x}_k + a\tilde{x}_\ell$ with a yet arbitrary parameter $a \in \Bbbk \setminus \{0\}$. It induces the following transformation of the terms $x^\mu \in \mathbb{T}$:

$$x^\mu = \sum_{j=0}^{\mu_k} \binom{\mu_k}{j} a^j \tilde{x}^{\mu - j_k + j_\ell} \,. \tag{3}$$

$\tilde{x}^\mu$ is on the right hand side the only term whose coefficient does not depend on the parameter a. All other terms appearing there are greater for a class respecting term order (and their coefficients are different powers of a). Let $\operatorname{le}_\prec h = \mu$. Thus $\mu = [0, \dots, 0, \mu_k, \dots, \mu_n]$ with $\mu_k > 0$. We consider now the multi index $\nu = \mu - (\mu_k)_k + (\mu_k)_\ell$; obviously, $\operatorname{cls} \nu > k$. Applying our transformation to the polynomial h leads to a polynomial $\tilde{h}$ containing the term $\tilde{x}^\nu$. Note that ν cannot be an element of $\operatorname{le}_\prec \mathcal{F}$. Indeed, if it was, it would be an element of the same set $(\mu_{\ell+1}, \dots, \mu_n)$ as μ. But this contradicts our assumption that ℓ is multiplicative for the multi index μ with respect to the Janet division, as by construction $\nu_\ell > \mu_\ell$.

Transforming all polynomials $f \in \mathcal{F}$ yields the set $\tilde{\mathcal{F}}$ on which we perform an involutive head autoreduction in order to obtain the set $\tilde{\mathcal{F}}^\triangle$. Under our assumption on the size of the ground field $\Bbbk$, we can choose the parameter a such that after the transformation each polynomial $\tilde{f} \in \tilde{\mathcal{F}}$ has at least the same class as the corresponding polynomial $f \in \mathcal{F}$, as our term order respects classes. This is a simple consequence of (3): cancellations of terms may occur only, if the parameter a is a zero of some polynomial (possibly one for each member of $\mathcal{F}$) with a degree not higher than $\deg \mathcal{F}$.

We know already that for the polynomial h considered above the transformation leads to a polynomial $\tilde{h}$ of greater class. We consider now all polynomials $f \in \mathcal{F}$ with $\operatorname{cls} f > k$ and h. After the change of variables all the transformed polynomials will thus have a class greater than k. Because of the special form of our transformation, the old leading exponent always remains in the support of each transformed polynomial and if exponents appear which are greater for our term order, then they are always accompanied by a coefficient depending on a. Furthermore, we noted above that ν was not contained in $\mathrm{le}_\prec\mathcal{F}$. As all our generators are of the same degree q, an involutive head autoreduction amounts to a simple Gaussian elimination. For a generic choice of the parameter a, it follows from our considerations above that even after the involutive head autoreduction each generator has at least the same class as in the original set $\mathcal{F}$ (and at least one a higher class).

Taking the remaining members of $\mathcal{F}$ into account may only increase the number of elements in $\tilde{\mathcal{F}}^\triangle$ having a class greater than k. But this implies that at least one of the values $\tilde{\beta}_q^{(k+1)}, \dots, \tilde{\beta}_q^{(n)}$ is larger than the corresponding value for the original set $\mathcal{F}$. By Remark 2.8, the Hilbert function of $\tilde{\mathcal{F}}^\triangle$ is then asymptotically greater than the one of $\mathcal{F}$ and our coordinates are not asymptotically regular. □

Corollary 2.14 *If the coordinates* $\mathbf{x}$ *are asymptotically regular for the finite, Pommaret head autoreduced set* $\mathcal{F} \subset \mathcal{P}$*, then with respect to a class respecting term order* $\prec$ *we have* $X_{P,\prec}(f) = X_{J,\mathcal{F},\prec}(f)$ *for all generators* $f \in \mathcal{F}$.

It is important to note that this corollary provides us only with a necessary but *not* with a sufficient criterion for asymptotic regularity of the coordinates $\mathbf{x}$. In other words, even if the Janet and the Pommaret division yield the same multiplicative variables for a given Pommaret head autoreduced set $\mathcal{F} \subset \mathcal{P}$, this fact does not imply that the used coordinates are asymptotically regular for $\mathcal{F}$.

Example 2.15 Let $\mathcal{F} = \{\underline{z^2} - y^2 - 2x^2,\ \underline{xz} + xy,\ \underline{yz} + y^2 + x^2\}$. The underlined terms are leading for the degree reverse lexicographic order. One easily checks that the Janet and the Pommaret division yield the same multiplicative variables. If we perform the transformation $\tilde{x} = z$, $\tilde{y} = y + z$ and $\tilde{z} = x$, then after an autoreduction we obtain the set $\tilde{\mathcal{F}}^\triangle = \{\underline{\tilde{z}^2} - \tilde{x}\tilde{y},\ \underline{\tilde{y}\tilde{z}},\ \underline{\tilde{y}^2}\}$. Again the Janet and the Pommaret division lead to the same multiplicative variables, but the Hilbert function $h_{\mathcal{F},P,\prec}$ is asymptotically smaller than $h_{\tilde{\mathcal{F}}^\triangle,P,\prec}$, as we find $\beta_2^{(2)} = 1 < 2 = \tilde{\beta}_2^{(2)}$. Thus the coordinates (x, y, z) are not asymptotically regular for $\mathcal{F}$.

The explanation of this phenomenon is very simple. Obviously our criterion depends only on the leading terms of the set $\mathcal{F}$. In other words, it analyses the monomial ideal $\langle \mathrm{lt}_\prec\mathcal{F}\rangle$. Here $\langle \mathrm{lt}_\prec\mathcal{F}\rangle = \langle xz, yz, z^2\rangle$ and one easily verifies that the used generating set is already a Pommaret basis. However, for $\mathcal{I} = \langle\mathcal{F}\rangle$ the leading ideal is $\mathrm{lt}_\prec\mathcal{I} = \langle x^3, xz, yz, z^2\rangle$ (one obtains a Janet basis for $\mathcal{I}$ by adding the polynomial x^3 to $\mathcal{F}$) and obviously it does not possess a finite Pommaret basis, as such a basis would have to contain all monomials x^3y^k with $k \in \mathbb{N}$ (or we exploit our criterion noting that y is a Janet but not a Pommaret multiplicative variable for x^3). Thus we have the opposite situation compared to Example 2.7: there $\mathrm{lt}_\prec\mathcal{I}$ had a finite Pommaret basis but $\langle \mathrm{lt}_\prec\mathcal{F}\rangle$ not; here it is the other way round. We will show later in Proposition 4.8 that whenever the monomial ideal $\langle \mathrm{lt}_\prec\mathcal{F}\rangle$ does not possess a finite Pommaret basis, then $\mathcal{F}$ possesses more Janet than Pommaret multiplicative variables. ◁

Eventually, we return to the problem of the existence of a finite Pommaret basis for every ideal $\mathcal{I} \subseteq \mathcal{P}$. As the Pommaret division is not Noetherian, Theorem 7.4 of Part I cannot be directly applied. However, with a little trick due to Gerdt [29] exploiting our above results on the relationship between the Pommaret and the Janet division we can achieve our goal at least for infinite fields.

Theorem 2.16 *Let $\prec$ be an arbitrary term order and $\Bbbk$ an infinite coefficient field. Then every polynomial ideal $\mathcal{I} \subseteq \mathcal{P}$ possesses a finite Pommaret basis for $\prec$ in suitably chosen variables $\mathbf{x}$.*

Proof As a first step we show that every ideal has a *Pommaret* head autoreduced *Janet* basis. Indeed, let us apply our polynomial completion algorithm (Algorithm 3 of Part I) for the Janet division with one slight modification: in the Lines /1/ and /9/ we perform the involutive head autoreductions with respect to the Pommaret division. It is obvious that if the algorithm terminates, the result is a basis with the wanted properties.

The Janet division is Noetherian (Lemma 2.14 of Part I). Thus without our modification the termination is obvious. With respect to the Janet division every set of multi indices is involutively autoreduced. Hence a Janet head autoreduction only takes care that no two elements of a set have the same leading exponents. But in Line /9/ we add a polynomial that is in involutive normal form so that no involutive head reductions are possible. As the Pommaret head autoreduction may only lead to a larger monoid ideal $\operatorname{le}_\prec \mathcal{H}_i$, the Noetherian argument in the proof of the termination of the algorithm remains valid after our modification.

Once the ascending ideal chain $\langle \operatorname{le}_\prec \mathcal{H}_1 \rangle \subseteq \langle \operatorname{le}_\prec \mathcal{H}_2 \rangle \subseteq \cdots$ has become stationary, the polynomial completion algorithm essentially reduces to the "monomial" one (Algorithm 2 of Part I). According to Corollary 2.11, the Pommaret head autoreductions may only increase the Janet spans $\langle \operatorname{le}_\prec \mathcal{H}_i \rangle_J$. Thus the termination of the monomial completion is not affected by our modification and the algorithm terminates for arbitrary input.

Let us now work with a generic coordinate system; i. e. we perform a coordinate transformation $\bar{\mathbf{x}} = A\mathbf{x}$ with an undetermined matrix A as in the proof of Proposition 2.9. By the considerations above, the modified algorithm will terminate, treating only a finite number of bases $\mathcal{H}_i$. According to Proposition 2.9, the coordinate systems that are asymptotically singular for at least one of them form a Zariski closed set. Thus generic coordinates are asymptotically regular for all sets $\operatorname{lt}_\prec \mathcal{H}_i$ and by Corollary 2.14[4] their Janet and their Pommaret spans coincide. But this observation implies that the result of the modified algorithm is not only a Janet but also a Pommaret basis. □

The argument at the end of this proof immediately implies the following analogue to Proposition 2.9.

Corollary 2.17 *The coordinate systems $\mathbf{x}$ which are δ-singular for a given ideal $\mathcal{I} \subseteq \mathcal{P}$ and a term order $\prec$ form a Zariski closed proper subset of $\Bbbk^{n \times n}$.*

[4] Note that it is not relevant here that the corollary assumes the use of a class respecting term order, since our argument deals only with monomial sets.

Remark 2.18 Combining the completion Algorithm 3 of Part I (or the optimised variants developed by Gerdt and Blinkov [31]) and the criterion for asymptotic singularity provided by Theorem 2.13, we can effectively determine δ-regular coordinates for any ideal $\mathcal{I} \subseteq \mathcal{P}$. Our approach is based on the observation that if our given coordinate system $\mathbf{x}$ is not δ-regular for $\mathcal{I}$, then any attempt to compute a Pommaret basis of $\mathcal{I}$ with the completion algorithm will sooner or later lead to a basis $\mathcal{H}$ of $\mathcal{I}$ for which the coordinates $\mathbf{x}$ are not asymptotically regular. Indeed, by the considerations in the proof of Theorem 2.16, the completion with respect to the Janet division (using Pommaret head autoreductions) terminates. Thus either the result is simultaneously a Pommaret basis of $\mathcal{I}$ (and the given coordinates $\mathbf{x}$ are already δ-regular for $\mathcal{I}$) or at some stage we encounter a basis $\mathcal{H}$ of $\mathcal{I}$ possessing more Janet than Pommaret multiplicative variables implying by Theorem 2.13 that the coordinates $\mathbf{x}$ are not asymptotically regular for $\mathcal{H}$.

There are (at least) two possibilities to exploit this observation in concrete computations. The first one consists of following the completion algorithm with the Pommaret division and checking before each iteration whether there are more Janet than Pommaret multiplicative variables. If this is the case, then we perform coordinate transformations of the form used in the proof of Theorem 2.13 until the Janet and the Pommaret division yield the same multiplicative variables. Then we continue with the completion. Alternatively, we compute a Pommaret head autoreduced Janet basis (which always exists by the considerations above) and check whether it is simultaneously a Pommaret basis. If this is the case, we again conclude that our coordinates $\mathbf{x}$ are δ-regular. Otherwise, we perform coordinate transformations as above and start again.

It is easy to provide for each approach examples where it fares better than the other one. The main disadvantage of the first approach is that it may perform transformations even if the coordinates $\mathbf{x}$ are δ-regular for the given ideal $\mathcal{I}$. Such redundant transformations will always occur, if we encounter a basis $\mathcal{H}$ such that the coordinates $\mathbf{x}$ are not δ-regular for the monoid ideal $\langle \operatorname{le}_\prec \mathcal{H} \rangle$ (this assertion follows from Proposition 4.8 below). As one can see from Example 2.7, sometimes the transformations are indeed necessary for the termination of the completion but sometimes they just make the computations more expensive.

In the second approach this problem does not appear, as we only check at the very end whether we actually have got a Pommaret basis. Thus we consider only $\operatorname{le}_\prec \mathcal{I}$ and not already some subideal contained in it. If the original coordinates $\mathbf{x}$ are δ-regular, then no transformation at all will be performed and we clearly fare better than with the first approach. On the other hand, if the coordinates $\mathbf{x}$ are not δ-regular, then we will not notice this fact before the end of the Janet completion. It will follow from our results in later sections that in such a situation, a Janet basis of $\mathcal{I}$ will typically be larger than a Pommaret basis in δ-regular coordinates; in particular, generally the Janet basis will contain elements of unnecessarily high degree. Thus in such situations the first approach will typically fare better, as it avoids a number of unnecessary normal form computations.

Note that at this point we are not able to prove that either strategy will lead to a Pommaret basis after a *finite* number of coordinate transformations. With the help of the theory developed in later sections, we will be able to provide a proof at least for the most important case of a class respecting order (Remark 9.11). The basic problem is that we do not know a bound for the degree of either a Janet or a Pommaret basis. It is clear that both every completion step and every coordinate transformation asymptotically increase the Hilbert function $h_{\mathcal{H},P,\prec}$ of the current basis $\mathcal{H}$. However, without a bound on the degree of the appearing bases, this information is not sufficient to conclude that either approach terminates in a finite number of steps. $\triangleleft$

Example 2.19 Let us apply the first approach to the Pommaret completion of the set $\mathcal{F} = \{\underline{z^2} - y^2 - 2x^2,\ \underline{xz} + xy,\ \underline{yz} + y^2 + x^2\}$ (with respect to the degree reverse lexicographic order). We have seen in Example 2.15 that the coordinates are not δ-regular for $\mathcal{I}$, although the Janet and the Pommaret span of $\mathcal{F}$ coincide. According to our algorithm we must first analyse the polynomial $y(xz + xy)$. Its involutive normal form with respect to $\mathcal{F}$ is $-\underline{x^3}$. If we determine the multiplicative variables for the enlarged set, they do not change for the old elements. For the new polynomial the Janet division yields $\{x, y\}$. But y is obviously not multiplicative for the Pommaret division. Thus our criterion tells us that the coordinates are not asymptotically regular for the enlarged basis and that the Pommaret completion may not terminate. Indeed, here it is easy to see that no matter how often we multiply the new polynomial by y, it will never become involutively head reducible and no finite Pommaret basis can exist for $\langle\mathcal{F}\rangle$.

In this example, the Janet completion (with or without Pommaret autoreductions) ends with the addition of this single obstruction to involution and we obtain as Janet basis the set

$$\mathcal{F}_J = \{\underline{z^2} - y^2 - 2x^2,\ \underline{xz} + xy,\ \underline{yz} + y^2 + x^2,\ \underline{x^3}\}\ . \tag{4}$$

In Example 2.15 we showed that the linear transformation $\tilde{x} = z$, $\tilde{y} = y + z$ and $\tilde{z} = x$ yields after an autoreduction the set $\tilde{\mathcal{F}}^{\triangle} = \{\underline{\tilde{z}^2} - \tilde{x}\tilde{y},\ \underline{\tilde{y}\tilde{z}},\ \underline{\tilde{y}^2}\}$. One easily checks that it is a Pommaret and thus also a Janet basis. This example clearly demonstrates that the Janet division also “feels” δ-singularity in the sense that in such coordinates it typically leads to larger bases of higher degree. $\triangleleft$

In Theorems 2.13 and 2.16 we assumed that we are working over an infinite field. A closer look at the proofs reveals that we could relax this assumption to “sufficiently large” where the required size of $\mathbb{k}$ is essentially determined by the degree and the size of the considered set $\mathcal{F}$. Thus in the case of a finite field, it may be necessary to enlarge $\mathbb{k}$ in order to guarantee the existence of a Pommaret basis. This problem is similar to the situation when one tries to put a zero-dimensional ideal in normal x_n-position [51, Def. 3.7.21].

In Part I we discussed the extension of the Mora normal form to involutive basis computation. Obviously, the above results remain valid, if we substitute the ordinary normal form by Mora’s version and hence we may also apply it to Pommaret bases with respect to semigroup orders.

Besides being necessary for the mere existence of a finite Pommaret basis, a second application of δ-regular coordinates is the construction of $\mathcal{I}$-regular sequences for a homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$. Recall that for any $\mathcal{P}$-module $\mathcal{M}$ a sequence $(f_1, \dots, f_r)$ of polynomials $f_i \in \mathcal{P}$ is called $\mathcal{M}$-regular, if the polynomials generate a proper ideal, f_1 is a non zero divisor for $\mathcal{M}$ and each f_i is a non zero divisor for $\mathcal{M}/\langle f_1,\dots,f_{i-1}\rangle\mathcal{M}$. The maximal length of an $\mathcal{M}$-regular sequence is the *depth* of the module. While the definition allows for arbitrary polynomials in such sequences, it suffices for computing the depth to consider only linear forms $f_i \in \mathcal{P}_1$. This fact follows, for example, from [24, Cor. 17.7] or [74, Lem. 4.1]. For this reason, the following proof treats only this case.

Proposition 2.20 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal and $\mathcal{H}$ a homogeneous Pommaret basis of it for a class respecting term order. Let $d = \min_{h\in\mathcal{H}} \operatorname{cls} h$. Then the variables $(x_1,\dots,x_d)$ form a maximal $\mathcal{I}$-regular sequence and thus $\operatorname{depth}\mathcal{I} = d$.*

Proof A Pommaret basis $\mathcal{H}$ induces a decomposition of $\mathcal{I}$ of the form

$$\mathcal{I} = \bigoplus_{h\in\mathcal{H}} \mathbb{k}[x_1,\dots,x_{\operatorname{cls} h}]\cdot h\ . \tag{5}$$

If $d = \min_{h\in\mathcal{H}} \operatorname{cls} h$ denotes the minimal class of a generator in $\mathcal{H}$, then (5) trivially implies that the sequence $(x_1,\dots,x_d)$ is $\mathcal{I}$-regular.

Let us try to extend this sequence by a variable x_k with $k > d$. We introduce $\mathcal{H}_d = \{h \in \mathcal{H} \mid \operatorname{cls} h = d\}$ and choose an element $\bar{h} \in \mathcal{H}_d$ of maximal degree. As we use a class respecting order, $\bar{h} \in \langle x_1,\dots,x_d\rangle$ by Lemma A.1 of Part I. By construction, x_k is non-multiplicative for $\bar{h}$ and for each $h \in \mathcal{H}$ a polynomial $P_h \in \mathbb{k}[x_1,\dots,x_{\operatorname{cls} h}]$ exists such that $x_k\bar{h} = \sum_{h\in\mathcal{H}} P_h h$. No polynomial h with $\operatorname{cls} h > d$ lies in $\langle x_1,\dots,x_d\rangle$ (obviously $\operatorname{lt}_\prec h \notin \langle x_1,\dots,x_d\rangle$). As the leading terms cannot cancel in the sum, $P_h \in \langle x_1,\dots,x_d\rangle$ for all $h \in \mathcal{H}\setminus\mathcal{H}_d$. Thus $x_k\bar{h} = \sum_{h\in\mathcal{H}_d} c_h h + g$ with $c_h \in \mathbb{k}$ and $g \in \langle x_1,\dots,x_d\rangle\mathcal{I}$. As $\mathcal{I}$ is a homogeneous ideal and as the degree of $\bar{h}$ is maximal in $\mathcal{H}_d$, all constants c_h must vanish.

It is not possible that $\bar{h} \in \langle x_1,\dots,x_d\rangle\mathcal{I}$, as otherwise $\bar{h}$ would be involutively head reducible by some other element of $\mathcal{H}$. Hence we have shown that any variable x_k with $k > d$ is a zero divisor in $\mathcal{I}/\langle x_1,\dots,x_d\rangle\mathcal{I}$ and the $\mathcal{I}$-regular sequence $(x_1,\dots,x_d)$ cannot be extended by any x_k with $k > d$. Obviously, the same argument applies to any linear combination of such variables x_k.

Finally, assume that the forms $y_1,\dots,y_{d+1} \in \mathcal{P}_1$ define an $\mathcal{I}$-regular sequence of length $d+1$. We extend them to a basis $\{y_1,\dots,y_n\}$ of the vector space $\mathcal{P}_1$ and perform the corresponding coordinate transformation $\mathbf{x} \mapsto \mathbf{y}$. Our basis $\mathcal{H}$ transforms into a set $\mathcal{H}_{\mathbf{y}}$ and after an involutive head autoreduction we obtain a set $\mathcal{H}_{\mathbf{y}}^{\triangle}$. In general, the coordinates $\mathbf{y}$ are not asymptotically regular for the latter. But there exist coordinates $\tilde{\mathbf{y}}$ of the form $\tilde{y}_k = y_k + \sum_{i=1}^{k-1} a_{ki}y_i$ with $a_{ki} \in \mathbb{k}$ such that if we transform $\mathcal{H}$ to them and perform afterwards an involutive head autoreduction, then they are asymptotically regular for the obtained set $\tilde{\mathcal{H}}_{\tilde{\mathbf{y}}}^{\triangle}$.

This fact implies that $\tilde{\mathcal{H}}_{\tilde{\mathbf{y}}}^{\triangle}$ is a Pommaret basis of the ideal $\tilde{\mathcal{I}} = \langle\tilde{\mathcal{H}}_{\tilde{\mathbf{y}}}^{\triangle}\rangle \subseteq \tilde{\mathcal{P}}$ it generates.[5] Thus $\min_{\tilde{h}\in\tilde{\mathcal{H}}_{\tilde{\mathbf{y}}}^{\triangle}} \operatorname{cls}\tilde{h} = d$ and, by the same argument as above, $\tilde{y}_{d+1}$

[5] By Definition 2.4 of asymptotic regularity, the involutive spans of the two sets $\mathcal{H}$ and $\tilde{\mathcal{H}}_{\tilde{\mathbf{y}}}^{\triangle}$ possess asymptotically the same Hilbert function. Since $\mathcal{H}$ is assumed to be a Pommaret basis of $\mathcal{I}$, this function is simultaneously the Hilbert function $h_{\mathcal{I}}$ of $\mathcal{I}$ implying that the involutive span of $\tilde{\mathcal{H}}_{\tilde{\mathbf{y}}}^{\triangle}$ is the full ideal $\tilde{\mathcal{I}}$.

is a zero divisor in $\tilde{\mathcal{I}}/\langle\tilde{y}_1,\dots,\tilde{y}_d\rangle\tilde{\mathcal{I}}$. Because of the special form of the transformation $\mathbf{y}\mapsto\tilde{\mathbf{y}}$, we have—considering everything as forms in $\mathcal{P}_1$—the equality $\langle\tilde{y}_1,\dots,\tilde{y}_d\rangle=\langle y_1,\dots,y_d\rangle$ and y_{d+1} must be a zero divisor in $\mathcal{I}/\langle y_1,\dots,y_d\rangle\mathcal{I}$. But this observation contradicts the assumption that $(y_1,\dots,y_{d+1})$ is an $\mathcal{I}$-regular sequence and thus indeed $\operatorname{depth}\mathcal{I}=d$. □

Remark 2.21 One may wonder to what extent this result really requires the Pommaret division. Given an arbitrary involutive basis $\mathcal{H}$ of $\mathcal{I}$, we may introduce the set $X_{\mathcal{I}}=\bigcap_{h\in\mathcal{H}}X_{L,\mathcal{H},\prec}(h)$; obviously, for a Pommaret basis $X_{\mathcal{I}}=\{x_1,\dots,x_d\}$ with $d=\min_{h\in\mathcal{H}}\operatorname{cls}h$. Again it is trivial to conclude from the induced direct decomposition of $\mathcal{I}$ that any sequence formed by elements of $X_{\mathcal{I}}$ is $\mathcal{I}$-regular. But in general we cannot claim that these are *maximal* $\mathcal{I}$-regular sequences and there does not seem to exist an obvious method to extend them. Thus only a lower bound for the depth is obtained this way.

As a simple example we consider the ideal $\mathcal{I}$ generated by $f_1=z^2-xy$, $f_2=yz-wx$ and $f_3=y^2-wz$. If we set $x_1=w$, $x_2=x$, $x_3=y$ and $x_4=z$, then it is straightforward to check that the set $\mathcal{F}=\{f_1,f_2,f_3\}$ is a Pommaret basis of $\mathcal{I}$ with respect to the degree reverse lexicographic order. By Proposition 2.20, (w,x,y) is a maximal $\mathcal{I}$-regular sequence and $\operatorname{depth}\mathcal{I}=3$.

If we set $x_1=w$, $x_2=z$, $x_3=y$ and $x_4=x$, then no finite Pommaret basis exists; these coordinates are not δ-regular. In order to obtain a Janet basis $\mathcal{F}_J$ of $\mathcal{I}$ (for the degree reverse lexicographic order with respect to the new ordering of the variables), we must enlarge $\mathcal{F}$ by $f_4=z^3-wx^2$ and $f_5=yz^3-wx^2z$. We find now $X_{\mathcal{I}}=\{w,x\}$, as

$$\begin{gathered}X_{J,\mathcal{F}_J,\prec_{\text{degrevlex}}}(f_1)=X_{J,\mathcal{F}_J,\prec_{\text{degrevlex}}}(f_2)=\{w,x\}\;,\\ X_{J,\mathcal{F}_J,\prec_{\text{degrevlex}}}(f_3)=\{w,z,y,x\}\;,\\ X_{J,\mathcal{F}_J,\prec_{\text{degrevlex}}}(f_4)=X_{J,\mathcal{F}_J,\prec_{\text{degrevlex}}}(f_5)=\{w,z,x\}\;.\end{gathered}\tag{6}$$

Thus $X_{\mathcal{I}}$ can be extended to a maximal $\mathcal{I}$-regular sequence by adding y. However, the Janet basis gives no indications, why y should be added. One could also conjecture that the minimal number of multiplicative variables for a generator gives the depth. But clearly this is also not true for the above Janet basis. Thus no obvious way seems to exist to deduce $\operatorname{depth}\mathcal{I}$ from $\mathcal{F}_J$. ◁

3 Combinatorial Decompositions

In the proof of Proposition 2.20 we could already see the power of the direct sum decompositions induced by (strong) involutive bases. In this section we want to study this aspect in more details. All results apply to arbitrary finitely generated polynomial modules. But for notational simplicity, we restrict to graded $\mathbb{k}$-algebras $\mathcal{A}=\mathcal{P}/\mathcal{I}$ with a homogeneous ideal $\mathcal{I}\subseteq\mathcal{P}$. If we speak of a basis of the ideal $\mathcal{I}$, we always assume that it is homogeneous, too.

The main motivation of Buchberger [16] for the introduction of Gröbner bases was to be able to compute effectively in such factor spaces. Indeed given a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I}$, the normal form with respect to $\mathcal{G}$ distinguishes a unique representative in each equivalence class. Our goal in this section is to show that Pommaret bases contain in addition much structural information about the algebra $\mathcal{A}$. More precisely, we want to compute fundamental invariants like the Hilbert polynomial (which immediately yields the Krull dimension and the multiplicity), the depth or the Castelnuovo-Mumford regularity (see Section 9). Our basic tools are combinatorial decompositions of the algebra $\mathcal{A}$ into direct sums of polynomial rings with a restricted number of variables.

Definition 3.1 *A* Stanley decomposition *of the graded* $\mathbb{k}$*-algebra* $\mathcal{A} = \mathcal{P}/\mathcal{I}$ *is an isomorphism of graded* $\mathbb{k}$*-linear spaces*

$$\mathcal{A} \cong \bigoplus_{t\in\mathcal{T}} \mathbb{k}[X_t]\cdot t \tag{7}$$

with a finite set $\mathcal{T} \subset \mathbb{T}$ *and sets* $X_t \subseteq \{x_1,\dots,x_n\}$.

The elements of the set X_t are again called the *multiplicative variables* of the generator t. As a first, trivial application of such decompositions we determine the Hilbert series and the (Krull) dimension.

Proposition 3.2 ([68]) *Let the graded algebra* $\mathcal{A}$ *possess the Stanley decomposition (7). Then its Hilbert series is*

$$\mathcal{H}_{\mathcal{A}}(\lambda) = \sum_{t\in\mathcal{T}} \frac{\lambda^{q_t}}{(1-\lambda)^{k_t}} \tag{8}$$

where $q_t = \deg t$ *and* $k_t = |X_t|$*. Thus the (Krull) dimension of* $\mathcal{A}$ *is given by* $D = \max_{t\in\mathcal{T}} k_t$ *and the multiplicity (or degree) by the number of terms* $t \in \mathcal{T}$ *with* $k_t = D$*.*

Vasconcelos [79, p. 23] calls Stanley decompositions "*an approach that is not greatly useful computationally but it is often nice theoretically*". One reason for his assessment is surely that the classical algorithm for their construction works only for monomial ideals and uses a recursion over the variables $x_1,\dots,x_n$. Thus for larger n it becomes quite inefficient. For a general ideal $\mathcal{I}$ one must first compute a Gröbner basis of $\mathcal{I}$ for some term order $\prec$ and then, exploiting the vector space isomorphism $\mathcal{P}/\mathcal{I} \cong \mathcal{P}/\mathrm{lt}_\prec\mathcal{I}$, one determines a Stanley decomposition. Its existence is guaranteed by the following result.

Proposition 3.3 *Let* $\mathcal{I} \subseteq \mathbb{N}_0^n$ *be a monoid ideal and* $\bar{\mathcal{I}} = \mathbb{N}_0^n \setminus \mathcal{I}$ *its complementary set. There exists a finite set* $\bar{\mathcal{N}} \subset \bar{\mathcal{I}}$ *and for each multi index* $\nu \in \bar{\mathcal{N}}$ *a set of indices* $N_\nu \subseteq \{1,\dots,n\}$ *such that*[6]

$$\bar{\mathcal{I}} = \bigcup_{\nu\in\bar{\mathcal{N}}} \left(\nu + \mathbb{N}^n_{N_\nu}\right) \tag{9}$$

and $(\nu + \mathbb{N}^n_{N_\nu}) \cap (\mu + \mathbb{N}^n_{N_\mu}) = \emptyset$ *for all* $\mu,\nu \in \bar{\mathcal{N}}$*.*

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[6] Recall from Part I the notation $\mathbb{N}^n_N = \{\nu \in \mathbb{N}^n_0 \mid \forall j \notin N : \nu_j = 0\}$.

A proof of this proposition may be found in the textbook [22, pp. 417–418] (there it is not shown that one can always construct a disjoint decomposition, but this extension is trivial). This proof is not completely constructive, as a certain degree q_0 is only defined by a Noetherian argument. But it is not difficult to see that we may take $q_0 = \max_{\nu\in\mathcal{N}} \nu_n$ where the set $\mathcal{N}$ is the minimal basis of the monoid ideal $\mathcal{I}$. Now one can straightforwardly transform the proof into a recursive algorithm for the construction of Stanley decompositions (see [67, Sect. 5.1] for the details). In fact, one obtains then exactly the algorithm proposed by Sturmfels and White [74, Lem. 2.4].

One must stress that the complementary decomposition (9) is not unique and different decompositions may use sets $\bar{\mathcal{N}}$ of different sizes. Given any involutive basis of the monoid ideal $\mathcal{I}$, it is trivial to determine a disjoint decomposition of $\mathcal{I}$ itself (Corollary 5.5 of Part I). However, there does not seem to exist an obvious way to obtain a *complementary decomposition* (9). The situation is different for Janet bases where already Janet himself presented a solution of this problem which can straightforwardly be extended to an algorithm.[7]

Proposition 3.4 ([48, §15]) *Let $\mathcal{N}_J$ be a Janet basis of the monoid ideal $\mathcal{I} \subseteq \mathbb{N}_0^n$. Then the set $\bar{\mathcal{N}} \subset \mathbb{N}_0^n$ in the decomposition (9) may be chosen such that for all $\nu \in \bar{\mathcal{N}}$ the equality $N_\nu = N_{J,\mathcal{N}_J\cup\{\nu\}}(\nu)$ holds.*

Remark 3.5 Janet did not formulate his algorithm in this algebraic language. He considered the problem of determining a formally well-posed initial value problem for an overdetermined system of partial differential equations [67, Sect. 9.3]. If one identifies this system with our ideal $\mathcal{I}$, his problem is equivalent to computing a Stanley decomposition of $\mathcal{P}/\mathcal{I}$. An algorithmic approach to formally well-posed initial value problem was also presented by Reid [60]. $\triangleleft$

According to Corollary 2.12, we may apply Janet's algorithm to Pommaret bases, too. But as the Pommaret division has such a simple global definition, it is almost trivial to provide an alternative decomposition depending only on the degree q of a Pommaret basis of the ideal $\mathcal{I}$ (we will see later in Sect. 9 that this degree is in fact an important invariant of $\mathcal{I}$). In general, this decomposition is larger than the one obtained with Janet's algorithm, but it has some advantages in theoretical applications.

Proposition 3.6 *The monoid ideal $\mathcal{I} \subseteq \mathbb{N}_0^n$ has a Pommaret basis of degree q, if and only if the sets $\bar{\mathcal{N}}_0 = \{\nu \in \bar{\mathcal{I}} \mid |\nu| < q\}$ and $\bar{\mathcal{N}}_1 = \{\nu \in \bar{\mathcal{I}} \mid |\nu| = q\}$ yield the disjoint decomposition*

$$\bar{\mathcal{I}} = \bar{\mathcal{N}}_0 \cup \bigcup_{\nu\in\bar{\mathcal{N}}_1} \mathcal{C}_P(\nu)\ . \tag{10}$$

Proof The definition of the Pommaret division implies the identity

$$(\mathbb{N}_0^n)_{\geq q} = \bigcup_{|\nu|=q} \mathcal{C}_P(\nu) \tag{11}$$

[7] For an alternative proof see [58].

from which one direction of the proposition follows trivially. Here $(\mathbb{N}_0^n)_{\geq q}$ denotes the set of all multi indices of length greater than or equal to q. By the definition of an involutive division, the union on the right hand side is disjoint.

For the converse, we claim that the set $\mathcal{H} = \{\mu \in \mathcal{I}_q\}$ is a Pommaret basis of monoid ideal $\mathcal{I}_{\geq q}$; this immediately implies our assertion by Lemma 2.2. Assume that $\mu \in \mathcal{H}$ with $\operatorname{cls}\mu = k$ and let $k < j \leq n$ be a non-multiplicative index for it. We must show that $\mu + 1_j \in \langle\mathcal{H}\rangle_P$. But this is trivial: we have $\mu + 1_j \in \mathcal{C}_P(\mu - 1_k + 1_j)$ and $\mu - 1_k + 1_j \in \mathcal{I}_q$, as otherwise we encounter the contradiction $\mu + 1_j \notin \mathcal{I}$ by (10). □

Example 3.7 The decomposition (10) is usually redundant. Considering for $\bar{\mathcal{N}}_1$ only multi indices of length q makes the formulation much easier but it is not optimal. Consider the trivial example $\mathcal{N}_P = \{[0,1]\}$. According to Proposition 3.6 we should set $\bar{\mathcal{N}}_0 = \{[0,0]\}$ and $\bar{\mathcal{N}}_1 = \{[1,0]\}$. But obviously $\bar{\mathcal{I}} = [0,0] + \mathbb{N}^2_{\{1\}}$. Janet's algorithm directly yields this more compact form. ◁

If $\mathcal{J} \subseteq \mathcal{P}$ is a polynomial ideal possessing a Pommaret basis for some term order $\prec$, then applying Proposition 3.6 to $\mathcal{I} = \operatorname{le}_\prec \mathcal{J}$ yields a Stanley decomposition of a special type: all sets X_t are of the form $X_t = \{x_1, x_2, \dots, x_{\operatorname{cls} t}\}$ where the number $\operatorname{cls} t$ is called the *class*[8] of the generator t. One speaks then of a *Rees decomposition* of $\mathcal{A} = \mathcal{P}/\mathcal{J}$ [59]. It is no coincidence that we use here the same terminology as in the definition of the Pommaret division: if $t = x^\mu$ with $\mu \in \bar{\mathcal{N}}_1$, then indeed its class is $\operatorname{cls}\mu$. Elimination of the redundancy in the decomposition (10) leads to the following result.

Corollary 3.8 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a polynomial ideal which has for some term order $\prec$ a Pommaret basis $\mathcal{H}$ such that $\min_{h\in\mathcal{H}} \operatorname{cls} \operatorname{le}_\prec h = d$. Then $\mathcal{P}/\mathcal{I}$ possesses a Rees decomposition where the minimal class of a generator is $d - 1$.*

Proof Obviously, it suffices to consider the monomial case and formulate the proof therefore in the multi index language of Proposition 3.6. Furthermore, for $d = 1$ there is nothing to be shown so that we assume from now on $d > 1$. Our starting point is the decomposition (10). For each $\nu \in \bar{\mathcal{N}}_1$ with $\operatorname{cls}\nu = k < d$ we introduce the multi index $\tilde{\nu} = \nu - (\nu_k)_k$, i. e. $\tilde{\nu}$ arises from ν by setting the kth entry to zero. Obviously, the k-dimensional cone $C_\nu = \tilde{\nu} + \mathbb{N}^n_{\{1,\dots,k\}}$ is still completely contained in the complement $\bar{\mathcal{I}}$ and $\mathcal{C}_P(\nu) \subset C_\nu$.

If we replace in (10) for any such ν the cone $\mathcal{C}_P(\nu)$ by C_ν, then we still have a decomposition of $\bar{\mathcal{I}}$, but no longer a disjoint one. We now show first that in the thus obtained decomposition all cones C with $0 < \dim C < d - 1$ can be dropped without loss. Indeed, for $k < d - 1$ we consider the multi index $\mu = \tilde{\nu} + (\nu_k)_{k+1}$. Obviously, $|\mu| = q$ and $\operatorname{cls}\mu = k + 1$; hence under the made assumptions $\mu \in \bar{\mathcal{N}}_1$. Furthermore, $\tilde{\mu} = \mu - (\mu_{k+1})_{k+1}$ is a divisor of $\tilde{\nu}$ (the two multi indices can differ at most in their $(k+1)$st entries and $\tilde{\mu}_{k+1} = 0$) and thus the inclusion $C_\nu \subset C_\mu = \tilde{\mu} + \mathbb{N}^n_{\{1,\dots,k+1\}}$ holds.

The remaining cones with $\dim C \geq d - 1$ are all disjoint. This is trivially true for all cones with $\dim C \geq d$, as these have not been changed. For the other

[8] Some authors prefer the term *level*.

ones, we note that if μ and ν are two multi indices with $\operatorname{cls}\mu = \operatorname{cls}\nu = d-1$ and $|\mu| = |\nu| = q$, then they must differ at some position ℓ with $\ell \geq d$. But this implies that the cones C_μ and C_ν are disjoint.

Thus there only remains to study the zero-dimensional cones consisting of the multi indices $\nu \in \bar{\mathcal{N}}_0$. If we set $\ell = q - |\nu|$, then $\mu = \nu + \ell_1 \in \bar{\mathcal{N}}_1$, since we assumed $d > 1$, and trivially $\nu \in C_\mu = (\mu - (\mu_1)_1) + \mathbb{N}^n_{\{1\}}$. By our considerations above the cone C_μ and thus ν is contained in some $(d-1)$-dimensional cone. Therefore we may also drop all zero-dimensional cones and obtain a Rees decomposition where all cones are at least $(d-1)$-dimensional. $\square$

Slightly generalising the notion of Rees decompositions, we speak of a *quasi-Rees decomposition*, if there exists a term $\bar{t} \in \mathcal{T}$ such that $\bigcup_{t\in\mathcal{T}} X_t = X_{\bar{t}}$, i. e. there exists a unique maximal set of multiplicative variables containing all other sets of multiplicative variables. Obviously, every Rees decomposition is a quasi-Rees decomposition, but not vice versa. We will see below that such decompositions possess special properties.

Sturmfels et al. [73] introduced the notion of *standard pairs* also leading to a kind of combinatorial decomposition, however not a disjoint one. They consider pairs (ν, N_ν) where $\nu \in \mathbb{N}^n_0$ is a multi index and $N_\nu \subseteq \{1,\dots,n\}$ a set of associated indices. Such a pair is called *admissible*, if $\operatorname{supp}\nu \cap N_\nu = \emptyset$, i. e. $\nu_i = 0$ for all $i \in N_\nu$. On the set of admissible pairs one defines a partial order: $(\nu, N_\nu) \leq (\mu, N_\mu)$, if and only if the restricted cone $\mu + \mathbb{N}^n_{N_\mu}$ is completely contained in $\nu + \mathbb{N}^n_{N_\nu}$. Obviously, this is equivalent to $\nu \mid \mu$ and any index i such that either $\mu_i > \nu_i$ or $i \in N_\mu$ is contained in N_ν.

Definition 3.9 *Let $\mathcal{I} \subseteq \mathbb{N}^n_0$ be an arbitrary monoid ideal. An admissible pair (ν, N_ν) is called* standard *for $\mathcal{I}$, if $\nu + \mathbb{N}^n_{N_\nu} \cap \mathcal{I} = \emptyset$ and (ν, N_ν) is minimal with respect to $<$ among all admissible pairs with this property.*

Any monoid ideal $\mathcal{I} \subset \mathbb{N}^n_0$ leads thus automatically to a uniquely determined set of standard pairs. These define both a decomposition of the complementary set $\overline{\mathcal{I}}$ into cones (though these will overlap in general) and a decomposition of the ideal $\mathcal{I}$ itself as an intersection of irreducible monomial ideals. The following result is contained in the proof of [73, Lemma 3.3].

Proposition 3.10 *Let $\mathcal{I} \subseteq \mathbb{N}^n_0$ be an arbitrary monoid ideal and denote the set of all associated standard pairs by $\mathcal{S}_\mathcal{I} = \bigl\{(\nu, N_\nu) \mid (\nu, N_\nu) \text{ standard for } \mathcal{I}\bigr\}$. Then the complementary set $\overline{\mathcal{I}}$ of $\mathcal{I}$ can be written in the form*

$$\overline{\mathcal{I}} = \bigcup_{(\nu,N_\nu)\in\mathcal{S}_\mathcal{I}} \nu + \mathbb{N}^n_{N_\nu} \tag{12}$$

and the ideal $\mathcal{I}$ itself can be decomposed as

$$\mathcal{I} = \bigcap_{(\nu,N_\nu)\in\mathcal{S}_\mathcal{I}} \langle (\nu_i + 1)_i \mid i \notin N_\nu \rangle\ . \tag{13}$$

According to Sturmfels et al. [73, Lemma 3.3] the number of standard pairs of a monomial ideal $\mathcal{I}$ equals the *arithmetic degree* of $\mathcal{I}$, a refinement of the classical concept of the degree of an ideal introduced by Bayer and Mumford [9]. We further note that the ideals on the right hand side of (13) are trivially irreducible, so that (13) indeed represents an irreducible decomposition of $\mathcal{I}$.

In general, this decomposition is highly redundant. Let $N \subseteq \{1,\dots,n\}$ be an arbitrary subset and consider all standard pairs (ν, N_ν) with $N_\nu = N$. Obviously, among these only the ones with multi indices ν which are maximal with respect to divisibility are relevant for the decomposition (13) and in fact restricting to the corresponding ideals yields the irredundant irreducible decomposition of $\mathcal{I}$ (which is unique according to [57, Thm. 5.27]). Their intersection defines a possible choice for the primary component for the prime ideal $\mathfrak{p}_N = \langle x_i \mid i \notin N\rangle$, so that we can also extract an irredundant primary decomposition from the standard pairs. As a trivial corollary of these considerations the standard pairs immediately yield the set $\operatorname{Ass}(\mathcal{P}/\mathcal{I})$ of associated prime ideals, as it consists of all prime ideals $\mathfrak{p}_N$ such that a standard pair (ν, N) exists.

Hoşten and Smith [47] discuss two algorithms for the direct construction of the set $\mathcal{S}_\mathcal{I}$ of all standard pairs given the minimal basis of $\mathcal{I}$. Alternatively, $\mathcal{S}_\mathcal{I}$ can easily be extracted from any complementary decomposition, as we show now. Thus once a Janet basis of $\mathcal{I}$ is known, we may use Janet's algorithm for the construction of a complementary decomposition and then obtain the standard pairs.

Let the finite set $\mathcal{T}_\mathcal{I} = \bigl\{(\nu, N_\nu) \mid \nu \in \mathbb{N}_0^n, N_\nu \subseteq \{1,\dots,n\}\bigr\}$ define a complementary decomposition of $\mathcal{I}$. If the pair $(\nu, N_\nu) \in \mathcal{T}_\mathcal{I}$ is not admissible, then we substitute it by the pair $(\bar\nu, N_\nu)$ where $\bar\nu_i = 0$ for all $i \in N_\nu$ and $\bar\nu_i = \nu_i$ else. Obviously, this operation produces an admissible pair and the thus obtained set $\overline{\mathcal{S}}_\mathcal{I}$ still defines a (generally no longer disjoint) decomposition of the complementary set $\overline{\mathcal{I}}$. Finally, we eliminate all pairs in $\overline{\mathcal{S}}_\mathcal{I}$ which are not minimal with respect to the partial order $\leq$ and obtain a set $\mathcal{S}_\mathcal{I}$.

Proposition 3.11 *Let $\mathcal{T}_\mathcal{I}$ be a finite complementary decomposition of the monoid ideal $\mathcal{I} \subseteq \mathbb{N}_0^n$. The thus constructed set $\mathcal{S}_\mathcal{I}$ consists of all standard pairs of $\mathcal{I}$.*

Proof It is trivial to see that the set $\overline{\mathcal{S}}_\mathcal{I}$ contains only admissible pairs and that $\nu + \mathbb{N}^n_{N_\nu} \subseteq \overline{\mathcal{I}}$ for any pair $(\nu, N_\nu) \in \overline{\mathcal{S}}_\mathcal{I}$. Thus there only remains to show that all standard pairs are contained in $\overline{\mathcal{S}}_\mathcal{I}$.

Let (μ, N_μ) be an admissible pair such that $\mu + \mathbb{N}^n_{N_\mu} \subseteq \overline{\mathcal{I}}$. Since the union of the cones $\nu + \mathbb{N}^n_{N_\nu}$ with $(\nu, N_\nu) \in \overline{\mathcal{S}}_\mathcal{I}$ still covers $\overline{\mathcal{I}}$, the finiteness of $\overline{\mathcal{S}}_\mathcal{I}$ implies the existence of a multi index $\overline{\mu} \in \mu + \mathbb{N}^n_{N_\mu}$ and a pair $(\nu, N_\nu) \in \overline{\mathcal{S}}_\mathcal{I}$ such that $\overline{\mu} + \mathbb{N}^n_{N_\mu} \subseteq \nu + \mathbb{N}^n_{N_\nu}$ (obviously, it is not possible to cover $\mu + \mathbb{N}^n_{N_\mu}$ with a finite number of lower-dimensional cones). As both (μ, N_μ) and (ν, N_ν) are admissible pairs, this entails that in fact $(\nu, N_\nu) \leq (\mu, N_\mu)$. Hence either $(\mu, N_\mu) \in \overline{\mathcal{S}}_\mathcal{I}$ or it is not a standard pair. $\square$

Remark 3.12 If we use the decomposition (10) derived from a Pommaret basis of degree q, then the determination of the set $\overline{\mathcal{S}}_\mathcal{I}$ is completely trivial. For all pairs $(\nu, N_\nu) \in \mathcal{T}_\mathcal{I}$ with $|\nu| < q$ we have $N_\nu = \emptyset$ and hence they are trivially admissible.

For all other pairs we find that $\operatorname{supp}\nu \cap N_\nu = \{\operatorname{cls}\nu\}$. Thus none of them is admissible, but they become admissible by simply setting the first non-vanishing entry of ν to zero. $\lhd$

Example 3.13 Consider the ideal $\mathcal{I} = \langle z^3, yz^2 - xz^2, y^2 - xy\rangle \subset \Bbbk[x,y,z]$. Both a Janet and Pommaret basis of $\mathcal{I}$ for the degree reverse lexicographic order is given by the set $\mathcal{H} = \{z^3, yz^2 - xz^2, y^2z - xyz, y^2 - xy\}$. Using Janet's algorithm, we obtain the set $\mathcal{T} = \{1, y, z, yz, z^2\}$ and the complementary decomposition

$$\mathcal{P}/\mathcal{I} \cong \Bbbk[x] \oplus \Bbbk[x]\cdot y \oplus \Bbbk[x]\cdot z \oplus \Bbbk[x]\cdot yz \oplus \Bbbk[x]\cdot z^2\ . \tag{14}$$

It follows from (14) that a complementary decomposition of the corresponding monoid ideal $\operatorname{le}_\prec\mathcal{I} = \langle [0,0,3], [0,1,2], [0,2,0]\rangle$ is given by

$$\begin{aligned} \mathcal{S}_{\mathcal{I}} = \Bigl\{&\bigl([0,0,0],\{1\}\bigr), \bigl([0,1,0],\{1\}\bigr), \bigl([0,0,1],\{1\}\bigr),\\ &\bigl([0,1,1],\{1\}\bigr), \bigl([0,0,2],\{1\}\bigr)\Bigr\} \end{aligned} \tag{15}$$

and one easily verifies that these are all standard pairs.

The complementary decomposition constructed via Proposition 3.6 is much larger. Besides many multi indices without any multiplicative indices, we obtain the following six multi indices for which 1 is the sole multiplicative index: $[3,0,0]$, $[2,1,0]$, $[2,0,1]$, $[1,1,1]$, $[1,2,0]$ and $[1,0,2]$. After setting the first entry to zero, we find precisely the multi indices appearing in (15) plus the multi index $[0,2,0]$. As $([0,1,0],\{1\}) < ([0,2,0],\{1\})$, the latter pair is not minimal. The same holds for all pairs corresponding to the multi indices without multiplicative indices and hence we also arrive at (15). $\lhd$

As an application of Rees decompositions, we will now show that given a Pommaret basis of the ideal $\mathcal{I}$, we can easily read off the dimension and the depth of the algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$. In principle, the determination of the dimension is of course already settled by Proposition 3.2 and the possibility to compute Stanley decompositions via Janet bases. However, in the case of a Pommaret basis a further useful characterisation of $\dim\mathcal{A}$ exists. It may be considered as a strengthening of the following general observation for quasi-Rees decompositions.

Lemma 3.14 *Let $\mathcal{G}$ be a Gröbner basis of the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ for some term order $\prec$ and assume that the finite set $\mathcal{T} \subset \mathbb{T}$ defines a quasi-Rees decomposition of $\mathcal{A} = \mathcal{P}/\mathcal{I}$ with the maximal set $X_{\bar t}$ of multiplicative variables for some term $\bar t \in \mathcal{T}$. If $q = 1 + \max_{t\in\mathcal{T}} \deg t$, then $\langle \mathcal{G}, X_{\bar t}\rangle_q = \mathcal{P}_q$ and no smaller set of variables or other set of variables of the same size has this property.*

Proof Assume first that the term $x^\mu \in \mathcal{P}_q \setminus \operatorname{lt}_\prec\mathcal{I}$ is not contained in the leading ideal. By definition of the degree q, we have $|\mu| > \deg t$ for all $t \in \mathcal{T}$. Hence x^μ must be properly contained in some cone of the quasi-Rees decomposition and can be written as a product mt with some $t \in \mathcal{T}$ and a term m in the variables $X_t \subseteq X_{\bar t}$ with $\deg m > 0$. This presentation implies $x^\mu \in \langle X_{\bar t}\rangle$.

If the term $x^\mu \in \mathcal{P}_q$ lies in $\operatorname{lt}_\prec\mathcal{I}$, we compute its normal form with respect to the Gröbner basis $\mathcal{G}$. If this normal form vanishes, then $x^\mu \in \langle\mathcal{G}\rangle$. Otherwise, it is a $\Bbbk$-linear combination of terms in $\mathcal{P}_q \setminus \mathrm{lt}_\prec\mathcal{I}$ and thus lies by the considerations above in $\langle X_{\bar{t}}\rangle$. Hence we may conclude that all terms of degree q lie in $\langle \mathcal{G}, X_{\bar{t}}\rangle$.

No set $\bar{X} \subset X$ of variables with $\bar{X} \neq X_t$ and $|\bar{X}| \leq |X_{\bar{t}}|$ can possess this property, as $\mathrm{lt}_\prec\mathcal{I} \cap \Bbbk[X_{\bar{t}}] = \{0\}$ and hence we always find a term $x^\mu \in (\Bbbk[X_{\bar{t}}])_q$ not contained in $\langle \mathcal{G}, \bar{X}\rangle$. Indeed, assume that a $x^\nu \in \mathrm{lt}_\prec\mathcal{I} \cap \Bbbk[X_{\bar{t}}]$ existed. Then obviously $\bar{t} \cdot x^\nu \in \mathrm{lt}_\prec\mathcal{I}$, contradicting the fact that $\mathcal{T}$ defines a complementary decomposition with multiplicative variables $X_{\bar{t}}$ for $\bar{t}$. □

Proposition 3.15 *Let $\mathcal{H}$ be a homogeneous Pommaret basis of the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ with $\deg\mathcal{H} = q$ for some term order $\prec$. Then the dimension D of the algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is*

$$D = \min\left\{ i \mid \langle \mathcal{H}, x_1, \dots, x_i\rangle_q = \mathcal{P}_q \right\}. \tag{16}$$

Proof The Hilbert polynomials of $\mathcal{A}$ and the truncation $\mathcal{A}_{\geq q}$ coincide. Thus it suffice to consider the latter algebra. By Lemma 2.2, a Pommaret basis of $\mathcal{I}_{\geq q}$ is given by the set $\mathcal{H}_q$ determined in (1). If D is the smallest number such that $\langle \mathcal{H}_q, x_1, \dots, x_D\rangle_q = \mathcal{P}_q$, then all multi indices ν with $|\nu| = q$ and $\operatorname{cls}\nu > D$ lie in $\mathrm{le}_\prec\mathcal{H}_q$ but a multi index μ exists such that $|\mu| = q$, $\operatorname{cls}\mu = D$ and $\mu \notin \mathrm{le}_\prec\mathcal{H}_q$. By Proposition 3.6, this observation entails that μ is a generator of class D of the complementary decomposition (10) and that the decomposition does not contain a generator of higher class. But this trivially implies that $\dim\mathcal{A} = D$. □

In a terminology apparently introduced by Gröbner [35, Section 131], a subset $X_\mathcal{I} \subseteq \{x_1, \dots, x_n\}$ is *independent modulo* the ideal $\mathcal{I}$, if $\mathcal{I} \cap \Bbbk[X_\mathcal{I}] = \{0\}$. If even $\mathrm{lt}_\prec\mathcal{I} \cap \Bbbk[X_\mathcal{I}] = \{0\}$ for some term order $\prec$, then one speaks of a *strongly* independent set for $\prec$. One can show that the maximal size of either an independent or a strongly independent set coincides with $\dim\mathcal{A}$. This approach to determining the dimension of an ideal has been taken up by Kredel and Weispfenning [50] using Gröbner bases (see also [11, Sects 6.3 & 9.3]).

Strong independence modulo $\mathcal{I}$ with respect to a term order $\prec$ is easy to verify effectively with the help of a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ for $\prec$: it follows immediately from the definition of a Gröbner basis that the set $X_\mathcal{I}$ is strongly independent, if and only if it satisfies $\mathrm{lt}_\prec\mathcal{G} \cap \Bbbk[X_\mathcal{I}] = \emptyset$. It is now a combinatorial (and thus sometimes quite expensive) exercise to determine effectively all maximal strongly independent sets modulo $\mathcal{I}$ and to compute their maximal size and hence $\dim\mathcal{A}$. The situation becomes much simpler, if $\mathcal{A} = \mathcal{P}/\mathcal{I}$ admits a quasi-Rees decomposition, as we will show now that in this case a unique maximal strongly independent set modulo $\mathcal{I}$ exists. This observation is based on the following result which is a variant of [62, Lemma 14].[9]

Lemma 3.16 *Let $\mathcal{I} \subset \mathcal{P}$ be an ideal and $\prec$ a term order. Assume that the finite set $\mathcal{T} \subset \mathbb{T}$ defines a quasi-Rees decomposition of the algebra $\mathcal{A}' = \mathcal{P}/\mathrm{lt}_\prec\mathcal{I}$ with the maximal set $X_{\bar{t}}$ of multiplicative variables for a term $\bar{t} \in \mathcal{T}$. Then the variable x_i*

[9] In this paper it is also shown how quasi-Rees decompositions can be effectively computed using Janet bases and coordinate transformations similar to the ones used by us for the construction of δ-regular coordinates.

is not contained in $X_{\bar{t}}$, if and only if the minimal basis of $\mathrm{lt}_{\prec}\mathcal{I}$ contains an element of the form $x_i^{e_i}$ for some exponent $e_i \in \mathbb{N}$.

Proof Assume first that $x_i \notin X_{\bar{t}}$; by definition of a quasi-Rees decomposition, $x_i \notin X_t$ for all $t \in \mathcal{T}$. Since $\mathcal{T}$ is a finite set, only finitely many terms of the form $t = x_i^{k_t}$ can be contained in it. If we choose k greater than all these values k_t, then $x_i^k \in \mathrm{lt}_{\prec}\mathcal{I}$ and the minimal basis of $\mathrm{lt}_{\prec}\mathcal{I}$ must contains an element $x_i^{e_i}$.

For the converse, assume that $x_i^{e_i}$ lies in the minimal basis of $\mathrm{lt}_{\prec}\mathcal{I}$. Then for any $t \in \mathcal{T}$ the term $t \cdot x_i^{e_i}$ lies in $\mathrm{lt}_{\prec}\mathcal{I}$ and thus x_i cannot be an element of X_t by definition of a complementary decomposition. □

Proposition 3.17 *Under the assumptions of Lemma 3.16, the set $X_{\bar{t}}$ is the unique maximal strongly independent set modulo the ideal $\mathcal{I}$.*

Proof We showed already in the proof of Lemma 3.14 that $\mathrm{lt}_{\prec}\mathcal{I} \cap \mathbb{k}[X_{\bar{t}}] = \{0\}$, i. e. that the set $X_{\bar{t}}$ is strongly independent modulo $\mathcal{I}$. It follows from Lemma 3.16 that no variable $x_i \notin X_{\bar{t}}$ can be contained in a strongly independent set modulo $\mathcal{I}$. Hence any such set must be a subset of $X_{\bar{t}}$. □

Corollary 3.18 *Let the chosen coordinates $\mathbf{x}$ be δ-regular for the ideal $\mathcal{I} \subset \mathcal{P}$, i. e. $\mathcal{I}$ possesses a Pommaret basis $\mathcal{H}$. Then $\{x_1, \dots, x_D\}$ with $D = \dim \mathcal{A}$ is the unique maximal strongly independent set modulo the ideal $\mathcal{I}$.*

Applying standard arguments of homological algebra to the exact sequence $0 \to \mathcal{I} \to \mathcal{P} \to \mathcal{P}/\mathcal{I} \to 0$, one easily shows that $\mathrm{depth}\,(\mathcal{P}/\mathcal{I}) = \mathrm{depth}\,\mathcal{I} - 1$. Hence Proposition 2.20 immediately implies the following result (one can also prove it directly along the lines of the proof of Proposition 2.20).

Proposition 3.19 *Let $\mathcal{H}$ be a homogeneous Pommaret basis of the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ for a class respecting term order and $d = \min_{h \in \mathcal{H}} \mathrm{cls}\, h$. Then the depth of $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is $\mathrm{depth}\,\mathcal{A} = d - 1$.*

Since $\{x_1, \dots, x_{d-1}\}$ is trivially a strongly independent set modulo $\mathcal{I}$, we obviously find that always $D \geq d-1$. Thus as a trivial corollary of Propositions 3.15 and 3.19, we find the well-known fact that for any graded algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ the inequality $\mathrm{depth}\,\mathcal{A} \leq \dim \mathcal{A}$ holds. In the limit case $\mathrm{depth}\,\mathcal{A} = \dim \mathcal{A}$, the algebra is by definition *Cohen-Macaulay* and we obtain the following characterisation of such algebras.

Theorem 3.20 *Let $\mathcal{H}$ be a Pommaret basis of degree q of the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ for a class respecting term order $\prec$ and set $d = \min_{h \in \mathcal{H}} \mathrm{cls}\, h$. The algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is Cohen-Macaulay, if and only if $\langle \mathcal{H}, x_1, \dots, x_{d-1} \rangle_q = \mathcal{P}_q$.*

An alternative characterisation, which is more useful for computations, is based on the existence of a special kind of Rees decomposition; one sometimes speaks of a *Hironaka decomposition*, a terminology introduced in [72, Sect. 2.3].

Corollary 3.21 *$\mathcal{A} = \mathcal{P}/\mathcal{I}$ is a Cohen-Macaulay algebra, if and only if a Rees decomposition of $\mathcal{A}$ exists where all generators have the same class.*

Proof One direction is trivial. If such a special decomposition exists with d the common class of all generators, then obviously both the dimension and the depth of $\mathcal{A}$ is d and thus $\mathcal{A}$ is Cohen-Macaulay.

For the converse, let us assume that $\mathcal{A}$ is a Cohen-Macaulay algebra and that $\dim\mathcal{A} = \operatorname{depth}\mathcal{A} = d$. Let $\mathcal{H}$ be a Pommaret basis of $\mathcal{I}$ with respect to the degree reverse lexicographic order. By Theorem 2.16, such a basis always exists in δ-regular variables $\mathbf{x}$. Proposition 3.19 implies that $\min_{h\in\mathcal{H}} \operatorname{cls} h = d+1$. We introduce the set $\bar{\mathcal{N}} = \{\nu \in \mathbb{N}_0^n \setminus \langle \operatorname{le}_\prec \mathcal{H}\rangle \mid \operatorname{cls}\nu > d\}$ (recall that by convention we defined $\operatorname{cls}[0,\dots,0] = n$ so that $[0,\dots,0] \in \bar{\mathcal{N}}$ whenever $\mathcal{I} \neq \mathcal{P}$). $\bar{\mathcal{N}}$ is finite, as all its elements satisfy $|\nu| < \deg\mathcal{H}$ by Theorem 3.20, and we claim that

$$\mathcal{A} \cong \bigoplus_{\nu\in\bar{\mathcal{N}}} \Bbbk[x_1,\dots,x_d]\cdot x^\nu\ . \tag{17}$$

In fact, (17) is precisely the decomposition obtained by applying Janet's algorithm (cf. Proposition 3.4). Consider any multi index $\nu \in \bar{\mathcal{N}}$; obviously, it is of the form $\nu = [0,\dots,0,\nu_{d+1},\dots,\nu_n]$ with $\sum_{i=d+1}^n \nu_i < q = \deg\mathcal{H}$. If we set $q' = q - \sum_{i=d+2}^n \nu_i$, then by Theorem 3.20 the multi index $[0,\dots,0,q',\nu_{d+2},\dots,\nu_n]$ lies in the monoid ideal $\langle \operatorname{le}_\prec \mathcal{H}\rangle$. But this fact implies the existence of a multi index $\nu' \in \operatorname{le}_\prec\mathcal{H}$ with $\nu' = [0,\dots,0,\nu'_{d+1},\nu_{d+2},\dots,\nu_n]$ with $\nu_{d+1} < \nu'_{d+1} \leq q'$. Hence the set $(\nu_{d+2},\dots,\nu_n)$ is considered in the assignment of multiplicative variables to the elements of $\mathcal{H}$ for the Janet division and it consists only of the multi index ν', as $\mathcal{H}$ is involutively head autoreduced (with respect to the Pommaret division). But this observation implies that Janet's algorithm chooses ν as an element of $\bar{\mathcal{N}}$ and assigns to it the multiplicative variables $x_1,\dots,x_d$.

Janet's algorithm cannot lead to a larger set $\bar{\mathcal{N}}$, as any further multi index would be of class less than or equal to d and thus be contained in $\Bbbk[x_1,\dots,x_d]\cdot 1$. But since we know that the sets are disjoint, this cannot happen and we obtain the decomposition (17). □

Example 3.22 Consider again the ideal $\mathcal{I} = \langle z^3, yz^2 - xz^2, y^2 - xy\rangle \subset \Bbbk[x,y,z]$ of Example 3.13. It follows from the Pommaret basis given there that both the depth and the dimension of $\mathcal{P}/\mathcal{I}$ is 1. Hence $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is Cohen-Macaulay and indeed (14) is a Hironaka decomposition. ◁

4 Noether Normalisation and Primary Decomposition

As a simple consequence of our results in the previous section, we show now that any complementary quasi-Rees decomposition of $\mathcal{I}$ induces a *Noether normalisation* [34, Def. 3.4.2] of $\mathcal{A}$ and that its maximal set $X_{\bar{t}}$ of multiplicative variables defines a homogeneous system of parameters for $\mathcal{A}$. A slightly less general form of the following result is contained in the proof of [62, Algo. 3] where it is exploited for the explicit construction of Noether normalisations using Janet bases.

Proposition 4.1 *Under the assumptions of Lemma 3.16, the restriction of the canonical projection $\pi : \mathcal{P} \to \mathcal{A}$ to $\Bbbk[X_{\bar{t}}]$ defines a Noether normalisation for $\mathcal{A}$.*

Proof By Proposition 3.17, the set $X_{\bar{t}}$ is strongly independent modulo $\mathcal{I}$ and thus also independent modulo $\mathcal{I}$, i. e. $\mathcal{I} \cap \Bbbk[X_{\bar{t}}] = \{0\}$ implying that the restriction of π to $\Bbbk[X_{\bar{t}}]$ is injective. Furthermore, it follows immediately from the definition of a complementary quasi-Rees decomposition that the algebra $\mathcal{A}$ is finitely generated as a module over the ring $\Bbbk[X_{\bar{t}}]$. $\square$

Remark 4.2 Recall from Lemma 3.16 that for any variable $x_i \notin X_{\bar{t}}$ the minimal basis of $\operatorname{lt}_\prec \mathcal{I}$ contains an element of the form $x_i^{e_i}$ for some exponent $e_i \in \mathbb{N}$. Thus any Gröbner basis of $\mathcal{I}$ for the chosen term order $\prec$ must contain an element $g_i \in \mathcal{G}$ with $\operatorname{lt}_\prec g_i = x_i^{e_i}$. Assume now that $\prec$ is the lexicographic order. Then g_i must be of the form $g_i = x_i^{e_i} + \sum_{j=0}^{e_i-1} P_{i,j} x_i^j$ with polynomials $P_{i,j} \in \Bbbk[x_1, \dots, x_{i-1}]$. Thus in this case we even obtain a general Noether normalisation. $\lhd$

Since according to [62, Algo. 3] every ideal $\mathcal{I} \subseteq \mathcal{P}$ admits a complementary quasi-Rees decomposition, we obtain as a trivial corollary the existence of a Noether normalisation for every affine algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ (alternatively, we may employ Theorem 2.16 and Proposition 3.6 asserting the existence of even a complementary Rees decomposition for every ideal $\mathcal{I}$). Comparing with the classical existence proof of Noether normalisations given e. g. in [34], we see that the search for variables admitting a quasi-Rees decomposition corresponds to putting the ideal $\mathcal{I}$ into *Noether position* [79, Def. 2.22].

However, a quasi-Rees decomposition is generally not yet a Rees decomposition and thus even if the variables are chosen in such a way that $\Bbbk[x_1, \dots, x_D]$ defines a Noether normalisation of $\mathcal{A}$, this fact is not sufficient for concluding that the ideal $\mathcal{I}$ possesses a Pommaret basis in these variables. As we will show now, the existence of a Pommaret basis is equivalent to a stronger property. Since under the assumptions of Proposition 3.15 $\Bbbk[x_1, \dots, x_D]$ also defines a Noether normalisation of $\mathcal{P}/\operatorname{lt}_\prec \mathcal{I}$, it suffices to consider monomial ideals.

Definition 4.3 *A monomial ideal $\mathcal{I} \subseteq \mathcal{P}$ is called* quasi-stable*, if it possesses a finite Pommaret basis.*

The reason for this terminology will become apparent in Section 8 when we consider stable ideals. We now give several equivalent algebraic characterisations of quasi-stable ideals which are independent of the theory of Pommaret bases. They will provide us with a further criterion for δ-regularity and also lead to a simple description of an irredundant primary decomposition of such ideals.

Proposition 4.4 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a monomial ideal with $\dim \mathcal{P}/\mathcal{I} = D$. Then the following six statements are equivalent.*

(i) *$\mathcal{I}$ is quasi-stable.*
(ii) *The variable x_1 is not a zero divisor for*[10] *$\mathcal{P}/\mathcal{I}^{\mathrm{sat}}$ and for all $1 \leq j < D$ the variable x_{j+1} is not a zero divisor for $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_j \rangle^{\mathrm{sat}}$.*
(iii) *We have $\mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_D^\infty$ and for all $D < j \leq n$ an exponent $k_j \geq 1$ exists such that $x_j^{k_j} \in \mathcal{I}$.*

[10] See Section 10 for a more detailed discussion of the saturation $\mathcal{I}^{\mathrm{sat}}$.

(iv) *For all* $1 \leq j \leq n$ *the equality* $\mathcal{I} : x_j^\infty = \mathcal{I} : \langle x_j, \dots, x_n \rangle^\infty$ *holds.*
(v) *For every associated prime ideal* $\mathfrak{p} \in \operatorname{Ass}(\mathcal{P}/\mathcal{I})$ *an integer* $1 \leq j \leq n$ *exists such that* $\mathfrak{p} = \langle x_j, \dots, x_n \rangle$.
(vi) *If* $x^\mu \in \mathcal{I}$ *and* $\mu_i > 0$ *for some* $1 \leq i < n$*, then for each* $0 < r \leq \mu_i$ *and* $i < j \leq n$ *an integer* $s \geq 0$ *exists such that* $x^{\mu - r_i + s_j} \in \mathcal{I}$.

Proof The equivalence of the statements (ii)–(v) was proven by Bermejo and Gimenez [12, Prop. 3.2]; the equivalence of (iv) and (vi) was shown by Herzog et al. [42, Prop. 2.2] (alternatively the equivalence of (i) and (vi) is an easy consequence of Lemma 2.3). Bermejo and Gimenez [12] called ideals satisfying any of these conditions *monomial ideals of nested type*; Herzog et al. [42] spoke of *ideals of Borel type* (yet another terminology used by Caviglia and Sbarra [18] is *weakly stable ideals*).[11] Thus it suffices to show that these concepts coincide with quasi-stability by proving the equivalence of (i) and (iii).

Assume first that the ideal $\mathcal{I}$ is quasi-stable with Pommaret basis $\mathcal{H}$. The existence of a term $x_j^{k_j} \in \mathcal{I}$ for all $D < j \leq n$ follows then immediately from Proposition 3.15. Now consider a term $x^\mu \in \mathcal{I} : x_k^\infty \setminus \mathcal{I}$ for some $1 \leq k \leq n$. By definition of such a colon ideal, there exists an integer ℓ such that $x_k^\ell x^\mu \in \mathcal{I}$ and hence a generator $x^\nu \in \mathcal{H}$ such that $x^\nu \mid_P x_k^\ell x^\mu$. If $\operatorname{cls} \nu > k$, then ν would also be an involutive divisor of μ contradicting the assumption $x^\mu \notin \mathcal{I}$. Thus we find $\operatorname{cls} \nu \leq k$ and $\nu_k > \mu_k$.

Next we consider for arbitrary exponents $m > 0$ the non-multiplicative products $x_{k+1}^m x^\nu \in \mathcal{I}$. For each m a generator $x^{\rho^{(m)}} \in \mathcal{H}$ exists which involutively divides $x_{k+1}^m x^\nu$. By the same reasoning as above, $\operatorname{cls} x^{\rho^{(m)}} > k+1$ is not possible, as the Pommaret basis $\mathcal{H}$ is by definition involutively autoreduced. This yields the estimate $\operatorname{cls} \nu \leq \operatorname{cls} x^{\rho^{(m)}} \leq k+1$.

We claim now that there exists an integer m_0 such that $\rho^{(m)} = \rho^{(m_0)}$ for all $m \geq m_0$ and $\operatorname{cls} x^{\rho^{(m_0)}} = k+1$. Indeed, if $\operatorname{cls} x^{\rho^{(m)}} < k+1$, then we must have $\rho_{k+1}^{(m)} = v_{k+1} + m$, since x_{k+1} is not multiplicative for $x^{\rho^{(m)}}$. Hence $x^{\rho^{(m)}}$ cannot be an involutive divisor of $x_{k+1}^{m+1} x^\nu$ and $\rho^{(m+1)} \notin \{\rho^{(1)}, \dots, \rho^{(m)}\}$. As the Pommaret basis $\mathcal{H}$ is a finite set, $\operatorname{cls} x^{\rho^{(m_0)}} = k+1$ for some value $m_0 > 0$. But then x_{k+1} is multiplicative for $x^{\rho^{(m_0)}}$ and thus $x^{\rho^{(m_0)}}$ is trivially an involutive divisor of $x_{k+1}^m x^\nu$ for all values $m \geq m_0$.

Note that, by construction, the generator $x^{\rho^{(m_0)}}$ is also an involutive divisor of $x_{k+1}^{m_0} x^\mu$, as x_k is multiplicative for it. Hence this term must lie in $\mathcal{I}$ and consequently $x^\mu \in \mathcal{I} : x_{k+1}^\infty$. Thus we may conclude that $\mathcal{I} : x_k^\infty \subseteq \mathcal{I} : x_{k+1}^\infty$ for all $1 \leq k < n$. This proves (iii).

For the converse assume that (iii) holds and let $\mathcal{B}$ be the minimal basis of the ideal $\mathcal{I}$. Let $x^\mu \in \mathcal{B}$ be an arbitrary term of class k. Then $x^\mu / x_k \in \mathcal{I} : x_k^\infty$. By assumption, this means that also $x^\mu / x_k \in \mathcal{I} : x_\ell^\infty$ for any non-multiplicative index ℓ. Hence for each term $x^\mu \in \mathcal{B}$ and for each value $\operatorname{cls}(x^\mu) < \ell \leq n$ there exists an integer $q_{\mu,\ell}$ such that $x_\ell^{q_{\mu,\ell}} x^\mu / x_k \notin \mathcal{I}$ but $x_\ell^{q_{\mu,\ell}+1} x^\mu / x_k \in \mathcal{I}$. For the

[11] As usual, one must revert the ordering of the variables $x_1, \dots, x_n$ in order to recover the results of the given references.

values $1 \leq \ell \leq \operatorname{cls} x^\mu$ we set $q_{\mu,\ell} = 0$. Observe that if $x^\nu \in \mathcal{B}$ is a minimal generator dividing $x_\ell^{q_{\mu,\ell}+1} x^\mu / x_k$, then we find for the inverse lexicographic order that $x^\nu \prec_{\text{invlex}} x^\mu$, since $\operatorname{cls}(x^\nu) \geq \operatorname{cls}(x^\mu)$ and $\nu_k < \mu_k$.

Consider now the set

$$\mathcal{H} = \left\{ x^{\mu+\rho} \mid x^\mu \in \mathcal{B} \wedge \forall 1 \leq \ell \leq n : 0 \leq \rho_\ell \leq q_{\mu,\ell} \right\} . \tag{18}$$

We claim that $\mathcal{H}$ is a weak involutive completion of $\mathcal{B}$ and thus a weak Pommaret basis of $\mathcal{I}$. In order to prove this assertion, we must show that each term $x^\lambda \in \mathcal{I}$ lies in the involutive cone of a member of $\mathcal{H}$.

As x^λ is assumed to be an element of $\mathcal{I}$, we can factor it as $x^\lambda = x^{\sigma^{(1)}} x^{\rho^{(1)}} x^{\mu^{(1)}}$ where $x^{\mu^{(1)}} \in \mathcal{B}$ is a minimal generator, $x^{\sigma^{(1)}}$ contains only multiplicative variables for $x^{\mu^{(1)}}$ and $x^{\rho^{(1)}}$ only non-multiplicative ones. If $x^{\mu^{(1)}+\rho^{(1)}} \in \mathcal{H}$, then we are done, as obviously $\operatorname{cls}\left(x^{\mu^{(1)}+\rho^{(1)}}\right) = \operatorname{cls}\left(x^{\mu^{(1)}}\right)$ and hence all variables contained in $x^{\sigma^{(1)}}$ are multiplicative for $x^{\mu^{(1)}+\rho^{(1)}}$, too.

Otherwise there exists at least one non-multiplicative variables x_ℓ such that $\rho_\ell^{(1)} > q_{\mu^{(1)},\ell}$. Any minimal generator $x^{\mu^{(2)}} \in \mathcal{B}$ dividing $x_\ell^{q_{\mu^{(1)},\ell}+1} x^{\mu^{(1)}} / x_k$ is also a divisor of x^λ and we find a second factorisation $x^\lambda = x^{\sigma^{(2)}} x^{\rho^{(2)}} x^{\mu^{(2)}}$ where again $x^{\sigma^{(2)}}$ consists only of multiplicative and $x^{\rho^{(2)}}$ only of non-multiplicative variables for $x^{\mu^{(2)}}$. If $x^{\mu^{(2)}+\rho^{(2)}} \in \mathcal{H}$, then we are done by the same argument as above; otherwise we iterate.

According to the observation made above, the sequence $(x^{\mu^{(1)}}, x^{\mu^{(2)}}, \dots)$ of minimal generators constructed this way is strictly descending with respect to the inverse lexicographic order. However, the minimal basis $\mathcal{B}$ is a finite set and thus the iteration cannot go on infinitely. As the iteration only stops, if there exists an involutive cone containing x^λ, the involutive span of $\mathcal{H}$ is indeed $\mathcal{I}$ and thus the ideal $\mathcal{I}$ quasi-stable. $\square$

Remark 4.5 Note that our considerations about standard pairs and the induced primary decomposition in the last section imply a simple direct proof of the implication “(i) $\Rightarrow$ (v)” in Proposition 4.4. If the ideal $\mathcal{I}$ is quasi-stable, then $\mathcal{I}$ admits a complementary Rees decomposition according to Proposition 3.6. Together with Propositions 3.10, 3.11 and Remark 3.12, this observation trivially implies that all associated prime ideals are of the form $\mathfrak{p} = \langle x_j, \dots, x_n \rangle$. $\triangleleft$

Lemma 4.6 *Let $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{P}$ be two quasi-stable ideals. Then the sum $\mathcal{I}_1 + \mathcal{I}_2$, the product $\mathcal{I}_1 \cdot \mathcal{I}_2$ and the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ are quasi-stable, too. If $\mathcal{I} \subseteq \mathcal{P}$ is a quasi-stable ideal, then the quotient $\mathcal{I} : \mathcal{J}$ is again quasi-stable for arbitrary monomial ideals $\mathcal{J} \subseteq \mathcal{P}$.*

Proof For the sum $\mathcal{I}_1 + \mathcal{I}_2$ the claim follows immediately from Remark 2.9 of Part I which states that the union $\mathcal{H}_1 \cup \mathcal{H}_2$ of (weak) Pommaret bases $\mathcal{H}_k$ of $\mathcal{I}_k$ is a weak Pommaret basis of the sum $\mathcal{I}_1 + \mathcal{I}_2$. Similarly, the case of both the product $\mathcal{I}_1 \cdot \mathcal{I}_2$ and the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ was settled in Remark 6.5 of Part I where for both ideals weak Pommaret bases were constructed.

For the last assertion we use Part (vi) of Proposition 4.4. If $\mathcal{J}$ is minimally generated by the monomials $m_1, \dots, m_r$, then $\mathcal{I} : \mathcal{J} = \bigcap_{k=1}^r \mathcal{I} : m_k$ and thus it suffice to consider the case that $\mathcal{J}$ is a principal ideal with generator x^ν. Assume that $x^\mu \in \mathcal{I} : x^\nu$ and that $\mu_i > 0$. Since $x^{\mu+\nu}$ lies in the quasi-stable ideal $\mathcal{I}$, we find for each $0 < r \leq \mu_i$ and $i < j \leq n$ and integer $s \geq 0$ exists such that $x^{\mu+\nu-r_i+s_j} \in \mathcal{I}$. As $r \leq \mu_i$, this trivially implies that $x^{\mu-r_i+s_j} \in \mathcal{I} : x^\nu$. $\square$

Remark 4.7 Alternative proofs for Lemma 4.6 were given by Cimpoeaş [21]. There it was also noted that its final statement trivially implies Part (v) of Proposition 4.4, as any associated prime ideal $\mathfrak{p}$ of a quasi-stable ideal $\mathcal{I}$ is of the form $\mathfrak{p} = \mathcal{I} : x^\nu$ for some monomial x^ν and thus is also quasi-stable. But the only quasi-stable prime ideals are obviously the ideals $\langle x_j, \dots, x_n\rangle$.

Above we actually proved that Part (iii) of Proposition 4.4 may be replaced by the equivalent statement $\mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_n^\infty$ which does not require a priori knowledge of D (the dimension D arises then trivially as the smallest value k such that $\mathcal{I} : x_k^\infty = \mathcal{P}$, i., e, for which $\mathcal{I}$ contains a minimal generator x_k^ℓ for some exponent $\ell > 0$). In this formulation it is straightforward to verify (iii) effectively: bases of the colon ideals $\mathcal{I} : x_k^\infty$ are easily obtained by setting $x_k = 1$ in any basis of $\mathcal{I}$ and for monomial ideals it is trivial to check inclusion, as one must only compare their minimal bases.

We furthermore note that if we have for some value $1 \leq k \leq n$ an ascending chain $\mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_k^\infty$, then for each $1 \leq j \leq k$ the minimal basis $\mathcal{B}_j$ of $\mathcal{I} : x_j^\infty$ lies in $\Bbbk[x_{j+1}, \dots, x_n]$. Indeed, no element of $\mathcal{B}_j$ can depend on x_j. Now assume that $x^\nu \in \mathcal{B}_j$ satisfies $\operatorname{cls}\nu = \ell < j$. Then $x_j^m x^\nu$ is a minimal generator of $\mathcal{I}$ for some suitable exponent $m \in \mathbb{N}_0$. This in turn implies that $x_j^m x^\nu / x_\ell^{\nu_\ell} \in \mathcal{I} : x_\ell^\infty \subseteq \mathcal{I} : x_j^\infty$ and hence $x^\nu / x_\ell^{\nu_\ell} \in \mathcal{I} : x_j^\infty$ which contradicts our assumption that x^ν was a minimal generator. $\triangleleft$

The above mentioned version of Proposition 4.4 (iii) provides us with a new, simple and effective criterion for δ-regularity of a monomial ideal. The following converse to Theorem 2.13 shows that for monomial ideals the notion of δ-regularity and asymptotic regularity (for Pommaret autoreduced bases) are equivalent. Obviously, this observation entails their equivalence for Pommaret autoreduced *Gröbner* bases of arbitrary ideals.

Proposition 4.8 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a monomial ideal and $\mathcal{B}$ a finite, Pommaret autoreduced monomial basis of it. If $\mathcal{I}$ is not quasi-stable, then for at least one generator in the basis $\mathcal{B}$ a variable exists which is Janet but not Pommaret multiplicative.*

Proof As the ideal $\mathcal{I}$ is not quasi-stable, there exists a minimal value k such that $\mathcal{I} : x_k^\infty \nsubseteq \mathcal{I} : x_{k+1}^\infty$. Let x^μ be a minimal generator of $\mathcal{I} : x_k^\infty$ which is not contained in $\mathcal{I} : x_{k+1}^\infty$. Then for a suitable exponent $m \in \mathbb{N}_0$ the term $x^{\bar\mu} = x_k^m x^\mu$ is a minimal generator of $\mathcal{I}$ and hence contained in $\mathcal{B}$.

We claim now that $\mathcal{B}$ contains a generator for which x_{k+1} is Janet but not Pommaret multiplicative. If $x_{k+1} \in X_{J,\mathcal{B}}(x^{\bar\mu})$, then we are done, as according to Remark 4.7 $\operatorname{cls}\bar\mu = k$ and hence $x_{k+1} \notin X_P(x^{\bar\mu})$. Otherwise $\mathcal{B}$ contains a term x^ν such that $\nu_\ell = \mu_\ell$ for $k+1 < \ell \leq n$ and $\nu_{k+1} > \mu_{k+1}$. If several generators with this property exist in $\mathcal{B}$, we choose one for which ν_{k+1} takes a maximal value so that we have $x_{k+1} \in X_{J,\mathcal{B}}(x^\nu)$ by definition of the Janet division. If $\operatorname{cls}\nu < k+1$, we are again done, as then $x_{k+1} \notin X_P(x^\nu)$. Now assume that $\operatorname{cls}\nu = k+1$ and consider the term $x^\rho = x^\nu/x_{k+1}^{\nu_{k+1}}$. Obviously, $x^\rho \in \mathcal{I} : x_{k+1}^\infty$ contradicting our assumption $x^\mu \notin \mathcal{I} : x_{k+1}^\infty$ since $x^\rho \mid x^\mu$. Hence this case cannot arise. □

We mentioned above that while δ-regular coordinates ensure that $\mathcal{I}$ is in Noether position the converse is not true. Based on Proposition 4.4 (v), one can formulate a converse for monomial ideals stating that a Pommaret basis of a monomial ideal induces not only a Noether normalisation of the ideal itself but simultaneously of all its primary components.

Corollary 4.9 ([12, Prop. 3.6]) *Let $\mathcal{I}$ be a monomial ideal with $\dim \mathcal{P}/\mathcal{I} = D$. Furthermore, let $\mathcal{I} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be an irredundant monomial primary decomposition with $D_j = \dim \mathcal{P}/\mathfrak{q}_j$ for $1 \leq j \leq r$. Then $\mathcal{I}$ is quasi-stable, if and only if $\mathbb{k}[x_1, \dots, x_D]$ defines a Noether normalisation of $\mathcal{P}/\mathcal{I}$ and $\mathbb{k}[x_1, \dots, x_{D_j}]$ one of $\mathcal{P}/\mathfrak{q}_j$ for each primary component $\mathfrak{q}_j$.*

We may also exploit Proposition 4.4 for actually deriving an irredundant primary decomposition $\mathcal{I} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ with monomial ideals $\mathfrak{q}_j$ for an arbitrary quasi-stable ideal $\mathcal{I}$.[12] Bermejo and Gimenez [12, Rem. 3.3] noted that their proof of the implication "(v) $\Rightarrow$ (iv)" in Proposition 4.4 has some simple consequences for the primary ideals $\mathfrak{q}_j$. Let again $D = \dim \mathcal{P}/\mathcal{I}$. Then $\mathfrak{p} = \langle x_{D+1}, \dots, x_n \rangle$ is the unique minimal prime ideal associated to $\mathcal{I}$ and the corresponding unique primary component is given by $\mathcal{I} : x_D^\infty$ (if $D = 0$, then obviously $\mathcal{I}$ is already a primary ideal). More generally, we find for any $1 \leq k \leq D$ that

$$\mathcal{I} : x_k^\infty = \bigcap_{\mathfrak{p}_j \subseteq \langle x_{k+1}, \dots, x_n \rangle} \mathfrak{q}_j \tag{19}$$

where $\mathfrak{p}_j = \sqrt{\mathfrak{q}_j}$ is the corresponding associated prime ideal. Based on these observations, an irredundant primary decomposition can be constructed by working backwards through the sequence $\mathcal{I} \subseteq \mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_n^\infty$.

Let $d = \operatorname{depth} \mathcal{P}/\mathcal{I}$, i.e. $d+1$ is the minimal class of a generator in the Pommaret basis $\mathcal{H}$ of $\mathcal{I}$ according to Proposition 3.19.[13] For $1 \leq k \leq D$ we set $s_k = \min\{s \mid \mathcal{I} : x_k^s = \mathcal{I} : x_k^{s+1}\}$, i.e. s_k is the highest x_k-degree of a minimal generator of $\mathcal{I}$. Then we introduce the ideals $\mathcal{J}_k = \mathcal{I} + \langle x_{k+1}^{s_{k+1}}, \dots, x_D^{s_D} \rangle$ and

$$\mathfrak{q}_k = \mathcal{J}_k : x_k^\infty = \mathcal{I} : x_k^\infty + \langle x_{k+1}^{s_{k+1}}, \dots, x_D^{s_D} \rangle \,. \tag{20}$$

It is easy to see that all the ideals $\mathcal{J}_k$ are again quasi-stable provided the ideal $\mathcal{I}$ is quasi-stable (this follows immediately from Proposition 4.4 and the fact that in this case $(\mathcal{I} : x_i^\infty) : x_j^\infty = \mathcal{I} : x_j^\infty$ for $i < j$). For notational simplicity we formally define $\mathcal{I} : x_0^\infty = \mathcal{I}$ and $\mathfrak{q}_0 = \mathcal{J}_0 = \mathcal{I} + \langle x_1^{s_1}, \dots, x_D^{s_D} \rangle$. Since obviously $\dim \mathcal{P}/\mathcal{J}_k = k$ for $0 \leq k \leq D$, it follows from the considerations above that $\mathfrak{q}_k$ is an $\langle x_{k+1}, \dots, x_n \rangle$-primary ideal.

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[12] The following construction is joint work with M. Hausdorf and M. Sahbi and has already appeared in [38].

[13] Note that for determining the depth d in the case of a quasi-stable ideal, it is not necessary to compute the Pommaret basis: since multiplication with a non-multiplicative variable never decreases the class, $d+1$ is also the minimal class of a minimal generator.

Proposition 4.10 ([38, Prop. 4.6]) *Let $\mathcal{I} \subseteq \mathcal{P}$ be a quasi-stable ideal. Then a monomial primary decomposition is given by $\mathcal{I} = \bigcap_{k=d}^{D} \mathfrak{q}_k$. Eliminating all primary ideals $\mathfrak{q}_k$ where $\mathcal{I} : x_k^\infty = \mathcal{I} : x_{k+1}^\infty$ makes it an irredundant decomposition.*

Proof We first show that the equality $\mathcal{I} : x_k^\infty = \bigcap_{\ell=k}^{D} \mathfrak{q}_\ell$ holds or equivalently that $\mathcal{I} : x_k^\infty = \mathfrak{q}_k \cap (\mathcal{I} : x_{k+1}^\infty)$ for $0 \leq k \leq n$; for $k = d$ this represents the first statement of the proposition, since obviously $\mathcal{I} : x_0^\infty = \cdots = \mathcal{I} : x_d^\infty = \mathcal{I}$. By definition of the value s_{k+1}, we have that [34, Lemma 3.3.6]

$$\mathcal{I} : x_k^\infty = \left(\mathcal{I} : x_k^\infty + \langle x_{k+1}^{s_{k+1}} \rangle\right) \cap \left((\mathcal{I} : x_k^\infty) : x_{k+1}^\infty\right) . \tag{21}$$

The second factor obviously equals $\mathcal{I} : x_{k+1}^\infty$. To the first one we apply the same construction and decompose

$$\begin{aligned} &\mathcal{I} : x_k^\infty + \langle x_{k+1}^{s_{k+1}} \rangle = \\ &\qquad = \left(\mathcal{I} : x_k^\infty + \langle x_{k+1}^{s_{k+1}}, x_{k+2}^{s_{k+2}} \rangle\right) \cap \left((\mathcal{I} : x_k^\infty + \langle x_{k+1}^{s_{k+1}} \rangle) : x_{k+2}^\infty\right) \\ &\qquad = \left(\mathcal{I} : x_k^\infty + \langle x_{k+1}^{s_{k+1}}, x_{k+2}^{s_{k+2}} \rangle\right) \cap \left(\mathcal{I} : x_{k+2}^\infty + \langle x_{k+1}^{s_{k+1}} \rangle\right) . \end{aligned} \tag{22}$$

Continuing in this manner, we arrive at a decomposition

$$\mathcal{I} : x_k^\infty = \mathfrak{q}_k \cap \cdots \cap (\mathcal{I} : x_{k+1}^\infty) \tag{23}$$

where the dots represent factors of the form $\mathcal{I} : x_\ell^\infty + \langle x_{k+1}^{s_{k+1}}, \dots, x_{\ell-1}^{s_{\ell-1}} \rangle$ with $\ell \geq k+2$. Since we assume that $\mathcal{I}$ is quasi-stable, $\mathcal{I} : x_{k+1}^\infty$ is contained in each of these factors and we may omit them which proves our claim.

In the thus obtained primary decomposition of $\mathcal{I}$ the radicals of all appearing primary ideals are pairwise different. Furthermore, it is obvious that $\mathfrak{q}_k$ is redundant whenever $\mathcal{I} : x_k^\infty = \mathcal{I} : x_{k+1}^\infty$. Thus there only remains to prove that all the other primary ideals $\mathfrak{q}_k$ are indeed necessary. Assume that $\mathcal{I} : x_k^\infty \subsetneq \mathcal{I} : x_{k+1}^\infty$ (which is in particular the case for $k < d$). Then there exists a minimal generator x^μ of $\mathcal{I} : x_{k+1}^\infty$ which is not contained in $\mathcal{I} : x_k^\infty$. Consider the monomial $x_k^{s_k} x^\mu$. It cannot lie in $\mathcal{I} : x_k^\infty$, as otherwise already $x^\mu \in \mathcal{I} : x_k^\infty$, and thus it also cannot be contained in $\mathfrak{q}_k$ (since we showed above that $\mathcal{I} : x_k^\infty = \mathfrak{q}_k \cap (\mathcal{I} : x_{k+1}^\infty)$). On the other hand we find that $x_k^{s_k} x^\mu \in \mathfrak{q}_\ell$ for all $\ell > k$ since then $\mathcal{I} : x_{k+1}^\infty \subseteq \mathfrak{q}_\ell$ and for all $\ell < k$ since then $\langle x_k^{s_k} \rangle \subseteq \mathfrak{q}_\ell$. Hence $\mathfrak{q}_k$ is not redundant. □

According to Lemma 4.6, the quotient ideals $\mathcal{I} : x_k^\infty$ are again quasi-stable. It is straightforward to obtain Pommaret bases for them. We disjointly decompose the monomial Pommaret basis $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_n$ where $\mathcal{H}_k$ contains all generators of class k. Furthermore, we write $\mathcal{H}'_k$ for the set obtained by setting $x_k = 1$ in each generator in $\mathcal{H}_k$.

Lemma 4.11 *For any $1 \leq k \leq n$ the set $\mathcal{H}' = \mathcal{H}'_k \cup \bigcup_{\ell=k+1}^{n} \mathcal{H}_\ell$ is a weak Pommaret basis of the colon ideal $\mathcal{I} : x_k^\infty$.*

Proof We first show that $\mathcal{H}'$ is an involutive set. By definition of the Pommaret division, it is obvious that the subset $\bigcup_{\ell=k+1}^{n} \mathcal{H}_\ell$ is involutive. Thus there only remains to consider the non-multiplicative products of the members of $\mathcal{H}'_k$. Take

$x^\mu \in \mathcal{H}'_k$ and let x_ℓ be a non-multiplicative variable for it. Obviously, there exists an $m > 0$ such that $x_k^m x^\mu \in \mathcal{H}_k$ and hence a generator $x^\nu \in \bigcup_{\ell=k}^n \mathcal{H}_\ell$ such that $x_\ell x_k^m x^\mu$ lies in the involutive cone $\mathcal{C}_P(x^\nu)$. Writing $x_\ell x_k^m x^\mu = x^{\rho+\nu}$, we distinguish two cases. If $\operatorname{cls}\nu > k$, then $\rho_k = m$ and we can divide by x_k^m in order to obtain an involutive standard representation of $x_\ell x^\mu$ with respect to $\mathcal{H}'$. If $\operatorname{cls}\nu = k$, then the multi index ρ is of the form r_k, i. e. only the kth entry is different from zero, and we even find that $x_\ell x^\mu = x^\nu / x_k^r \in \mathcal{H}'_k$.

Thus there only remains to prove that $\mathcal{H}'$ is actually a generating set for $\mathcal{I} : x_k^\infty$. For this we first note that the Pommaret basis of a quasi-stable ideal contains a generator of class k only, if there is a minimal generator of class k, as applying the monomial completion Algorithm 2 of Part I to the minimal basis adds only non-multiplicative multiples of the minimal generators (and these are trivially of the same class). By Remark 4.5, all minimal generators of $\mathcal{I} : x_k^\infty$ have at least class $k+1$. Thus setting $x_k = 1$ in any member of $\bigcup_{\ell=1}^{k-1} \mathcal{H}_\ell$ can never produce a minimal generator of $\mathcal{I} : x_k^\infty$ and thus $\mathcal{H}'$ is a weak involutive completion of the minimal basis of $\mathcal{I} : x_k^\infty$. According to Proposition 2.8 of Part I, an involutive autoreduction yields a strong basis. $\square$

The ideals $\langle x_{k+1}^{s_{k+1}}, \dots, x_D^{s_D} \rangle$ are obviously irreducible and for $k \geq d$ exactly of the form that they possess a Pommaret basis as discussed in Remark 2.13 of Part I. There we also gave an explicit Pommaret basis for such an ideal. Since according to Remark 2.9 of Part I the union of two (weak) Pommaret bases of two monomial ideals $\mathcal{I}_1, \mathcal{I}_2$ yields a weak Pommaret basis of $\mathcal{I}_1 + \mathcal{I}_2$, we obtain this way easily weak Pommaret bases for all primary ideals $\mathfrak{q}_k$ appearing in the irredundant decomposition of Proposition 4.10.

Thus the crucial information for obtaining an irredundant primary decomposition of a quasi-stable ideal $\mathcal{I}$ is where "jumps" are located, i. e. where $\mathcal{I} : x_k^\infty \subsetneq \mathcal{I} : x_{k+1}^\infty$. Since these ideals are quasi-stable, the positions of the jumps are determined by their depths. A chain with all the jumps is obtained by the following simple recipe: set $\mathcal{I}_0 = \mathcal{I}$ and define $\mathcal{I}_{k+1} = \mathcal{I}_k : x_{d_k}^\infty$ where $d_k = \operatorname{depth} \mathcal{I}_k$. This construction leads to the so-called *sequential chain* of $\mathcal{I}$:

$$\mathcal{I}_0 = \mathcal{I} \subsetneq \mathcal{I}_1 \subsetneq \cdots \subsetneq \mathcal{I}_r = \mathcal{P};. \tag{24}$$

Remark 4.12 With the help of the sequential chain (24) one can also show straightforwardly that any quasi-stable ideal is *sequentially Cohen-Macaulay* [42, Cor. 2.5] (recall that the algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ is sequentially Cohen-Macaulay [70], if a chain $\mathcal{I}_0 = \mathcal{I} \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_r = \mathcal{P}$ exists such that all quotients $\mathcal{I}_{k+1}/\mathcal{I}_k$ are Cohen-Macaulay and their dimensions are ascending: $\dim(\mathcal{I}_k/\mathcal{I}_{k-1}) < \dim(\mathcal{I}_{k+1}/\mathcal{I}_k)$).

Indeed, consider the ideal $\mathcal{J}_k = \mathcal{I}_k \cap \mathbb{k}[x_{d_k}, \dots, x_n]$. By Remark 4.7, the minimal generators of $\mathcal{J}_k$ are the same as the minimal ones of $\mathcal{I}_k$; furthermore, by Proposition 4.4(iv) $\mathcal{J}_k^{\mathrm{sat}} = \mathcal{J}_k : x_{d_k}^\infty$. Hence we find that $\mathcal{I}_{k+1} = \langle \mathcal{J}_k^{\mathrm{sat}} \rangle_{\mathcal{P}}$ and

$$\mathcal{I}_{k+1}/\mathcal{I}_k \cong (\mathcal{J}_k^{\mathrm{sat}}/\mathcal{J}_k)[x_1, \dots, x_{d_k-1}] \,. \tag{25}$$

Since the factor ring $\mathcal{J}_k^{\mathrm{sat}}/\mathcal{J}_k$ is trivially finite (as a $\mathbb{k}$-linear space), the quotient $\mathcal{I}_{k+1}/\mathcal{I}_k$ is thus a $(d_k - 1)$-dimensional Cohen-Macaulay module. $\triangleleft$

5 Syzygies for Involutive Bases

Gröbner bases are a very useful tool in syzygy theory. A central result is *Schreyer's Theorem* [1, 64] that the standard representations of the S-polynomials between the elements of a Gröbner basis directly determine a Gröbner basis of the first syzygy module with respect to an appropriately chosen term order. Now we study the use of involutive bases in this context.

In Part I we introduced involutive bases only for ideals, but the extension to submodules of free modules $\mathcal{P}^m$ is trivial. We represent elements of $\mathcal{P}^m$ as vectors $\mathbf{f} = (f_1, \dots, f_m)$ with $f_\alpha \in \mathcal{P}$. The standard basis of $\mathcal{P}^m$ consists of the unit vectors $\mathbf{e}_\alpha$ with $e_{\alpha\beta} = \delta_{\alpha\beta}$ and $1 \leq \alpha \leq m$; thus $\mathbf{f} = f_1\mathbf{e}_1 + \cdots + f_m\mathbf{e}_m$. Now a term $\mathbf{t}$ is a vector of the form $\mathbf{t} = t\mathbf{e}_\alpha$ for some α and with $t \in \mathbb{T}$ a term in $\mathcal{P}$. We denote the set of all terms by $\mathbb{T}^m$; it is a monoid module over $\mathbb{T}$.

Let $\mathcal{H} \subset \mathcal{P}^m$ be a finite set, $\prec$ a term order on $\mathbb{T}^m$ and L an involutive division on $\mathbb{N}_0^n$. We divide $\mathcal{H}$ into m disjoint sets $\mathcal{H}_\alpha = \bigl\{\mathbf{h} \in \mathcal{H} \mid \mathrm{lt}_\prec \mathbf{h} = t\mathbf{e}_\alpha,\ t \in \mathbb{T}\bigr\}$. This leads naturally to m sets $\mathcal{N}_\alpha = \bigl\{\mu \in \mathbb{N}_0^n \mid x^\mu \mathbf{e}_\alpha \in \mathrm{lt}_\prec \mathcal{H}_\alpha\bigr\}$. If $\mathbf{h} \in \mathcal{H}_\alpha$, we assign the multiplicative variables $X_{L,\mathcal{H},\prec}(\mathbf{h}) = \bigl\{x_i \mid i \in N_{L,\mathcal{N}_\alpha}(\mathrm{le}_\prec \mathbf{h})\bigr\}$. The involutive span $\langle\mathcal{H}\rangle_{L,\prec}$ is defined by an obvious generalisation of the old definition in Part I.

Let $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ be an involutive basis of the submodule $\mathcal{M} \subseteq \mathcal{P}^m$. Take an arbitrary element $\mathbf{h}_\alpha \in \mathcal{H}$ and choose an arbitrary non-multiplicative variable $x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$ of it. By the results of Part I, we can determine with an involutive normal form algorithm for each generator $\mathbf{h}_\beta \in \mathcal{H}$ a unique polynomial $P_\beta^{(\alpha;k)} \in \Bbbk[X_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)]$ such that $x_k\mathbf{h}_\alpha = \sum_{\beta=1}^s P_\beta^{(\alpha;k)}\mathbf{h}_\beta$. To this relation corresponds the syzygy

$$\mathbf{S}_{\alpha;k} = x_k\mathbf{e}_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha;k)}\mathbf{e}_\beta \in \mathcal{P}^s\ . \tag{26}$$

We denote the set of all thus obtained syzygies by

$$\mathcal{H}_{\mathrm{Syz}} = \bigl\{\mathbf{S}_{\alpha;k} \mid 1 \leq \alpha \leq s;\ x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)\bigr\}\ . \tag{27}$$

Lemma 5.1 *Let $\mathcal{H}$ be an involutive basis for the involutive division L and the term order $\prec$. If $\mathbf{S} = \sum_{\beta=1}^s S_\beta\mathbf{e}_\beta$ is an arbitrary syzygy in the module $\mathrm{Syz}(\mathcal{H})$ with $S_\beta \in \Bbbk[X_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)]$ for all $1 \leq \beta \leq s$, then $\mathbf{S} = 0$.*

Proof By definition of a syzygy, $\sum_{\beta=1}^s S_\beta\mathbf{h}_\beta = 0$. As the involutive basis $\mathcal{H}$ is involutively head autoreduced, each element $\mathbf{f} \in \langle\mathcal{H}\rangle$ possesses a unique involutive standard representation. In particular, this holds for $0 \in \langle\mathcal{H}\rangle$. Thus either $\mathbf{S} = 0$ or $S_\beta \notin \Bbbk[X_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)]$ for at least one β. □

A fundamental ingredient of Schreyer's Theorem is the term order $\prec_\mathcal{F}$ on $\mathbb{T}^s$ induced by an arbitrary finite set $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_s\} \subset \mathcal{P}^m$ and an arbitrary term order $\prec$ on $\mathbb{T}^m$: given two terms $\mathbf{s} = s\mathbf{e}_\sigma$ and $\mathbf{t} = t\mathbf{e}_\tau$, we set $\mathbf{s} \prec_\mathcal{F} \mathbf{t}$, if either $\mathrm{lt}_\prec(s\mathbf{f}_\sigma) \prec \mathrm{lt}_\prec(t\mathbf{f}_\tau)$ or $\mathrm{lt}_\prec(s\mathbf{f}_\sigma) = \mathrm{lt}_\prec(t\mathbf{f}_\tau)$ and $\tau < \sigma$.

Corollary 5.2 *If $\mathcal{H} \subset \mathcal{P}$ is an involutive basis, then the set $\mathcal{H}_{\mathrm{Syz}}$ generates the syzygy module* $\mathrm{Syz}(\mathcal{H})$.

Proof Let $\mathbf{S} = \sum_{\beta=1}^{s} S_\beta \mathbf{e}_\beta$ by an arbitrary non-vanishing syzygy in $\mathrm{Syz}(\mathcal{H})$. By Lemma 5.1, at least one of the coefficients S_β must contain a term x^μ with a non-multiplicative variable $x_j \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)$. Let $cx^\mu \mathbf{e}_\beta$ be the maximal such term with respect to the term order $\prec_{\mathcal{H}}$ and j the maximal non-multiplicative index with $\mu_j > 0$. Then we eliminate this term by computing $\mathbf{S}' = \mathbf{S} - cx^{\mu-1_j}\mathbf{S}_{\beta;j}$. If $\mathbf{S}' \neq 0$, we iterate. Since all new terms introduced by the subtraction are smaller than the eliminated term with respect to $\prec_{\mathcal{H}}$, we must reach zero after a finite number of steps. Thus this computation leads to a representation of $\mathbf{S}$ as a linear combination of elements of $\mathcal{H}_{\mathrm{Syz}}$. □

Let $\mathcal{H} = \{\mathbf{h}_1, \cdots, \mathbf{h}_s\}$ be an involutive basis and thus a Gröbner basis for the term order $\prec$. Without loss of generality we may assume that $\mathcal{H}$ is a monic basis. Set $\mathbf{t}_\alpha = \mathrm{lt}_\prec \mathbf{h}_\alpha$ and $\mathbf{t}_{\alpha\beta} = \mathrm{lcm}(\mathbf{t}_\alpha, \mathbf{t}_\beta)$. We have for every S-polynomial a standard representation $\mathbf{S}_\prec(\mathbf{h}_\alpha, \mathbf{h}_\beta) = \sum_{\gamma=1}^{s} f_{\alpha\beta\gamma}\mathbf{h}_\gamma$ where the polynomials $f_{\alpha\beta\gamma} \in \mathcal{P}$ satisfy $\mathrm{lt}_\prec\bigl(\mathbf{S}_\prec(\mathbf{h}_\alpha, \mathbf{h}_\beta)\bigr) \succeq \mathrm{lt}_\prec(f_{\alpha\beta\gamma}\mathbf{h}_\gamma)$ for $1 \leq \gamma \leq s$. Setting $\mathbf{f}_{\alpha\beta} = \sum_{\gamma=1}^{s} f_{\alpha\beta\gamma}\mathbf{e}_\gamma$, we introduce for $\alpha \neq \beta$ the syzygy

$$\mathbf{S}_{\alpha\beta} = \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\alpha}\mathbf{e}_\alpha - \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\beta}\mathbf{e}_\beta - \mathbf{f}_{\alpha\beta}\ . \tag{28}$$

Schreyer's Theorem asserts that the set $\mathcal{H}_{\mathrm{Schreyer}} = \{\mathbf{S}_{\alpha\beta} \mid 1 \leq \alpha < \beta \leq s\}$ of all these syzygies is a Gröbner basis of the first syzygy module $\mathrm{Syz}(\mathcal{H})$ for the induced term order $\prec_{\mathcal{H}}$.

We denote by $\tilde{\mathbf{S}}_{\alpha\beta} = \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\alpha}\mathbf{e}_\alpha - \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\beta}\mathbf{e}_\beta$ the syzygy of the leading terms corresponding to $\mathbf{S}_{\alpha\beta}$ and if $\mathcal{S} \subseteq \mathcal{H}_{\mathrm{Schreyer}}$ is a set of syzygies, $\tilde{\mathcal{S}}$ contains the corresponding syzygies of the leading terms.

Lemma 5.3 *Let $\mathcal{S} \subseteq \mathcal{H}_{\mathrm{Schreyer}}$ be such that $\tilde{\mathcal{S}}$ generates* $\mathrm{Syz}(\mathrm{lt}_\prec\mathcal{H})$. *Then $\mathcal{S}$ generates* $\mathrm{Syz}(\mathcal{H})$. *Assume furthermore that the three pairwise distinct indices α, β, γ are such that*[14] $\mathbf{S}_{\alpha\beta}, \mathbf{S}_{\beta\gamma}, \mathbf{S}_{\alpha\gamma} \in \mathcal{S}$ *and* $\mathbf{t}_\gamma \mid \mathbf{t}_{\alpha\beta}$. *Then the smaller set $\mathcal{S} \setminus \{\mathbf{S}_{\alpha\beta}\}$ still generates* $\mathrm{Syz}(\mathcal{H})$.

Proof It is a classical result in the theory of Gröbner bases that $\tilde{\mathcal{S}} \setminus \{\tilde{\mathbf{S}}_{\alpha\beta}\}$ still generates $\mathrm{Syz}(\mathrm{lt}_\prec\mathcal{H})$. In fact, this is the basic property underlying Buchberger's second criterion for avoiding redundant S-polynomials. Thus it suffices to show the first assertion; the second one is a simple corollary.

Let $\mathbf{R} = \sum_{\alpha=1}^{s} R_\alpha \mathbf{e}_\alpha \in \mathrm{Syz}(\mathcal{H})$ be an arbitrary syzygy of the full generators and set $\mathbf{t}_{\mathbf{R}} = \max_\prec \bigl\{\mathrm{lt}_\prec(R_\alpha \mathbf{h}_\alpha) \mid 1 \leq \alpha \leq s\bigr\}$. Then

$$\tilde{\mathbf{R}} = \sum_{\mathrm{lt}_\prec(R_\alpha\mathbf{h}_\alpha)=\mathbf{t}_R} \mathrm{lt}_\prec(R_\alpha\mathbf{h}_\alpha) \in \mathrm{Syz}(\mathrm{lt}_\prec\mathcal{H})\ . \tag{29}$$

According to our assumption $\tilde{\mathcal{S}}$ is a generating set of $\mathrm{Syz}(\mathrm{lt}_\prec\mathcal{H})$, so that we may write $\tilde{\mathbf{R}} = \sum_{\tilde{\mathbf{S}}\in\tilde{\mathcal{S}}} a_{\tilde{\mathbf{S}}}\tilde{\mathbf{S}}$ for some coefficients $a_{\tilde{\mathbf{S}}} \in \mathcal{P}$. Let us now consider the

[14] If $\alpha > \beta$, then we understand that $\mathbf{S}_{\beta\alpha} \in \mathcal{S}$ etc.

syzygy $\mathbf{R}' = \mathbf{R} - \sum_{\mathbf{S}\in\mathcal{S}} a_{\tilde{\mathbf{S}}}\mathbf{S}$. Obviously, $\mathbf{t}_{\mathbf{R}'} \prec \mathbf{t}_{\mathbf{R}}$. By iteration we obtain thus in a finite number of steps a representation $\mathbf{R} = \sum_{\mathbf{S}\in\mathcal{S}} b_{\mathbf{S}}\mathbf{S}$ and thus $\mathcal{S}$ generates the module $\mathrm{Syz}(\mathcal{H})$. □

As a consequence of this simple lemma, we can now show that each involutive basis yields immediately a Gröbner basis of the first syzygy module. In fact, this basis is automatically computed during the determination of the involutive basis with the completion Algorithm 3 of Part I. This is completely analogously to the automatic determination of $\mathcal{H}_{\mathrm{Schreyer}}$ with the Buchberger algorithm.

Theorem 5.4 *Let $\mathcal{H}$ be an involutive basis for the involutive division L and the term order $\prec$. Then the set $\mathcal{H}_{\mathrm{Syz}}$ is a Gröbner basis of the syzygy module $\mathrm{Syz}(\mathcal{H})$ for the term order $\prec_{\mathcal{H}}$.*

Proof Without loss of generality, we may assume that $\mathcal{H}$ is a monic basis, i. e. all leading coefficients are 1. Let $\mathbf{S}_{\alpha;k} \in \mathcal{H}_{\mathrm{Syz}}$. As $\mathcal{H}$ is an involutive basis, the unique polynomials $P_\beta^{(\alpha;k)}$ in (26) satisfy $\mathrm{lt}_\prec(P_\beta^{(\alpha;k)}\mathbf{h}_\beta) \preceq \mathrm{lt}_\prec(x_k\mathbf{h}_\alpha)$ and there exists only one index $\bar\beta$ such that $\mathrm{lt}_\prec(P_{\bar\beta}^{(\alpha;k)}\mathbf{h}_{\bar\beta}) = \mathrm{lt}_\prec(x_k\mathbf{h}_\alpha)$. It is easy to see that we have $\mathbf{S}_{\alpha;k} = \mathbf{S}_{\alpha\bar\beta}$. Thus $\mathcal{H}_{\mathrm{Syz}} \subseteq \mathcal{H}_{\mathrm{Schreyer}}$.

Let $\mathbf{S}_{\alpha\beta} \in \mathcal{H}_{\mathrm{Schreyer}} \setminus \mathcal{H}_{\mathrm{Syz}}$ be an arbitrary syzygy. We prove first that the set $\mathcal{H}_{\mathrm{Schreyer}} \setminus \{\mathbf{S}_{\alpha\beta}\}$ still generates $\mathrm{Syz}(\mathcal{H})$. Any syzygy in $\mathcal{H}_{\mathrm{Schreyer}}$ has the form $\mathbf{S}_{\alpha\beta} = x^\mu\mathbf{e}_\alpha - x^\nu\mathbf{e}_\beta + \mathbf{R}_{\alpha\beta}$. By construction, one of the monomials x^μ and x^ν must contain a non-multiplicative variable x_k for $\mathbf{h}_\alpha$ or $\mathbf{h}_\beta$, respectively. Without loss of generality, we assume that $x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$ and $\mu_k > 0$. This implies that $\mathcal{H}_{\mathrm{Syz}}$ contains the syzygy $\mathbf{S}_{\alpha;k}$. As shown above, a unique index $\gamma \neq \beta$ exists such that $\mathbf{S}_{\alpha;k} = \mathbf{S}_{\alpha\gamma}$.

Let $\mathbf{S}_{\alpha\gamma} = x_k\mathbf{e}_\alpha - x^\rho\mathbf{e}_\gamma + \mathbf{R}_{\alpha\gamma}$. By construction, $x^\rho\mathbf{t}_\gamma = x_k\mathbf{t}_\alpha$ divides $x^\mu\mathbf{t}_\alpha = \mathbf{t}_{\alpha\beta}$. Thus $\mathbf{t}_\gamma \mid \mathbf{t}_{\alpha\beta}$ and by Lemma 5.3 the set $\mathcal{H}_{\mathrm{Schreyer}} \setminus \{\mathbf{S}_{\alpha\beta}\}$ still generates $\mathrm{Syz}(\mathcal{H})$. If we try to iterate this argument, we encounter the following problem. In order to be able to eliminate $\mathbf{S}_{\alpha\beta}$ we need both $\mathbf{S}_{\alpha\gamma}$ and $\mathbf{S}_{\beta\gamma}$ in the remaining set. For $\mathbf{S}_{\alpha\gamma} \in \mathcal{H}_{\mathrm{Syz}}$, this is always guaranteed. But we know nothing about $\mathbf{S}_{\beta\gamma}$ and, if it is not an element of $\mathcal{H}_{\mathrm{Syz}}$, it could have been removed in an earlier iteration.

We claim that with respect to the term order $\prec_{\mathcal{H}}$ the term $\mathrm{lt}_{\prec_{\mathcal{H}}}\mathbf{S}_{\alpha\beta}$ is greater than both $\mathrm{lt}_{\prec_{\mathcal{H}}}\mathbf{S}_{\alpha\gamma}$ and $\mathrm{lt}_{\prec_{\mathcal{H}}}\mathbf{S}_{\beta\gamma}$. Without loss of generality, we may assume for simplicity that $\alpha < \beta < \gamma$, as the syzygies $\mathbf{S}_{\alpha\beta}$ and $\mathbf{S}_{\beta\alpha}$ differ only by a sign. Thus $\mathrm{lt}_{\prec_{\mathcal{H}}}\mathbf{S}_{\alpha\beta} = \frac{\mathbf{t}_{\alpha\beta}}{\mathbf{t}_\alpha}\mathbf{e}_\alpha$ and similarly for $\mathbf{S}_{\alpha\gamma}$ and $\mathbf{S}_{\beta\gamma}$. Furthermore, $\mathbf{t}_\gamma \mid \mathbf{t}_{\alpha\beta}$ trivially implies $\mathbf{t}_{\alpha\gamma} \mid \mathbf{t}_{\alpha\beta}$ and hence $\mathbf{t}_{\alpha\gamma} \prec \mathbf{t}_{\alpha\beta}$ for any term order $\prec$. Obviously, the same holds for $\mathbf{t}_{\beta\gamma}$. Now a straightforward application of the definition of the term order $\prec_{\mathcal{H}}$ proves our claim.

Thus if we always remove the syzygy $\mathbf{S}_{\alpha\beta} \in \mathcal{H}_{\mathrm{Schreyer}} \setminus \mathcal{H}_{\mathrm{Syz}}$ whose leading term is maximal with respect to the term order $\prec_{\mathcal{H}}$, it can never happen that the syzygy $\mathbf{S}_{\beta\gamma}$ required for the application of Lemma 5.3 has already been eliminated earlier and $\mathcal{H}_{\mathrm{Syz}}$ is a generating set of $\mathrm{Syz}(\mathcal{H})$.

It is a simple corollary of Schreyer's theorem that $\mathcal{H}_{\mathrm{Syz}}$ is even a Gröbner basis of $\mathrm{Syz}(\mathcal{H})$. Indeed, we know that $\mathcal{H}_{\mathrm{Schreyer}}$ is a Gröbner basis of $\mathrm{Syz}(\mathcal{H})$ for the term order $\prec_{\mathcal{H}}$ and it follows from our considerations above that whenever we remove a syzygy $\mathbf{S}_{\alpha\beta}$ we still have in the remaining set at least one syzygy whose leading term divides $\mathrm{lt}_{\prec_{\mathcal{H}}}\mathbf{S}_{\alpha\beta}$. Thus we find

$$\langle \mathrm{lt}_{\prec_{\mathcal{H}}}(\mathcal{H}_{\mathrm{Syz}})\rangle = \langle \mathrm{lt}_{\prec_{\mathcal{H}}}(\mathcal{H}_{\mathrm{Schreyer}})\rangle = \mathrm{lt}_{\prec_{\mathcal{H}}}\mathrm{Syz}(\mathcal{H}) \tag{30}$$

which proves our assertion. $\square$

This result is not completely satisfying, as it only yields a Gröbner and not an involutive basis of the syzygy module. The latter seems to be hard to achieve for arbitrary divisions L. For some divisions it is possible with a little effort. The key idea is that in the order $\prec_{\mathcal{H}}$ the numbering of the generators in $\mathcal{H}$ is important and we must choose the right one. For this purpose we generalise a construction of Plesken and Robertz [58] for the special case of a Janet basis.

We associate a directed graph with each involutive basis $\mathcal{H}$. Its vertices are given by the elements in $\mathcal{H}$. If $x_j \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h})$ for some generator $\mathbf{h} \in \mathcal{H}$, then, by definition of an involutive basis, $\mathcal{H}$ contains a unique generator $\bar{\mathbf{h}}$ such that $\mathrm{le}_{\prec}\bar{\mathbf{h}}$ is an involutive divisor of $\mathrm{le}_{\prec}(x_j\mathbf{h})$. In this case we include a directed edge from $\mathbf{h}$ to $\bar{\mathbf{h}}$. The thus defined graph is called the *L-graph* of the basis $\mathcal{H}$.

Lemma 5.5 *If the division L is continuous, then the L-graph of any involutive set $\mathcal{H} \subset \mathcal{P}$ is acyclic.*

Proof The leading exponents of the vertices of a path in an L-graph define a sequence as in the definition of a continuous division. If the path is a cycle, then the sequence contains identical elements contradicting the continuity of L. $\square$

We order the elements of $\mathcal{H}$ as follows: whenever the L-graph of $\mathcal{H}$ contains a path from $\mathbf{h}_\alpha$ to $\mathbf{h}_\beta$, then we must have $\alpha < \beta$. Any ordering satisfying this condition is called an *L-ordering*. Note that by the lemma above for a continuous division L-orderings always exist (although they are in general not unique).

For the Pommaret division P it is easy to describe explicitly a P-ordering without using the P-graph: we require that if either $\mathrm{cls}\,\mathbf{h}_\alpha < \mathrm{cls}\,\mathbf{h}_\beta$ or $\mathrm{cls}\,\mathbf{h}_\alpha = \mathrm{cls}\,\mathbf{h}_\beta = k$ and and the last non-vanishing entry of $\mathrm{le}_{\prec}\mathbf{h}_\alpha - \mathrm{le}_{\prec}\mathbf{h}_\beta$ is negative, then we must have $\alpha < \beta$. Thus we sort the generators $\mathbf{h}_\alpha$ first by their class and within each class lexicographically (according to our definition in Appendix A of Part I). It is straightforward to verify that this defines indeed a P-ordering.

Example 5.6 Let us consider the ideal $\mathcal{I} \subset \Bbbk[x,y,z]$ generated by the six polynomials $h_1 = x^2$, $h_2 = xy$, $h_3 = xz - y$, $h_4 = y^2$, $h_5 = yz - y$ and $h_6 = z^2 - z + x$. One easily verifies that they form a Pommaret basis $\mathcal{H}$ for the degree reverse lexicographic order. The corresponding P-graph has the following form

$$\begin{tikzcd}
 & h_2 \arrow[r] \arrow[dr] & h_4 \arrow[d] \\
h_1 \arrow[ur] \arrow[dr] & & h_5 \arrow[d] \\
 & h_3 \arrow[r] \arrow[ur] & h_6
\end{tikzcd} \tag{31}$$

One clearly sees that the generators are already P-ordered, namely according to the description above. ◁

The decisive observation about an L-ordering is that we can now easily determine the leading terms of all syzygies $\mathcal{S}_{\alpha;k} \in \mathcal{H}_{\mathrm{Syz}}$ for the Schreyer order $\prec_{\mathcal{H}}$.

Lemma 5.7 *Let the elements of the involutive basis $\mathcal{H} \subset \mathcal{P}$ be ordered according to an L-ordering. Then the syzygies $\mathcal{S}_{\alpha;k}$ satisfy $\mathrm{lt}_{\prec_{\mathcal{H}}}\mathcal{S}_{\alpha;k} = x_k\mathbf{e}_\alpha$.*

Proof By the properties of the involutive standard representation, we have in (26) $\mathrm{lt}_\prec(P_\beta^{(\alpha;k)}\mathbf{h}_\beta) \preceq \mathrm{lt}_\prec(x_k\mathbf{h}_\alpha)$ for all β and only one index $\bar\beta$ exists for which $\mathrm{lt}_\prec(P_{\bar\beta}^{(\alpha;k)}\mathbf{h}_{\bar\beta}) = \mathrm{lt}_\prec(x_k\mathbf{h}_\alpha)$. Thus $\mathrm{le}_\prec\mathbf{h}_{\bar\beta}$ is an involutive divisor of $\mathrm{le}_\prec(x_k\mathbf{h}_\alpha)$ and the L-graph of $\mathcal{H}$ contains an edge from $\mathbf{h}_\alpha$ to $\mathbf{h}_{\bar\beta}$. In an L-ordering, this implies $\alpha < \bar\beta$. Now the assertion follows immediately from the definition of the term order $\prec_{\mathcal{H}}$. □

There remains the problem of controlling the multiplicative variables associated to these leading terms by the involutive division L. For arbitrary divisions it does not seem possible to make any statement. Thus we simply define a class of involutive divisions with the desired properties and show afterwards that at least the Janet and the Pommaret division belong to this class.

Definition 5.8 *An involutive division L is of* Schreyer type *for the term order $\prec$, if for any set $\mathcal{H}$ which is involutive with respect to L and $\prec$ all sets $\bar X_{L,\mathcal{H},\prec}(\mathbf{h})$ with $\mathbf{h} \in \mathcal{H}$ are again involutive.*

Lemma 5.9 *Both the Janet and the Pommaret division are of Schreyer type for any term order $\prec$.*

Proof For the Janet division any set of variables, i. e. monomials of degree one, is involutive. Indeed, let $\mathcal{F}$ be such a set and $x_k \in \mathcal{F}$, then

$$X_{J,\mathcal{F}}(x_k) = \{x_i \mid x_i \notin \mathcal{F} \vee i \leq k\} \tag{32}$$

which immediately implies the assertion. For the Pommaret division sets of non-multiplicative variables are always of the form $\mathcal{F} = \{x_k, x_{k+1}, \dots, x_n\}$ and such a set is trivially involutive. □

An example of an involutive division which is not of Schreyer type is the *Thomas division* T [76] defined as follows: let $\mathcal{N} \subset \mathbb{N}_0^n$ be a finite set and $\nu \in \mathcal{N}$ an arbitrary element; then $i \in N_{T,\mathcal{N}}(\nu)$, if and only if $\nu_i = \max_{\mu\in\mathcal{N}} \mu_i$ (obviously, one may consider the Janet division as a kind of refinement of the Thomas division). One easily sees that no set consisting only of variables can be involutive for the Thomas division so that it cannot be of Schreyer type.

Theorem 5.10 *Let L be a continuous involutive division of Schreyer type for the term order $\prec$ and $\mathcal{H}$ an L-ordered involutive basis of the polynomial module $\mathcal{M}$ with respect to L and $\prec$. Then $\mathcal{H}_{\mathrm{Syz}}$ is an involutive basis of $\mathrm{Syz}(\mathcal{H})$ with respect to L and the term order $\prec_{\mathcal{H}}$.*

Proof By Lemma 5.7, the leading term of $\mathbf{S}_{\alpha;k} \in \mathcal{H}_{\mathrm{Syz}}$ is $x_k\mathbf{e}_\alpha$ and we have one such generator for each non-multiplicative variable $x_k \in \bar{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$. Since we assume that L is of Schreyer type for $\prec$, these leading terms form an involutive set. As we know already from Theorem 5.4 that $\mathcal{H}_{\mathrm{Syz}}$ is a Gröbner basis of $\mathrm{Syz}(\mathcal{H})$, the assertion follows trivially. □

Note that under the made assumptions it follows immediately from the simple form of the leading terms that $\mathcal{H}_{\mathrm{Syz}}$ is a minimal Gröbner basis of $\mathrm{Syz}(\mathcal{H})$.

Example 5.11 We continue with Example 5.6. As all assumption of Theorem 5.10 are satisfied, the eight syzygies

$$\mathbf{S}_{1;3} = z\mathbf{e}_1 - x\mathbf{e}_3 - \mathbf{e}_2 \ , \tag{33a}$$
$$\mathbf{S}_{2;3} = z\mathbf{e}_2 - x\mathbf{e}_5 - \mathbf{e}_2 \ , \tag{33b}$$
$$\mathbf{S}_{3;3} = z\mathbf{e}_3 - x\mathbf{e}_6 + \mathbf{e}_5 - \mathbf{e}_3 + \mathbf{e}_1 \ , \tag{33c}$$
$$\mathbf{S}_{4;3} = z\mathbf{e}_4 - y\mathbf{e}_5 - \mathbf{e}_4 \ , \tag{33d}$$
$$\mathbf{S}_{5;3} = z\mathbf{e}_5 - y\mathbf{e}_6 + \mathbf{e}_2 \ , \tag{33e}$$
$$\mathbf{S}_{1;2} = y\mathbf{e}_1 - x\mathbf{e}_2 \ , \tag{33f}$$
$$\mathbf{S}_{2;2} = y\mathbf{e}_2 - x\mathbf{e}_4 \ , \tag{33g}$$
$$\mathbf{S}_{3;2} = y\mathbf{e}_3 - x\mathbf{e}_5 + \mathbf{e}_4 - \mathbf{e}_2 \tag{33h}$$

form a Pommaret basis of the syzygy module $\mathrm{Syz}(\mathcal{H})$ with respect to the induced term order $\prec_{\mathcal{H}}$. Indeed, as

$$z\mathbf{S}_{1;2} = y\mathbf{S}_{1;3} - x\mathbf{S}_{2;3} + x\mathbf{S}_{4;2} + \mathbf{S}_{2;2} \ , \tag{34a}$$
$$z\mathbf{S}_{2;2} = y\mathbf{S}_{2;3} - x\mathbf{S}_{4;3} + \mathbf{S}_{2;2} \ , \tag{34b}$$
$$z\mathbf{S}_{3;2} = y\mathbf{S}_{3;3} - x\mathbf{S}_{5;3} - \mathbf{S}_{2;3} + \mathbf{S}_{4;3} + \mathbf{S}_{3;2} - \mathbf{S}_{1;2} \ , \tag{34c}$$

all products of the generators with their non-multiplicative variables possess an involutive standard representation. ◁

6 Free Resolutions I: The Polynomial Case

As Theorem 5.10 yields again an involutive basis of the syzygy module, we may apply it repeatedly and construct this way a resolution for any polynomial submodule $\mathcal{M} \subseteq \mathcal{P}^m$ given an involutive basis of it for an involutive division of Schreyer type. We specialise now to Pommaret bases where one can even make a number of statements about the size of the resolution. In particular, we immediately obtain a stronger form of Hilbert's Syzygy Theorem as a corollary (in fact, we will see later that we get the strongest possible form, as the arising free resolution is always of minimal length).

Theorem 6.1 *Let $\mathcal{H}$ be a Pommaret basis of the polynomial submodule $\mathcal{M} \subseteq \mathcal{P}^m$. If we denote by $\beta_0^{(k)}$ the number of generators $\mathbf{h} \in \mathcal{H}$ such that $\mathrm{cls}\,\mathrm{le}_\prec\mathbf{h} = k$ and set $d = \min\{k \mid \beta_0^{(k)} > 0\}$, then $\mathcal{M}$ possesses a finite free resolution*

$$0 \longrightarrow \mathcal{P}^{r_{n-d}} \longrightarrow \cdots \longrightarrow \mathcal{P}^{r_1} \longrightarrow \mathcal{P}^{r_0} \longrightarrow \mathcal{M} \longrightarrow 0 \tag{35}$$

of length $n - d$ where the ranks of the free modules are given by

$$r_i = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_0^{(k)} \ . \tag{36}$$

Proof According to Theorem 5.10, $\mathcal{H}_{\mathrm{Syz}}$ is a Pommaret basis of $\mathrm{Syz}(\mathcal{H})$ for the term order $\prec_{\mathcal{H}}$. Applying the theorem again, we can construct a Pommaret basis of the second syzygy module $\mathrm{Syz}^2(\mathcal{H})$ and so on. In the proof of Theorem 5.10 we showed that $\mathrm{le}_{\prec_{\mathcal{H}}} \mathbf{S}_{\alpha;k} = x_k \mathbf{e}_\alpha$. Hence $\mathrm{cls}\, \mathbf{S}_{\alpha;k} = k > \mathrm{cls}\, \mathbf{h}_\alpha$ and if d is the minimal class of a generator in $\mathcal{H}$, then the minimal class in $\mathcal{H}_{\mathrm{Syz}}$ is $d+1$. This yields the length of the resolution (35), as a Pommaret basis with $d = n$ generates a free submodule.

The ranks of the modules follow from a rather straightforward combinatorial calculation. Let $\beta_i^{(k)}$ denote the number of generators of class k of the ith syzygy module $\mathrm{Syz}^i(\mathcal{H})$. By definition of the generators $\mathbf{S}_{\alpha;k}$, we find $\beta_i^{(k)} = \sum_{j=1}^{k-1} \beta_{i-1}^{(j)}$, as each generator of class less than k in the Pommaret basis of $\mathrm{Syz}^{i-1}(\mathcal{H})$ contributes one generator of class k to the basis of $\mathrm{Syz}^i(\mathcal{H})$. A simple induction allows us to express the $\beta_i^{(k)}$ in terms of the $\beta_0^{(k)}$:

$$\beta_i^{(k)} = \sum_{j=1}^{k-i} \binom{k-j-1}{i-1} \beta_0^{(j)} \ . \tag{37}$$

The ranks of the modules in (35) are given by $r_i = \sum_{k=1}^{n} \beta_i^{(k)}$; entering (37) yields via a classical identity for binomial coefficients (36). □

Remark 6.2 Theorem 6.1 remains valid for any involutive basis $\mathcal{H}$ with respect to a continuous division of Schreyer type, if we define $\beta_0^{(k)}$ (respectively $\beta_i^{(k)}$ in the proof) as the number of generators with k multiplicative variables, since Theorem 5.10 holds for any such basis. Indeed, after the first step we always analyse monomial sets of the form $\{x_{i_1}, x_{i_2}, \dots, x_{i_{n-k}}\}$ with $i_1 < i_2 < \cdots < i_{n-k}$. By assumption, these sets are involutive and this is only possible, if one of the generators possesses n multiplicative variables, another one $n-1$ and so on until the last generator which has only $n-k$ multiplicative variables (this fact follows for example from Proposition 3.2 on the form of the Hilbert series). Hence the basic recursion relation $\beta_i^{(k)} = \sum_{j=1}^{k-1} \beta_{i-1}^{(j)}$ and all subsequent combinatorial computations remain valid for any division of Schreyer type.

For the special case of the Janet division, Plesken and Robertz [58] proved directly the corresponding statement. Here it is straightforward to determine explicitly the multiplicative variables for any syzygy: if h_α is a generator in the Janet basis $\mathcal{H}$ with the non-multiplicative variables $\bar{X}_{J,\mathcal{H},\prec}(h_\alpha) = \{x_{i_1}, x_{i_2}, \dots, x_{i_{n-k}}\}$ where $i_1 < i_2 < \cdots < i_{n-k}$, then

$$X_{J,\mathcal{H}_{\mathrm{Syz}},\prec}(\mathbf{S}_{\alpha;i_j}) = \{x_1, \dots, x_n\} \setminus \{x_{i_{j+1}}, x_{i_{j+2}}, \dots, x_{i_{n-k}}\} \ , \tag{38}$$

as one easily verifies. ◁

As in general the resolution (35) is not minimal, the ranks r_i appearing in it cannot be identified with the Betti numbers of the module $\mathcal{M}$. However, they obviously represent an upper bound for them. With a little bit more effort one can easily derive similar bounds even for the multigraded Betti numbers; we leave this task as an exercise for the reader.

We may explicitly write the syzygy resolution (35) as a complex. Let $\mathcal{W}$ be a free $\mathcal{P}$-module with basis $\{w_1, \dots, w_p\}$, i. e. its rank is given by the size of the Pommaret basis $\mathcal{H}$. Let $\mathcal{V}$ be a further free $\mathcal{P}$-module with basis $\{v_1, \dots, v_n\}$, i. e. its rank is determined by the number of variables in $\mathcal{P}$, and denote by $\Lambda\mathcal{V}$ the exterior algebra over $\mathcal{V}$. We set $\mathcal{C}_i = \mathcal{W} \otimes_{\mathcal{P}} \Lambda^i \mathcal{V}$ for $0 \leq i \leq n$. If $\mathbf{k} = (k_1, \dots, k_i)$ is a sequence of integers with $1 \leq k_1 < k_2 < \cdots < k_i \leq n$ and $v_{\mathbf{k}}$ denotes the wedge product $v_{k_1} \wedge \cdots \wedge v_{k_i}$, then a basis of this free $\mathcal{P}$-module is given by the set of all tensor products $w_\alpha \otimes v_{\mathbf{k}}$. Finally, we introduce the submodule $\mathcal{S}_i \subset \mathcal{C}_i$ generated by all those basis elements where $k_1 > \operatorname{cls} \mathbf{h}_\alpha$. Note that the rank of $\mathcal{S}_i$ is precisely r_i as defined by (36).

We denote the elements of the Pommaret basis of $\operatorname{Syz}^i(\mathcal{H})$ by $\mathbf{S}_{\alpha;\mathbf{k}}$ with the inequalities $\operatorname{cls} \mathbf{h}_\alpha < k_1 < \cdots < k_i$. An involutive normal form computation determines for every non-multiplicative index $n \geq k_{i+1} > k_i = \operatorname{cls} \mathbf{S}_{\alpha;\mathbf{k}}$ unique polynomials $P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k},k_{i+1})} \in \Bbbk[x_1, \dots, x_{\ell_i}]$ such that

$$x_{k_{i+1}} \mathbf{S}_{\alpha;\mathbf{k}} = \sum_{\beta=1}^{p} \sum_{\boldsymbol{\ell}} P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k},k_{i+1})} \mathbf{S}_{\beta;\boldsymbol{\ell}} \tag{39}$$

where the second sum is over all integer sequences $\boldsymbol{\ell} = (\ell_1, \dots, \ell_i)$ satisfying $\operatorname{cls} \mathbf{h}_\beta < \ell_1 < \cdots < \ell_i \leq n$. Now we define the $\mathcal{P}$-module homomorphisms $\epsilon : \mathcal{S}_0 \to \mathcal{M}$ and $\delta : \mathcal{S}_{i+1} \to \mathcal{S}_i$ by $\epsilon(w_\alpha) = \mathbf{h}_\alpha$ and

$$\delta(w_\alpha \otimes v_{\mathbf{k},k_{i+1}}) = x_{k_{i+1}} w_\alpha \otimes v_{\mathbf{k}} - \sum_{\beta,\boldsymbol{\ell}} P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k},k_{i+1})} w_\beta \otimes v_{\boldsymbol{\ell}} \,. \tag{40}$$

We extend the differential δ to a map $\mathcal{C}_{i+1} \to \mathcal{C}_i$ as follows. If $k_i \leq \operatorname{cls} \mathbf{h}_\alpha$, then we set $\delta(w_\alpha \otimes v_{\mathbf{k}}) = 0$. Otherwise let j be the smallest value such that $k_j > \operatorname{cls} \mathbf{h}_\alpha$ and set (by slight abuse of notation)

$$\delta(w_\alpha \otimes v_{k_1} \wedge \cdots \wedge v_{k_i}) = v_{k_1} \wedge \cdots \wedge v_{k_{j-1}} \wedge \delta(w_\alpha \otimes v_{k_j} \wedge \cdots \wedge v_{k_i}) \,. \tag{41}$$

Thus the factor $v_{k_1} \wedge \cdots \wedge v_{k_{j-1}}$ remains simply unchanged and does not affect the differential. This definition makes, by construction, $(\mathcal{C}_*, \delta)$ to a complex and $(\mathcal{S}_*, \delta)$ to an exact subcomplex which (augmented by the map $\epsilon : \mathcal{S}_0 \to \mathcal{M}$) is isomorphic to the syzygy resolution (35).

Example 6.3 We continue with the ideal of Example 5.6 and 5.11, respectively. As here $d = 1$, the resolution has length 2 in this case. Using the notation introduced above, the module $\mathcal{S}_0$ is then generated by $\{w_1, \dots, w_6\}$, the module $\mathcal{S}_1$ by the eight elements $\{w_1 \otimes v_3, \dots, w_5 \otimes v_3, w_1 \otimes v_2, \dots, w_3 \otimes v_2\}$ (the first three generators in the Pommaret basis $\mathcal{H}$ are of class 1, the next two of class 2 and the final one of class 3) and the module $\mathcal{S}_2$ by $\{w_1 \otimes v_2 \wedge v_3, \dots, w_3 \otimes v_2 \wedge v_3\}$

corresponding to the three first syzygies of class 2. It follows from the expressions (33) and (34), respectively, for the first and second syzygies that the differential δ is here defined by the relations

$$
\begin{aligned}
\delta(w_1 \otimes v_3) &= zw_1 - xw_3 - w_2 \ , && \text{(42a)}\\
\delta(w_2 \otimes v_3) &= zw_2 - xw_5 - w_2 \ , && \text{(42b)}\\
\delta(w_3 \otimes v_3) &= zw_3 - xw_6 + w_5 - w_3 + w_1 \ , && \text{(42c)}\\
\delta(w_4 \otimes v_3) &= zw_4 - yw_5 - w_4 \ , && \text{(42d)}\\
\delta(w_5 \otimes v_3) &= zw_5 - yw_6 + w_2 \ , && \text{(42e)}\\
\delta(w_3 \otimes v_2) &= yw_3 - xw_5 + w_4 - w_2 \ , && \text{(42f)}\\
\delta(w_2 \otimes v_2) &= yw_2 - xw_4 \ , && \text{(42g)}\\
\delta(w_1 \otimes v_2) &= yw_1 - xw_2 \ , && \text{(42h)}\\
\delta(w_1 \otimes v_2 \wedge v_3) &= zw_1 \otimes v_2 - yw_1 \otimes v_3 + xw_2 \otimes v_3 - && \text{(42i)}\\
& xw_3 \otimes v_2 - w_2 \otimes v_2 \ . &&\\
\delta(w_2 \otimes v_2 \wedge v_3) &= zw_2 \otimes v_2 - yw_2 \otimes v_3 + xw_4 \otimes v_3 - w_2 \otimes v_2 \ , && \text{(42j)}\\
\delta(w_3 \otimes v_2 \wedge v_3) &= zw_3 \otimes v_2 - yw_3 \otimes v_3 + xw_5 \otimes v_3 + && \text{(42k)}\\
& w_2 \otimes v_3 - w_4 \otimes v_3 - w_3 \otimes v_2 + w_1 \otimes v_2 \ , &&
\end{aligned}
$$

It represents a straightforward albeit rather tedious task to verify explicitly the exactness of the thus constructed complex $(\mathcal{S}_*, \delta)$. $\triangleleft$

In the case that $m = 1$ and thus $\mathcal{M}$ is actually an ideal in $\mathcal{P}$, it is tempting to try to equip the complex $(\mathcal{C}_*, \delta)$ with the structure of a differential algebra. We first introduce a multiplication $\times$ on $\mathcal{W}$. If h_α and h_β are two elements of the Pommaret basis $\mathcal{H}$, then their product possesses a unique involutive standard representation $h_\alpha h_\beta = \sum_{\gamma=1}^{p} P_{\alpha\beta\gamma} h_\gamma$ and we define

$$w_\alpha \times w_\beta = \sum_{\gamma=1}^{p} P_{\alpha\beta\gamma} w_\gamma \tag{43}$$

and continue $\mathcal{P}$-linearly on $\mathcal{W}$. This multiplication can be extended to the whole complex $\mathcal{C}_*$ by defining for arbitrary elements $w, \bar{w} \in \mathcal{W}$ and $\omega, \bar{\omega} \in \Lambda\mathcal{V}$

$$(w \otimes \omega) \times (\bar{w} \otimes \bar{\omega}) = (w \times \bar{w}) \otimes (\omega \wedge \bar{\omega}) \ . \tag{44}$$

The distributivity of $\times$ is obvious from its definition. For obtaining a differential algebra, the product $\times$ must furthermore be associative and satisfy the graded Leibniz rule $\delta(a \times b) = \delta(a) \times b + (-1)^{|a|} a \times \delta(b)$ where $|a|$ denotes the form degree of a. While in general both conditions are not met, a number of special situations exist where one indeed obtains a differential algebra.

Let us first consider the associativity. It suffices to study it at the level of $\mathcal{W}$ where we find that

$$w_\alpha \times (w_\beta \times w_\gamma) = \sum_{\delta,\epsilon=1}^{p} P_{\beta\gamma\delta} P_{\alpha\delta\epsilon} w_\epsilon \ , \tag{45a}$$

$$(w_\alpha \times w_\beta) \times w_\gamma = \sum_{\delta,\epsilon=1}^{p} P_{\alpha\beta\delta} P_{\gamma\delta\epsilon} w_\epsilon \ . \tag{45b}$$

One easily checks that both $\sum_{\delta,\epsilon=1}^{p} P_{\beta\gamma\delta} P_{\alpha\delta\epsilon} h_\epsilon$ and $\sum_{\delta,\epsilon=1}^{p} P_{\alpha\beta\delta} P_{\gamma\delta\epsilon} h_\epsilon$ are standard representations of the product $h_\alpha h_\beta h_\gamma$ for the Pommaret basis $\mathcal{H}$. However, we cannot conclude that they are involutive standard representations, as we do not know whether $P_{\beta\gamma\delta}$ and $P_{\alpha\beta\delta}$, respectively, are multiplicative for h_ϵ. If this was the case, then the associativity would follow immediately from the uniqueness of involutive standard representations.

For the graded Leibniz rule the situation is similar but more involved. In the next section we will discuss it in more details for the monomial case. In the end, it boils down to analysing standard representations for products of the form $x_k h_\alpha h_\beta$. Again there exist two different ways for obtaining them and a sufficient condition for the satisfaction of the Leibniz rule is that both lead always to the unique involutive standard representation.

Example 6.4 Let us analyse the by now familiar ideal $\mathcal{I} \subset \Bbbk[x, y, z]$ generated by $h_1 = y^2 - z$, $h_2 = yz - x$ and $h_3 = z^2 - xy$. We showed already in Part I (Example 5.10) that these polynomials form a Pommaret basis of $\mathcal{I}$ for the degree reverse lexicographic term order. The Pommaret basis of the first syzygy module consists of $\mathbf{S}_{1;3} = z\mathbf{e}_1 - y\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{S}_{2;3} = z\mathbf{e}_2 - y\mathbf{e}_3 - x\mathbf{e}_1$. As both generators are of class 3, this is a free module and the resolution stops here.

In a straightforward calculation one obtains for the multiplication $\times$ the following defining relations:

$$w_1^2 = w_3 - yw_2 + y^2 w_1 \ , \quad w_1 \times w_2 = -yw_3 + y^2 w_2 - xw_1 \ , \tag{46a}$$

$$w_1 \times w_3 = (y^2 - z)w_3 \ , \quad w_2^2 = y^2 w_3 - xw_2 + xyw_1 \ , \tag{46b}$$

$$w_2 \times w_3 = (yz - x)w_3 \ , \quad w_3^2 = (z^2 - xy)w_3 \ . \tag{46c}$$

Note that all coefficients of w_1 and w_2 are contained in $\Bbbk[x, y]$ and are thus multiplicative for all generators. This observation immediately implies that our multiplication is associative, as any way to evaluate the product $w_\alpha \times w_\beta \times w_\gamma$ leads to the unique involutive standard representation of $h_\alpha h_\beta h_\gamma$.

As furthermore in the only two non-multiplicative products $zh_2 = yh_3 + xh_1$ and $zh_1 = yh_2 + h_3$ all coefficients on the right hand sides lie in $\Bbbk[x, y]$, too, it follows from the same line of reasoning that the differential satisfies the Leibniz rule and we have a differential algebra. $\lhd$

The situation is not always as favourable as in this example. The next example shows that in general we cannot expect to obtain a differential algebra (in fact, not even an associative algebra).

Example 6.5 We continue with the ideal of Examples 5.6, 5.11 and 6.3. Evaluation of the defining relation (43) is particularly simple for the products of the form $w_i \times w_6 = h_i w_6$, as all variables are multiplicative for the generator h_6. Two further products are $w_5^2 = y^2 w_6 - y w_5 - x w_4$ and $w_3 \times w_5 = xyw_6 - yw_5 - xw_2$. In a straightforward computation one finds

$$(w_3 \times w_5) \times w_5 - w_3 \times w_5^2 = x^2 w_4 - xyw_2 \ , \tag{47}$$

so that the multiplication is not associative. Note that the difference corresponds to the syzygy $x^2 h_4 - xyh_2 = 0$. This result is not surprising, as it encodes the difference between two standard representations of $h_3 h_5^2$. The reason for the non-associativity lies in the coefficient y of w_5 in the power w_5^2; it is non-multiplicative for h_2 and the generator w_2 appears in the product $w_3 \times w_5$. Hence computing $w_3 \times w_5^2$ does not lead to an involutive standard representation of $h_3 h_5^2$ whereas the product $(w_3 \times w_5) \times w_5$ does. $\triangleleft$

7 Free Resolutions II: The Monomial Case

For monomial modules it is possible to obtain a closed form of the differential (40) based only on the set $\mathcal{H}$ and thus to generalise results by Eliahou and Kervaire [27] for stable ideals. The existence of a Pommaret basis is now a non-trivial assumption, as the property of being monomial is not invariant under coordinate transformations. Thus we always assume in the sequel that we are dealing with a quasi-stable submodule $\mathcal{M} \subseteq \mathcal{P}^m$. Let $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_p\}$ with $\mathbf{h}_\alpha \in \mathbb{T}^m$ be its monomial Pommaret basis (by Proposition 2.11 of Part I, it is unique). Furthermore, we introduce the function $\Delta(\alpha, k)$ determining the unique generator in the Pommaret basis $\mathcal{H}$ such that $x_k \mathbf{h}_\alpha = t_{\alpha,k} \mathbf{h}_{\Delta(\alpha,k)}$ with a term $t_{\alpha,k} \in \Bbbk[X_P(\mathbf{h}_{\Delta(\alpha,k)})]$.

Lemma 7.1 *The function Δ and the terms $t_{\alpha,k}$ satisfy the following relations.*

(i) *The inequality $\operatorname{cls} \mathbf{h}_\alpha \leq \operatorname{cls} \mathbf{h}_{\Delta(\alpha,k)} \leq k$ holds for all non-multiplicative indices $k > \operatorname{cls} \mathbf{h}_\alpha$.*
(ii) *Let $k_2 > k_1 > \operatorname{cls} \mathbf{h}_\alpha$ be two non-multiplicative indices. If $\operatorname{cls} \mathbf{h}_{\Delta(\alpha,k_2)} \geq k_1$, then $\Delta\bigl(\Delta(\alpha,k_1),k_2\bigr) = \Delta(\alpha,k_2)$ and $x_{k_1} t_{\alpha,k_2} = t_{\alpha,k_1} t_{\Delta(\alpha,k_1),k_2}$. Otherwise we have the two equations $\Delta\bigl(\Delta(\alpha,k_1),k_2\bigr) = \Delta\bigl(\Delta(\alpha,k_2),k_1\bigr)$ and $t_{\alpha,k_1} t_{\Delta(\alpha,k_1),k_2} = t_{\alpha,k_2} t_{\Delta(\alpha,k_2),k_1}$.*

Proof Part (i) is trivial. The inequality $\operatorname{cls} \mathbf{h}_\alpha \leq \operatorname{cls} \mathbf{h}_{\Delta(\alpha,k)}$ follows from the definition of Δ and the Pommaret division. If $\operatorname{cls} \mathbf{h}_{\Delta(\alpha,k)} > k$, then $\mathbf{h}_{\Delta(\alpha,k)}$ would be an involutive divisor of $\mathbf{h}_\alpha$ which contradicts the fact that any involutive basis is involutively head autoreduced.

For Part (ii) we compute the involutive standard representation of $x_{k_1} x_{k_2} \mathbf{h}_\alpha$. There are two ways to do it. We may either write

$$x_{k_1} x_{k_2} \mathbf{h}_\alpha = x_{k_2} t_{\alpha,k_1} \mathbf{h}_{\Delta(\alpha,k_1)} = t_{\alpha,k_1} t_{\Delta(\alpha,k_1),k_2} \mathbf{h}_{\Delta(\Delta(\alpha,k_1),k_2)} \ , \tag{48}$$

which is an involutive standard representation by Part (i), or start with

$$x_{k_1} x_{k_2} \mathbf{h}_\alpha = x_{k_1} t_{\alpha,k_2} \mathbf{h}_{\Delta(\alpha,k_2)} \tag{49}$$

requiring a case distinction. If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_2)} \geq k_1$, this is an involutive standard representation and its uniqueness implies our claim. Otherwise we rewrite multiplicatively $x_{k_1}\mathbf{h}_{\Delta(\alpha,k_2)} = t_{\Delta(\alpha,k_2),k_1}\mathbf{h}_{\Delta(\Delta(\alpha,k_2),k_1)}$ in order to obtain the involutive standard representation. Again our assertion follows from its uniqueness. □

Using this lemma, we can now provide a closed form for the differential δ which does not require involutive normal form computations in the syzygy modules $\operatorname{Syz}^i(\mathcal{H})$ (which are of course expensive to perform) but is solely based on information already computed during the determination of $\mathcal{H}$. For its proof we must introduce some additional notations and conventions. If again $\mathbf{k} = (k_1,\dots,k_i)$ is an integer sequence with $1 \leq k_1 < \cdots < k_i \leq n$, then we write $\mathbf{k}_j$ for the same sequence of indices but with k_j eliminated. Its first entry is denoted by $(\mathbf{k}_j)_1$; hence $(\mathbf{k}_j)_1 = k_1$ for $j > 1$ and $(\mathbf{k}_j)_1 = k_2$ for $j = 1$. The syzygy $\mathbf{S}_{\alpha;\mathbf{k}}$ is only defined for $\operatorname{cls}\mathbf{h}_\alpha < k_1$. We extend this notation by setting $\mathbf{S}_{\alpha;\mathbf{k}} = 0$ for $\operatorname{cls}\mathbf{h}_\alpha \geq k_1$. This convention will simplify some sums in the sequel.

Theorem 7.2 *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a quasi-stable submodule and $\mathbf{k} = (k_1,\dots,k_i)$. Then the differential δ of the complex $\mathcal{C}_*$ may be written in the form*

$$\delta(w_\alpha \otimes v_{\mathbf{k}}) = \sum_{j=1}^{i}(-1)^{i-j}\bigl(x_{k_j}w_\alpha - t_{\alpha,k_j}w_{\Delta(\alpha,k_j)}\bigr) \otimes v_{\mathbf{k}_j}\ . \tag{50}$$

Proof All summands where k_j is multiplicative for $\mathbf{h}_\alpha$ vanish which trivially implies (41). Thus we restrict to the case $\operatorname{cls}\mathbf{h}_\alpha < k_1$ where (50) is equivalent to

$$\mathbf{S}_{\alpha;\mathbf{k}} = \sum_{j=1}^{i}(-1)^{i-j}\bigl(x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j} - t_{\alpha,k_j}\mathbf{S}_{\Delta(\alpha,k_j);\mathbf{k}_j}\bigr)\ . \tag{51}$$

Some of the terms $\mathbf{S}_{\Delta(\alpha,k);\mathbf{k}_j}$ might vanish by our above introduced convention. The equation (51) is trivial for $i = 1$ (with $\mathbf{S}_\alpha = \mathbf{h}_\alpha$) and a simple corollary of Lemma 7.1 (ii) for $i = 2$.

For $i > 2$ things become messy. We proceed by induction on i. In our approach, the syzygy $\mathbf{S}_{\alpha;\mathbf{k}}$ arises from the non-multiplicative product $x_{k_i}\mathbf{S}_{\alpha;\mathbf{k}_i}$. Thus we must compute now the involutive normal form of this product. By our induction hypothesis we may write

$$x_{k_i}\mathbf{S}_{\alpha;\mathbf{k}_i} = \sum_{j=1}^{i-1}(-1)^{i-1-j}\bigl(x_{k_j}x_{k_i}\mathbf{S}_{\alpha;\mathbf{k}_{ji}} - x_{k_i}t_{\alpha,k_j}\mathbf{S}_{\Delta(\alpha,k_j);\mathbf{k}_{ji}}\bigr)\ . \tag{52}$$

As x_{k_i} is always non-multiplicative, using again the induction hypothesis, each summand may be replaced by the corresponding syzygy—but only at the expense of the introduction of many additional terms. The main task in the proof will be to show that most of them cancel. However, the cancellations occur in a rather complicated manner with several cases, so that no simple way for proving (51) seems to exist. We obtain the following lengthy expression:

$$
\begin{aligned}
x_{k_i}\mathbf{S}_{\alpha;\mathbf{k}_i} &= \sum_{j=1}^{i-1}(-1)^{i-1-j}\Big[x_{k_j}\mathbf{S}^{\textcircled{1}}_{\alpha;\mathbf{k}_j} - t_{\alpha,k_j}\mathbf{S}^{\textcircled{2}}_{\Delta(\alpha,k_j);\mathbf{k}_j}\Big]\\
&+\sum_{j=1}^{i-1}x_{k_j}\left[\sum_{\ell=1}^{j-1}(-1)^{\ell+j+1}x_{k_\ell}\mathbf{S}^{\textcircled{3}}_{\alpha;\mathbf{k}_{\ell j}} - \sum_{\ell=j+1}^{i-1}(-1)^{\ell+j+1}x_{k_\ell}\mathbf{S}^{\textcircled{4}}_{\alpha;\mathbf{k}_{j\ell}}\right]\\
&-\sum_{j=1}^{i-1}\left[\sum_{\ell=1}^{j-1}(-1)^{\ell+j+1}x_{k_j}t_{\alpha,k_\ell}\mathbf{S}^{\textcircled{5}}_{\Delta(\alpha,k_\ell);\mathbf{k}_{\ell j}} - \right.\\
&\qquad\qquad\left.\sum_{\ell=j+1}^{i-1}(-1)^{\ell+j+1}x_{k_j}t_{\alpha,k_\ell}\mathbf{S}^{\textcircled{6}}_{\Delta(\alpha,k_\ell);\mathbf{k}_{j\ell}}\right]\\
&+\sum_{j=1}^{i-2}(-1)^{i-1-j}x_{k_j}t_{\alpha,k_i}\mathbf{S}^{\textcircled{7}}_{\Delta(\alpha,k_i);\mathbf{k}_{ji}} + x_{k_{i-1}}t_{\alpha,k_i}\mathbf{S}^{\textcircled{8}}_{\Delta(\alpha,k_i);\mathbf{k}_{i-1,i}}\\
&-\sum_{j=1}^{i-1}t_{\alpha,k_j}\left[\sum_{\ell=1}^{j-1}(-1)^{\ell+j+1}x_{k_\ell}\mathbf{S}^{\textcircled{9}}_{\Delta(\alpha,k_j);\mathbf{k}_{\ell j}} - \right.\\
&\qquad\qquad\left.\sum_{\ell=j+1}^{i-1}(-1)^{\ell+j+1}x_{k_\ell}\mathbf{S}^{\textcircled{10}}_{\Delta(\alpha,k_j);\mathbf{k}_{j\ell}}\right]\\
&+\sum_{j=1}^{i-1}t_{\alpha,k_j}\left[\sum_{\ell=1}^{j-1}(-1)^{\ell+j+1}t_{\Delta(\alpha,k_j),k_\ell}\mathbf{S}^{\textcircled{11}}_{\Delta(\Delta(\alpha,k_j),k_\ell);\mathbf{k}_{\ell j}} - \right.\\
&\qquad\qquad\left.\sum_{\ell=j+1}^{i-1}(-1)^{\ell+j+1}t_{\Delta(\alpha,k_j),k_\ell}\mathbf{S}^{\textcircled{12}}_{\Delta(\Delta(\alpha,k_j),k_\ell);\mathbf{k}_{j\ell}}\right]\\
&-\sum_{j=1}^{i-1}(-1)^{i-1-j}t_{\alpha,k_j}t_{\Delta(\alpha,k_j),k_i}\mathbf{S}^{\textcircled{13}}_{\Delta(\Delta(\alpha,k_j),k_i);\mathbf{k}_{ji}}\;.
\end{aligned}
\tag{53}
$$

Note that the terms ⑦, ⑧ and ⑬, respectively, correspond to the special case $\ell = i$ (and $j = i-1$) in the sums ⑥ and ⑫, respectively. We list them separately, as they must be treated differently. The existence of any summand where the coefficient contains a term $t_{\cdot,\cdot}$ is bound on conditions.

With the exception of the coefficient $x_{k_{i-1}}$ in the term ⑧, all coefficients are already multiplicative. Thus this term must be further expanded using the induction hypothesis for the last time:

$$
\begin{aligned}
x_{k_{i-1}}t_{\alpha,k_i}\mathbf{S}_{\Delta(\alpha,k_i);\mathbf{k}_{i-1,i}} &= t_{\alpha,k_i}\mathbf{S}^{\textcircled{14}}_{\Delta(\alpha,k_i);\mathbf{k}_i}\\
&-\sum_{j=1}^{i-2}(-1)^{i-1-j}x_{k_j}t_{\alpha,k_i}\mathbf{S}^{\textcircled{15}}_{\Delta(\alpha,k_i);\mathbf{k}_{ji}}\\
&+\sum_{j=1}^{i-1}(-1)^{i-1-j}t_{\alpha,k_i}t_{\Delta(\alpha,k_i),k_j}\mathbf{S}^{\textcircled{16}}_{\Delta(\Delta(\alpha,k_i),k_j);\mathbf{k}_{ji}}\;.
\end{aligned}
\tag{54}
$$

The left hand side of (53) and the terms ①, ② and ⑭ represent the syzygy $\mathbf{S}_{\alpha,\mathbf{k}}$ we are looking for. We must thus show that all remaining terms vanish. In order to simplify the discussion of the double sums, we swap j and ℓ in ③, ⑤, ⑨ and ⑪ so that everywhere $j < \ell$. It is now easy to see that ③ and ④ cancel; each summand of ③ also appears in ④ but with the opposite sign. Note, however, that the same argument does not apply to ⑪ and ⑫, as the existence of these terms is bound to different conditions!

For the other cancellations, we must distinguish several cases depending on the classes of the generators in the Pommaret basis $\mathcal{H}$. We first study the double sums and thus assume that $1 \leq j < i$.

– If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_j)} < (\mathbf{k}_j)_1$, the terms ⑤ and ⑩ are both present and cancel each other. We must now make a second case distinction on the basis of $\mathbf{h}_{\Delta(\alpha,k_\ell)}$.
  – If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_\ell)} < (\mathbf{k}_j)_1$, then the terms ⑥ and ⑨ are also present and cancel each other. Furthermore, both ⑪ and ⑫ exist and cancel due to the second case of Lemma 7.1 (ii).
  – If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_\ell)} \geq (\mathbf{k}_j)_1$, then none of the four terms ⑥, ⑨, ⑪ and ⑫ exists. For the latter two terms, this fact is a consequence of the first case of Lemma 7.1 (ii).
– If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_j)} \geq (\mathbf{k}_j)_1$, then neither ⑤ nor ⑩ nor ⑫ exists. For the remaining double sums, we must again consider the class of $\mathbf{h}_{\Delta(\alpha,k_\ell)}$.
  – If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_\ell)} < (\mathbf{k}_j)_1$, then the terms ⑥ and ⑨ exist and cancel each other. The term ⑪ does not exist, as Lemma 7.1 implies the inequalities $\operatorname{cls}\mathbf{h}_{\Delta(\Delta(\alpha,k_\ell),k_j)} = \operatorname{cls}\mathbf{h}_{\Delta(\Delta(\alpha,k_j),k_\ell)} \geq \operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_j)} \geq (\mathbf{k}_j)_1$.
  – If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_\ell)} \geq (\mathbf{k}_j)_1$, then neither ⑥ nor ⑨ exist and the term ⑪ is not present either; this time the application of Lemma 7.1 (ii) yields the chain of inequalities $\operatorname{cls}\mathbf{h}_{\Delta(\Delta(\alpha,k_\ell),k_j)} \geq \operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_\ell)} \geq (\mathbf{k}_j)_1$.

For the remaining terms everything depends on the class of $\mathbf{h}_{\Delta(\alpha,k_i)}$ controlling in particular the existence of the term ⑧.

– If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_i)} < k_1 \leq (\mathbf{k}_j)_1$, then the term ⑧ exists and generates the terms ⑮ and ⑯. Under this condition, the term ⑦ is present, too, and because of Lemma 7.1 (ii) it cancels ⑮. Again by Lemma 7.1 (ii), the conditions for the existence of ⑬ and ⑯ are identical and they cancel each other.
– If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_i)} \geq k_1$, then ⑧ and consequently ⑮ and ⑯ are not present. The analysis of ⑦ and ⑬ requires a further case distinction.
  – Under the made assumption, the case $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_i)} < (\mathbf{k}_j)_1$ can occur only for $j = 1$ as otherwise $(\mathbf{k}_j)_1 = k_1$. Because of Lemma 7.1 (ii), the terms ⑦ and ⑬ exist for $j = 1$ and cancel each other.
  – If $\operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_i)} \geq (\mathbf{k}_j)_1$, then ⑦ does not exist. The term ⑬ is also not present, but there are two different possibilities: depending on which case of Lemma 7.1 (ii) applies, we either find $\operatorname{cls}\mathbf{h}_{\Delta(\Delta(\alpha,k_j),k_i)} = \operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_i)}$ or $\operatorname{cls}\mathbf{h}_{\Delta(\Delta(\alpha,k_j),k_i)} = \operatorname{cls}\mathbf{h}_{\Delta(\Delta(\alpha,k_i),k_j)} \geq \operatorname{cls}\mathbf{h}_{\Delta(\alpha,k_i)}$; but in any case the class is too high.

Thus we have shown that indeed all terms vanish with the exception of ①, ② and ⑭ which are needed for the syzygy $\mathbf{S}_{\alpha,\mathbf{k}}$. This proves our claim. □

Remark 7.3 In the last section we introduced for any involutive basis $\mathcal{H}$ with respect to a division L its L-graph. We augment now this graph by weights for the edges. Recall that we have a directed edge from $\mathbf{h}$ to $\bar{\mathbf{h}}$, if $\mathrm{le}_\prec \bar{\mathbf{h}}$ is an involutive divisor of $\mathrm{le}_\prec(x_k\mathbf{h})$ for some non-multiplicative variable $x_k \in \bar{X}_{\mathcal{H},L,\prec}(\mathbf{h})$. If $\mathrm{le}_\prec(x_k\mathbf{h}) = \mathrm{le}_\prec\bar{\mathbf{h}} + \mu$, then we assign the weight x^μ to this edge. For a monomial Pommaret basis the corresponding P-graph has then a directed edge from $\mathbf{h}_\alpha$ to $\mathbf{h}_{\Delta(\alpha,k)}$ with weight $t_{\alpha,k}$ for every non-multiplicative variable $x_k \in \bar{X}_P(\mathbf{h}_\alpha)$. Thus we may say that by Theorem 7.2 the whole complex $(\mathcal{C}_*, \delta)$ (and the isomorphic syzygy resolution of $\langle\mathcal{H}\rangle$) is encoded in the weighted P-graph of $\mathcal{H}$. $\lhd$

As in the previous section, we may introduce for monomial ideals, i. e. for $m = 1$, the product $\times$. The right hand side of its defining equation (43) simplifies for a monomial basis $\mathcal{H}$ to

$$w_\alpha \times w_\beta = m_{\alpha,\beta} w_{\Gamma(\alpha,\beta)} \tag{55}$$

where the function $\Gamma(\alpha,\beta)$ determines the unique generator $h_{\Gamma(\alpha,\beta)}$ such that $h_\alpha h_\beta = m_{\alpha,\beta} h_{\Gamma(\alpha,\beta)}$ with a term $m_{\alpha,\beta} \in \mathbb{k}[X_P(h_{\Gamma(\alpha,\beta)})]$. Corresponding to Lemma 7.1, we obtain now the following result.

Lemma 7.4 *The function Γ and the terms $m_{\alpha,\beta}$ satisfy the following relations.*

(i) $\mathrm{cls}\, h_{\Gamma(\alpha,\beta)} \geq \max\{\mathrm{cls}\, h_\alpha, \mathrm{cls}\, h_\beta\}$.
(ii) $\Gamma\bigl(\Gamma(\alpha,\beta),\gamma\bigr) = \Gamma\bigl(\alpha,\Gamma(\beta,\gamma)\bigr)$ *and* $m_{\alpha,\beta} m_{\Gamma(\alpha,\beta),\gamma} = m_{\beta,\gamma} m_{\Gamma(\beta,\gamma),\alpha}$.
(iii) $\Gamma\bigl(\Delta(\alpha,k),\beta\bigr) = \Delta\bigl(\Gamma(\alpha,\beta),k\bigr)$ *and* $t_{\alpha,k} m_{\Delta(\alpha,k),\beta} = t_{\Gamma(\alpha,\beta),k} m_{\alpha,\beta}$.

Proof Part (i) is obvious from the definition of the function Γ. Part (ii) and (iii), respectively, follow from the analysis of the two different ways to compute the involutive standard representation of $h_\alpha h_\beta h_\gamma$ and $x_k h_\alpha h_\beta$, respectively. We omit the details, as they are completely analogous to the proof of Lemma 7.1. $\square$

Theorem 7.5 *Let $\mathcal{H}$ be the Pommaret basis of the quasi-stable ideal $\mathcal{I} \subseteq \mathcal{P}$. Then the product $\times$ defined by (55) makes the complex $(\mathcal{C}_*, \delta)$ to a differential algebra.*

Proof This is a straightforward consequence of Lemma 7.4. Writing out the relations one has to check, one easily finds that Part (ii) ensures the associativity of $\times$ and Part (iii) the satisfaction of the graded Leibniz rule. $\square$

8 Minimal Resolutions and Projective Dimension

Recall that for a graded polynomial module $\mathcal{M}$ a graded free resolution is *minimal*, if all entries of the matrices corresponding to the maps $\phi_i : \mathcal{P}^{r_i} \to \mathcal{P}^{r_{i-1}}$ are of positive degree, i. e. no constant coefficients appear. Up to isomorphisms, the minimal resolution is unique and its length is an important invariant, the *projective dimension* $\mathrm{proj}\dim\mathcal{M}$ of the module. If the module $\mathcal{M}$ is graded, then the resolution (35) is obviously graded, too. However, in general, it is not minimal. Our first goal consists of finding conditions under which (35) is minimal. A simple criterion for minimality that can be directly checked on the Pommaret basis $\mathcal{H}$ of $\mathcal{M}$ is provided by the next result.

Lemma 8.1 *The resolution (35) is minimal, if and only if all first syzygies* $\mathbf{S}_{\alpha;k}$ *are free of constant terms.*

Proof One direction is of course trivial. Since (35) was obtained by iterating Theorem 5.10, it suffices for proving the converse to show that under the made assumption all second syzygies $\mathbf{S}_{\alpha;k_1,k_2}$ are free of constant terms. But this is easy to see: we have $\mathbf{S}_{\alpha;k_1,k_2} = x_{k_2}\mathbf{e}_{\alpha;k_1} - x_{k_1}\mathbf{e}_{\alpha;k_2} + \sum_{\gamma,\ell} c_{\gamma;\ell}\mathbf{e}_{\gamma,\ell}$ where every non-vanishing coefficient $c_{\gamma;\ell}$ is divisible by a coefficient $P_\beta^{(\alpha;k)}$ with $k = k_1$ or $k = k_2$ appearing in the first syzygy $\mathbf{S}_{\alpha;k}$ and thus is of positive degree. □

A minimal resolution of the graded module $\mathcal{M}$ is *linear*, if all maps appearing in it are linear in the sense that the entries of the matrices describing them are zero or homogeneous polynomials of degree 1. The graded module $\mathcal{M}$ is called *componentwise linear*, if for every degree $d \geq 0$ the module $\mathcal{M}_{\langle d\rangle} = \langle \mathcal{M}_d\rangle$ generated by the component $\mathcal{M}_d$ of degree d has a linear resolution (in other words, if the only non-vanishing Betti numbers of $\mathcal{M}_{\langle d\rangle}$ are $\beta_{i,i+d}$ for $i = 0, 1, \dots$) [41].

Theorem 8.2 *If the resolution (35) is minimal, then the graded module* $\mathcal{M} \subseteq \mathcal{P}^m$ *is componentwise linear.*

Proof Let $\mathcal{H}$ be the Pommaret basis of $\mathcal{M}$ and $d \geq 0$ an arbitrary degree. As in Lemma 2.2, it is easy to see that the set

$$\mathcal{G}_d = \left\{x^\mu h \mid h \in \mathcal{H} \wedge |\mu| + \deg h = d \wedge \forall j > \operatorname{cls} h : \mu_j = 0\right\} \tag{56}$$

defines a $\Bbbk$-linear basis of the homogeneous component $\mathcal{M}_d$ and thus generates the module $\mathcal{M}_{\langle d\rangle}$. Consider now a product $x_j\bar{g}$ for some generator $\bar{g} = x^\mu\bar{h} \in \mathcal{G}_d$ where $j > k = \operatorname{cls}\bar{g}$ so that x_j is non-multiplicative for $\bar{g}$. If $j \leq \operatorname{cls}\bar{h}$, then $\mathcal{G}_d$ also contains the generator $\tilde{g} = x^{\mu-1_k+1_j}\bar{h}$ and we have $x_j\bar{g} = x_k\tilde{g}$ where the latter product is multiplicative.

Otherwise, the variable x_j is non-multiplicative for $\bar{h}$, too, and the resolution (35) contains a first syzygy corresponding to an involutive standard representation $x_j\bar{h} = \sum_{h\in\mathcal{H}} P_h h$. If $|\mu| > 0$, then we can lift this equation to a standard representation $x_j\bar{g} = \sum_{h\in\mathcal{H}} P_h x^\mu h$. However, in general it will no longer be an involutive one, as the term x^μ may depend on variables which are non-multiplicative for some generators $h \in \mathcal{H}$. In this case, we must rewrite the right hand side using further first syzygies from (35). It is not difficult to see that after a finite number of such steps we also arrive at an involutive standard representation

$$x_j\bar{g} = \sum_{h\in\mathcal{H}} P_h h \tag{57}$$

where for notational simplicity we still denote the coefficients by P_h.

Assume now that the resolution (35) is minimal. Obviously, all first syzygies and thus also the coefficients P_h in (57) are then free of constant terms. But this observation implies that we can transform (57) into an involutive standard representation $x_j\bar{g} = \sum_{g\in\mathcal{G}_d} Q_g g$ with respect to $\mathcal{G}_d$ and hence this set is a Pommaret basis of the module $\mathcal{M}_{\langle d\rangle}$ by Corollary 7.3 of Part I. As all elements of $\mathcal{G}_d$ are of degree d, it follows immediately from the form of (35) evaluated for $\mathcal{G}_d$ that $\mathcal{M}_{\langle d\rangle}$ has a linear resolution and thus $\mathcal{M}$ is componentwise linear. □

Example 8.3 The converse of Theorem 8.2 is not true, as the following trivial counterexample demonstrates. Consider the monomial ideal $\mathcal{I} = \langle x, y^2 \rangle \subset \mathbb{k}[x,y]$. It is componentwise linear: $\mathcal{I}_{\langle 1 \rangle} = \langle x \rangle$ is as principal ideal a free module; all ideals $\mathcal{I}_{\langle d \rangle}$ for $d > 1$ are simply generated by all monomials of degree d and thus possess trivially a linear resolution. For the natural ordering of variables $x_1 = x$ and $x_2 = y$, the Pommaret basis of $\mathcal{I}$ is $\mathcal{H} = \{x, xy, y^2\}$ and since the arising resolution contains the first syzygy $y\mathbf{e}_1 - \mathbf{e}_2$, it is not minimal. Comparing with the proof above, we see that $\mathcal{G}_1 = \{x\}$ is not a Pommaret basis of $\mathcal{I}_{\langle 1 \rangle}$ (actually, $\mathcal{I}_{\langle 1 \rangle}$ does not even possess a finite Pommaret basis, as it is not quasi-stable).

Note, however, that the situation is different, if we change the ordering of the variables to $x_1 = y$ and $x_2 = x$. In this case the minimal basis $\mathcal{H}' = \{x, y^2\}$ is already a Pommaret basis of $\mathcal{I}$ and the arising resolution is trivially minimal, as the only syzygy is $y^2\mathbf{e}_1 - x\mathbf{e}_2$. We will see later (Theorem 9.12) that this observation is no accident but that *generically* the resolution (35) is minimal for componentwise linear modules (and thus generically (36) yields the Betti numbers of componentwise linear modules). $\triangleleft$

For quasi-stable monomial modules $\mathcal{M}$ a simple combinatorial characterisation exists when our resolution is minimal. We will also provide a simple alternative characterisation via Pommaret bases.

Definition 8.4 ([27]) *A (possibly infinite) set $\mathcal{N} \subseteq \mathbb{N}_0^n$ is called* stable*, if for each multi index $\nu \in \mathcal{N}$ all multi indices $\nu - 1_k + 1_j$ with $k = \operatorname{cls}\nu < j \leq n$ are also contained in $\mathcal{N}$. A monomial submodule $\mathcal{M} \subseteq \mathcal{P}^m$ is* stable*, if each of the sets $\mathcal{N}_\alpha = \{\mu \mid x^\mu \mathbf{e}_\alpha \in \mathcal{M}\} \subseteq \mathbb{N}_0^n$ with $1 \leq \alpha \leq m$ is stable.*

Remark 8.5 The stable modules are of considerable interest, as they contain as a subset the *Borel-fixed* modules, i. e. modules $\mathcal{M} \subseteq \mathcal{P}^m$ which remain invariant under the natural action of the Borel group.[15] Indeed, one can show that (for a ground field of characteristic 0) a module is Borel-fixed, if and only if it can be generated by a set $\mathcal{S}$ of monomials such that whenever $x^\nu \mathbf{e}_j \in \mathcal{S}$ then also $x^{\nu - 1_k + 1_j}\mathbf{e}_j \in \mathcal{S}$ for all $\operatorname{cls}\nu \leq k < j \leq n$ [24, Thm. 15.23]. Generically, the leading terms of any polynomial module form a Borel-fixed module [28] [24, Thm. 15.20]. Note that while stability is obviously independent of the characteristic of the ground field, the same does not hold for the notion of a Borel-fixed module. $\triangleleft$

Any monomial submodule has a unique minimal basis. For stable submodules it must coincide with its Pommaret basis. This result represents a very simple and effective characterisation of stable submodules. Furthermore, it shows that any stable submodule is trivially quasi-stable and thus explains the terminology introduced in Definition 4.3.

Proposition 8.6 ([54, Lem. 2.13][16]) *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a monomial submodule. $\mathcal{M}$ is stable, if and only if its minimal basis $\mathcal{H}$ is simultaneously a Pommaret basis.*

[15] Classically, the Borel group consists of upper triangular matrices. In our "inverse" conventions we must take lower triangular matrices.

[16] See also the remark after [27, Lem. 1.2].

Mall [54, Thm. 2.15] proved that for any ideal $\mathcal{I} \subseteq \mathcal{P}$ the reduced Gröbner basis is simultaneously a Pommaret basis, if and only if the leading ideal $\mathrm{lt}_\prec \mathcal{I}$ is stable. Combining this result with the above mentioned fact that the generic initial ideal is always Borel-fixed and thus stable, one obtains the following theorem.

Theorem 8.7 *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a graded submodule in generic position and $\mathcal{G}$ the reduced Gröbner basis of $\mathcal{M}$ for an arbitrary term order $\prec$. If $\operatorname{char} \Bbbk = 0$, then $\mathcal{G}$ is also the minimal Pommaret basis of $\mathcal{M}$ for $\prec$.*

Remark 8.8 It follows from Lemma 2.3, that if $\mathcal{H}$ is a Pommaret basis of the submodule $\mathcal{M} \subseteq \mathcal{P}^m$ of degree q, then $(\mathrm{lt}_\prec \mathcal{M})_{\geq q}$ is a stable monomial submodule, as $\mathrm{lt}_\prec \mathcal{H}_q$ is obviously its minimal basis and simultaneously a Pommaret basis. $\triangleleft$

Our next result implies that for stable modules the resolution (35) is isomorphic to the minimal resolution constructed by Eliahou and Kervaire [27, Thm. 2.1]. In fact, if one formulates (35) as a complex as described in Section 6, then one finds that in this special case the two resolutions are identical.

Theorem 8.9 *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a quasi-stable module. Then the syzygy resolution given by (35) is minimal, if and only if $\mathcal{M}$ is stable.*

Proof According to Lemma 8.1, the resolution (35) is minimal, if and only if all first syzygies are free of constant terms. For a monomial module this is the case, if and only if already the minimal basis is the Pommaret basis, since otherwise the Pommaret basis contains generators $\mathbf{h}_1$, $\mathbf{h}_2$ related by $\mathbf{h}_2 = x_j \mathbf{h}_1$ for some non-multiplicative variable x_j leading to a first syzygy $x_j \mathbf{e}_1 - \mathbf{e}_2$ with a constant term. Now our claim follows from Proposition 8.6. $\square$

Example 8.10 One might be tempted to conjecture that this result extended to polynomial modules, i. e. that (35) was minimal for polynomial modules $\mathcal{M}$ with stable leading module $\mathrm{lt}_\prec \mathcal{M}$. Unfortunately, this is not true. Consider the homogeneous ideal $\mathcal{I} \subset \Bbbk[x, y, z]$ generated by $h_1 = z^2 + xy$, $h_2 = yz - xz$, $h_3 = y^2 + xz$, $h_4 = x^2 z$ and $h_5 = x^2 y$. One easily checks that these elements form a Pommaret basis $\mathcal{H}$ for the degree reverse lexicographic term order and that $\mathrm{lt}_\prec \mathcal{M}$ is a stable module. A Pommaret basis of $\mathrm{Syz}(\mathcal{H})$ is given by

$$\mathbf{S}_{2;3} = z\mathbf{e}_2 + (x-y)\mathbf{e}_1 + x\mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 \ , \tag{58a}$$
$$\mathbf{S}_{3;3} = z\mathbf{e}_3 - x\mathbf{e}_1 - (x+y)\mathbf{e}_2 - \mathbf{e}_4 + \mathbf{e}_5 \ , \tag{58b}$$
$$\mathbf{S}_{4;3} = z\mathbf{e}_4 - x^2\mathbf{e}_1 + x\mathbf{e}_5 \ , \tag{58c}$$
$$\mathbf{S}_{5;3} = z\mathbf{e}_5 - x^2\mathbf{e}_2 - x\mathbf{e}_4 \ , \tag{58d}$$
$$\mathbf{S}_{4;2} = (y-x)\mathbf{e}_4 - x^2\mathbf{e}_2 \ , \tag{58e}$$
$$\mathbf{S}_{5;2} = y\mathbf{e}_5 - x^2\mathbf{e}_3 + x\mathbf{e}_4 \ . \tag{58f}$$

As the first two generators show, the resolution (35) is not minimal. $\triangleleft$

Given an arbitrary graded free resolution, it is a standard task to reduce it to the minimal resolution using just some linear algebra (see for example [23, Chapt. 6, Theorem 3.15] for a detailed discussion). Thus for any concrete module $\mathcal{M}$ it is straightforward to obtain from (35) the minimal resolution. However, even in the monomial case it seems highly non-trivial to find a closed form description of the outcome of the minimisation process. Nevertheless, the resolution (35) contains so much structure that certain statements are possible.

Theorem 8.11 *Let $\mathcal{H}$ be a Pommaret basis of the graded module $\mathcal{M} \subseteq \mathcal{P}^m$ for a class respecting term order and set $d = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls}\mathbf{h}$. Then $\operatorname{proj}\dim\mathcal{M} = n-d$.*

Proof Consider the resolution (35) which is of length $n-d$. The last map in it is defined by the syzygies $\mathbf{S}_{\alpha;(d+1,\dots,n)}$ originating in the generators $\mathbf{h}_\alpha \in \mathcal{H}$ with $\operatorname{cls}\mathbf{h}_\alpha = d$. Choose now among these generators an element $\mathbf{h}_\gamma$ of maximal degree (recall that the same choice was crucial in the proof of Proposition 2.20). Then the syzygy $\mathbf{S}_{\gamma;(d+1,\dots,n)}$ cannot contain any constant coefficient, as the coefficients of all basis vectors $\mathbf{e}_{\beta;\mathbf{k}}$ where the last entry of $\mathbf{k}$ is n must be contained in $\langle x_1,\dots,x_{n-1}\rangle$ and the coefficients of the basis vectors $\mathbf{e}_{\alpha;(d+1,\dots,n-1)}$ cannot be constant for degree reasons.

If we start now a minimisation process at the end of the resolution, then it will never introduce a constant term into the syzygy $\mathbf{S}_{\gamma;(d+1,\dots,n)}$ and thus it will never be eliminated. It is also not possible that it is reduced to zero, as the last map in a free resolution is obviously injective. This implies that the last term of the resolution will not vanish during the minimisation and the length of the minimal resolution, i. e. $\operatorname{proj}\dim\mathcal{M}$, is still $n-d$. □

The graded *Auslander-Buchsbaum formula* [24, Exercise 19.8] is now a trivial corollary of this theorem and Proposition 2.20 on the depth. Note that, in contrast to other proofs, our approach is constructive in the sense that we automatically have an explicit regular sequence of maximal length and a (partially) explicit free resolution of minimal length.

Corollary 8.12 (Auslander-Buchsbaum) *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a graded polynomial module with $\mathcal{P} = \Bbbk[x_1,\dots,x_n]$. Then $\operatorname{depth}\mathcal{M} + \operatorname{proj}\dim\mathcal{M} = n$.*

As for a monomial module no term order is needed, we obtain as a further simple corollary the following relation between $\operatorname{proj}\dim\mathcal{M}$ and $\operatorname{proj}\dim(\operatorname{lt}_\prec\mathcal{M})$.

Corollary 8.13 *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a graded module and $\prec$ an arbitrary term order for which $\mathcal{M}$ possesses a Pommaret basis. Then $\operatorname{proj}\dim\mathcal{M} \leq \operatorname{proj}\dim(\operatorname{lt}_\prec\mathcal{M})$. If $\prec$ is a class respecting term order, then we even have equality.*

Proof Let $\mathcal{H}$ be the Pommaret basis of the module $\mathcal{M}$ for the term order $\prec$ and set $d = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls}(\operatorname{lt}_\prec\mathbf{h})$. Then it follows immediately from Theorem 8.11 that $\operatorname{proj}\dim(\operatorname{lt}_\prec\mathcal{M}) = n-d$. On the other hand, Theorem 6.1 guarantees the existence of the free resolution (35) of length $n-d$ for $\mathcal{M}$ so that this value is an upper bound for $\operatorname{proj}\dim\mathcal{M}$. For a class respecting term order we have equality by Theorem 8.11. □

9 Castelnuovo-Mumford Regularity

For notational simplicity we restrict again to ideals instead of submodules. In many situations it is of interest to obtain a good estimate on the degree of an ideal basis. Up to date, no satisfying answer is known to this question. Somewhat surprisingly, the stronger problem of bounding not only the degree of a basis of $\mathcal{I}$ but also of its syzygies can be treated effectively.

Definition 9.1 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal. $\mathcal{I}$ is called* q-regular, *if its ith syzygy module is generated by elements of degree less than or equal to $q + i$. The* Castelnuovo-Mumford regularity $\operatorname{reg} \mathcal{I}$ *is the least q for which $\mathcal{I}$ is q-regular.*

Among other applications, the Castelnuovo-Mumford regularity $\operatorname{reg} \mathcal{I}$ is a useful measure for the complexity of Gröbner basis computations [9]. The question of effectively computing $\operatorname{reg} \mathcal{I}$ has recently attracted some interest. In this section we show that $\operatorname{reg} \mathcal{I}$ is trivially determined by a Pommaret basis with respect to the degree reverse lexicographic order and provide alternative proofs to some characterisations of the Castelnuovo-Mumford regularity proposed in the literature.

Theorem 9.2 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal. The Castelnuovo-Mumford regularity of $\mathcal{I}$ is q, if and only if $\mathcal{I}$ has in some coordinates a homogeneous Pommaret basis of degree q with respect to the degree reverse lexicographic order.*

Proof Let $\mathbf{x}$ be some δ-regular coordinates for the ideal $\mathcal{I}$ so that it possess a Pommaret basis $\mathcal{H}$ of degree q with respect to the degree reverse lexicographic order in these coordinates. Then the ith module of the syzygy resolution (35) induced by the basis $\mathcal{H}$ is obviously generated by elements of degree less than or equal to $q + i$. Thus we have the trivial estimate $\operatorname{reg} \mathcal{I} \leq q$ and there only remains to show that it is in fact an equality.

For this purpose, consider a generator $h_\gamma \in \mathcal{H}$ of degree q which is of minimal class among all elements of this maximal degree q in $\mathcal{H}$. If $\operatorname{cls} h_\gamma = n$, then h_γ cannot be removed from $\mathcal{H}$ without loosing the basis property, as the leading term of no other generator of class n can divide $\operatorname{lt}_\prec h_\gamma$ and, since the degree reverse lexicographic order is class respecting, all other generators do not contain any terms of class n. Hence we trivially find $\operatorname{reg} \mathcal{I} = q$ in this case.

If $\operatorname{cls} h_\gamma = n - i$ for some $i > 0$, then the resolution (35) contains at the ith position the syzygy $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$ of degree $q + i$. Assume now that we minimise the resolution step by step starting at the end. We claim that the syzygy $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$ is not eliminated during this process.

There are two possibilities how $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$ could be removed during the minimisation. The first one is that a syzygy at the next level of the resolution contained the term $\mathbf{e}_{\gamma;(n-i+1,\dots,n)}$ with a constant coefficient. Any such syzygy is of the form (39) with $\operatorname{cls} h_\alpha < n - i$ and $\operatorname{cls} h_\alpha < k_1 < \cdots < k_i < n$ and its leading term is $x_{k_{i+1}} \mathbf{e}_{\alpha;\mathbf{k}}$ with $k_{i+1} > k_i$. However, since $\operatorname{cls}(x_{k_1} \cdots x_{k_{i+1}} h_\alpha) < n - i$ and $\operatorname{cls}(x_{n-i+1} \cdots x_n h_\gamma) = n - i$, it follows from our use of the degree reverse lexicographic order (since we assume that everything is homogeneous, both polynomial have the same degree) and the definition of the induced Schreyer term orders, that the term $\mathbf{e}_{\gamma;(n-i+1,\dots,n)}$ is greater than the leading term $x_{k_{i+1}}\mathbf{e}_{\alpha;\mathbf{k}}$ of any syzygy $\mathbf{S}_{\alpha;(k_1,\dots,k_{i+1})}$ at the level $i+1$ and thus cannot appear.

The second possibility is that $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$ itself contained a constant coefficient at some vector $\mathbf{e}_{\beta;\boldsymbol{\ell}}$. However, this required $\deg h_\beta = \deg h_\gamma + 1$ which is again a contradiction.[17] As the minimisation process never introduces new constant coefficients, the syzygy $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$ may only be modified but not eliminated. Furthermore, the modifications cannot make $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$ to the zero syzygy, as otherwise a basis vector of the next level was in the kernel of the differential. However, this is not possible, as we assume that the tail of the resolution is already minimised and by the exactness of the sequence any kernel member must be a linear combination of syzygies. Hence the final minimal resolution will contain at the ith position a generator of degree $q+i$ and $\operatorname{reg}\mathcal{I} = q$. □

To some extent this result was to be expected. By Theorem 8.7, the reduced Gröbner basis is generically also a Pommaret basis and, according to Bayer and Stillman [10], this basis has for the degree reverse lexicographic order generically the degree $\operatorname{reg}\mathcal{I}$. Thus the only surprise is that Theorem 9.2 does not require that the leading ideal is stable and the Pommaret basis $\mathcal{H}$ is not necessarily a reduced Gröbner basis (if the ideal $\mathcal{I}$ has a Pommaret basis of degree q, then the truncated ideal $(\operatorname{le}_\prec\mathcal{I})_{\geq q}$ is always stable by Remark 8.8 and thus the set $\mathcal{H}_q$ defined by (1) is the reduced Gröbner basis of $\mathcal{I}_{\geq q}$).

Note furthermore that Theorem 9.2 implies a remarkable fact: given arbitrary coordinates $\mathbf{x}$, the ideal $\mathcal{I}$ either does not possess a finite Pommaret basis for the degree reverse lexicographic order or, if such a basis exists, it is of degree $\operatorname{reg}\mathcal{I}$. Hence using Pommaret bases, it becomes trivial to determine the Castelnuovo-Mumford regularity: it is just the degree of the basis.

Remark 9.3 The proof of Theorem 9.2 also provides us with information about the positions where in the minimal resolution the maximal degree is attained. We only have to look for all elements of maximal degree in the Pommaret basis; their classes correspond to these positions. ◁

Remark 9.4 Recall from Remark 6.2 that Theorem 6.1 remains valid for any involutive basis $\mathcal{H}$ with respect to a continuous division of Schreyer type (with an obvious modification of the definition of the numbers $\beta_0^{(k)}$) and that it is independent of the used term order. It follows immediately from the form of the resolution (35), i. e. from the form of the maps in it given by the respective involutive bases according to Theorem 5.10, that always the estimate $\operatorname{reg}\mathcal{I} \leq \deg\mathcal{H}$ holds and thus any such basis provides us with a bound for the Castelnuovo–Mumford regularity.

This observation also implies that an involutive basis with respect to a division of Schreyer type and an arbitrary term order can never be of lower degree than the

[17] For later use we note the following fact about this argument. If $\mathbf{e}_{\beta;\boldsymbol{\ell}}$ is a constant term in the syzygy $\mathbf{S}_{\gamma;(n-i+1,\dots,n)}$, then it must be smaller than the leading term and hence $\operatorname{lt}_\prec(x_{\ell_1}\cdots x_{\ell_i}h_\beta) \prec \operatorname{lt}_\prec(x_{k_1}\cdots x_{k_{i+1}}h_\alpha)$ implying that $\operatorname{cls} h_\beta \leq \operatorname{cls} h_\gamma$. Thus it suffices, if h_γ is of maximal degree among all generators $h_\beta \in \mathcal{H}$ with $\operatorname{cls} h_\beta \leq \operatorname{cls} h_\gamma$. For the special case that h_γ is of minimal class, we exploited this observation already in the proof of Theorem 8.11.

Pommaret basis for the degree reverse lexicographic order. The latter one is thus in this sense optimal. As a concrete example consider again the ideal mentioned in Remark 2.21: in "good" coordinates a Pommaret basis of degree 2 exists for it and after a simple permutation of the variables its Janet basis is of degree 4. $\lhd$

In analogy to Corollary 8.13 comparing the projective dimension of a module $\mathcal{M}$ and its leading module $\mathrm{lt}_\prec\mathcal{M}$ with respect to an arbitrary term order $\prec$, we may derive a similar estimate for the Castelnuovo-Mumford regularity.

Corollary 9.5 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal and $\prec$ an arbitrary term order such that a Pommaret basis $\mathcal{H}$ of $\mathcal{I}$ exists. Then $\operatorname{reg}\mathcal{I} \leq \operatorname{reg}(\mathrm{lt}_\prec\mathcal{I}) = \deg\mathcal{H}$. If $\prec$ is the degree reverse lexicographic order, then even $\operatorname{reg}\mathcal{I} = \operatorname{reg}(\mathrm{lt}_\prec\mathcal{I})$.*

Proof It follows from Theorem 9.2 that $\operatorname{reg}(\mathrm{lt}_\prec\mathcal{I}) = \deg\mathcal{H}$. On the other hand, the form of the resolution (35) implies trivially that $\operatorname{reg}\mathcal{I} \leq \deg\mathcal{H}$. For the degree reverse lexicographic order Theorem 9.2 entails that $\operatorname{reg}\mathcal{I} = \deg\mathcal{H}$, too. $\square$

Combining the above results with Remark 8.8 and Proposition 8.6 immediately implies the following generalisation of a result by Eisenbud, Reeves and Totaro [26, Prop. 10] for Borel-fixed monomial ideals.

Proposition 9.6 *Let $\mathcal{I}$ be a quasi-stable ideal generated in degrees less than or equal to q. The ideal $\mathcal{I}$ is q-regular, if and only if the truncation $\mathcal{I}_{\geq q}$ is stable.*

Remark 9.7 Bayer et al. [8] introduced a refinement of the Castelnuovo–Mumford regularity: the *extremal Betti numbers*. Recall that the (graded) Betti number β_{ij} of the ideal I is defined as the number of minimal generators of degree $i+j$ of the ith module in the minimal free resolution of $\mathcal{I}$ (thus $\operatorname{reg}\mathcal{I}$ is the maximal value j such that $\beta_{i,i+j} > 0$ for some i). A Betti number $\beta_{ij} > 0$ is called extremal, if $\beta_{k\ell} = 0$ for all $k \geq i$ and $\ell > j$. There always exists a least one extremal Betti number: if we take the maximal value i for which $\beta_{i,i+\operatorname{reg}\mathcal{I}} > 0$, then $\beta_{i,i+\operatorname{reg}\mathcal{I}}$ is extremal. In general, there may exist further extremal Betti numbers. Bayer et al. [8, Thm. 1.6] proved that for any ideal $\mathcal{I}$ both the positions and the values of the extremal Betti numbers coincides with those of its generic initial ideal with respect to the degree reverse lexicographic order.

Our proof of Theorem 9.2 allows us to make the same statement for the ordinary initial ideal for $\prec_{\text{degrevlex}}$—provided the coordinates are δ-regular. Furthermore, it shows that the extremal Betti numbers of $\mathcal{I}$ can be immediately read off the Pommaret basis $\mathcal{H}$ of $\mathcal{I}$. Finally, if we introduce "pseudo-Betti numbers" for the (in general non-minimal) resolution (35), then the positions and values of the extremal ones coincide with the true extremal Betti numbers of $\mathcal{I}$.

Take the generator h_γ used in the proof of Theorem 9.2. If $\operatorname{cls} h_\gamma = n - i_1$ and $\deg h_\gamma = q_1$, then the considerations in the proof imply immediately that β_{i_1,q_1+i_1} is an extremal Betti number and its value is given by the number of generators of degree q_1 and class $n - i_1$ in the Pommaret basis $\mathcal{H}$. If $i_1 = \operatorname{depth}\mathcal{I}$, then this is the only extremal Betti number. Otherwise, let q_2 be the maximal degree of a generator $h \in \mathcal{H}$ with $\operatorname{cls} h < n - i_1$ and assume that $n - i_2$ is the minimal class of such a generator. Then the arguments used in the proof of Theorem 9.2 show that

β_{i_2,q_2+i_2} is also an extremal Betti number and that its value is given by the number of generators of degree q_2 and class $n-i_2$ in the Pommaret basis $\mathcal{H}$. Continuing in this manner, we obtain all extremal Betti numbers. Since all our considerations depend only on the leading terms of the generators, we find for the leading ideal $\operatorname{lt}_\prec\mathcal{I}$ exactly the same situation. $\triangleleft$

Bayer and Stillman [10] gave the following characterisation of q-regularity for which we now provide a new proof. Note the close relationship between their first condition and the idea of assigning multiplicative variables.

Theorem 9.8 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal which can be generated by elements of degree less than or equal to q. Then $\mathcal{I}$ is q-regular, if and only if for some value $0 \leq d \leq n$ linear forms $y_1,\dots,y_d \in \mathcal{P}_1$ exist such that*

$$\bigl(\langle\mathcal{I},y_1,\dots,y_{j-1}\rangle : y_j\bigr)_q = \langle\mathcal{I},y_1,\dots,y_{j-1}\rangle_q\ , \quad 1 \leq j \leq d\ , \tag{59a}$$

$$\langle\mathcal{I},y_1,\dots,y_d\rangle_q = \mathcal{P}_q\ . \tag{59b}$$

Proof Assume first that the conditions (59) are satisfied for some linear forms $y_1,\dots,y_d \in \mathcal{P}_1$ and choose variables $\mathbf{x}$ such that $x_i = y_i$ for $1 \leq i \leq d$. Let the finite set $\mathcal{H}_q$ be a basis of $\mathcal{I}_q$ as a vector space in triangular form with respect to the degree reverse lexicographic order, i. e. $\operatorname{lt}_\prec h_1 \neq \operatorname{lt}_\prec h_2$ for all $h_1,h_2 \in \mathcal{H}_q$. We claim that $\mathcal{H}_q$ is a Pommaret basis of the truncation $\mathcal{I}_{\geq q}$ implying that the full ideal $\mathcal{I}$ possesses a Pommaret basis of degree $q' \leq q$ and hence by Theorem 9.2 that $\operatorname{reg}\mathcal{I} \leq q$.

Let us write $\mathcal{H}_q = \bigl\{h_{k,\ell} \mid 1 \leq k \leq n,\ 1 \leq \ell \leq \ell_k\bigr\}$ where $\operatorname{cls} h_{k,\ell} = k$. A basis of the vector space $\langle\mathcal{I},x_1,\dots,x_j\rangle_q$ is then given by all $h_{k,\ell}$ with $k > j$ and all terms in $\langle x_1,\dots,x_j\rangle_q$. We will now show that

$$\mathcal{H}_{q+1} = \bigl\{x_j h_{k,\ell} \mid 1 \leq j \leq k,\ 1 \leq k \leq n,\ 1 \leq \ell \leq \ell_k\bigr\} \tag{60}$$

is a basis of $\mathcal{I}_{q+1}$ as a vector space. This implies that $\mathcal{H}_q$ is locally involutive for the Pommaret division and thus involutive by Corollary 7.3 of Part I. Since, by assumption, $\mathcal{I}$ is generated in degrees less than or equal to q, we have furthermore $\langle\mathcal{H}_q\rangle = \mathcal{I}_{\geq q}$ so that indeed $\mathcal{H}_q$ is a Pommaret basis of the ideal $\mathcal{I}_{\geq q}$.

Let $f \in \mathcal{I}_{q+1}$ and $\operatorname{cls} f = j$. By the properties of the degree reverse lexicographic order this implies that $f = x_j\hat{f} + g$ with $\hat{f} \in \bigl(\mathbb{k}[x_j,\dots,x_n] \setminus \{0\}\bigr)_q$ and $g \in \bigl(\langle x_1,\dots,x_{j-1}\rangle\bigr)_{q+1}$ (cf. Lemma A.1 of Part I). We distinguish two cases. The condition (59b) implies that $\bigl(\langle\mathcal{I},x_1,\dots,x_d\rangle\bigr)_q = \mathcal{P}_q$. Thus if $j > d$, we may write $\hat{f} = \sum_{k=d+1}^{n}\sum_{\ell=1}^{\ell_k} c_{k,\ell}h_{k,\ell} + \hat{g}$ with $c_{k,\ell} \in \mathbb{k}$ and $\hat{g} \in \bigl(\langle x_1,\dots,x_d\rangle\bigr)_q$. We set $\hat{f}_0 = \sum_{k=j}^{n}\sum_{\ell=1}^{\ell_k} c_{k,\ell}h_{k,\ell}$ and $\hat{f}_1 = \sum_{k=d+1}^{j-1}\sum_{\ell=1}^{\ell_k} c_{k,\ell}h_{k,\ell} + \hat{g}$. Obviously, $\hat{f} \in \bigl(\langle\mathcal{I},x_1,\dots,x_{j-1}\rangle : x_j\bigr)_q$. If $j \leq d$, then the condition (59a) implies that actually $\hat{f} \in \langle\mathcal{I},x_1,\dots,x_{j-1}\rangle_q$. Hence in this case we may decompose $\hat{f} = \hat{f}_0 + \hat{f}_1$ with $\hat{f}_0 = \sum_{k=j}^{n}\sum_{\ell=1}^{\ell_k} c_{k,\ell}h_{k,\ell}$ and $\hat{f}_1 \in \bigl(\langle x_1,\dots,x_{j-1}\rangle\bigr)_q$.

It is trivial that $\langle\mathcal{H}_{q+1}\rangle \subseteq \mathcal{I}_{q+1}$ (here we mean the linear span over $\mathbb{k}$ and not over $\mathcal{P}$). We show by an induction over j that $\mathcal{I}_{q+1} \subseteq \langle\mathcal{H}_{q+1}\rangle$. If $j = 1$, then

$f = x_1\hat{f}$ with $\hat{f} \in \mathcal{I}_q$. Thus $f \in \langle\mathcal{H}_{q+1}\rangle$. If $j > 1$, we write $f = f_0 + f_1$ with $f_0 = x_j\hat{f}_0$ and $f_1 = x_j\hat{f}_1 + g$ where $\hat{f}_0$ and $\hat{f}_1$ have been defined above. By construction, $f_0 \in \langle\mathcal{H}_{q+1}\rangle$, as x_j is multiplicative for all generators contained in $\hat{f}_0$, and $f_1 \in \mathcal{I}_{q+1}$ with $\operatorname{cls} f_1 < j$. According to our inductive hypothesis this implies that $f_1 \in \langle\mathcal{H}_{q+1}\rangle$, too. Hence $\langle\mathcal{H}_{q+1}\rangle = \mathcal{I}_{q+1}$.

Assume conversely that the ideal $\mathcal{I}$ is q-regular. Then, by Theorem 9.2, it possesses a Pommaret basis $\mathcal{H}$ of degree $\operatorname{reg}\mathcal{I} \leq q$ with respect to the degree reverse lexicographic order. We set $d = \dim \mathcal{P}/\mathcal{I}$ and claim that for the choice $y_i = x_i$ for $1 \leq i \leq d$ the conditions (59) are satisfied. For the second equality (59b) this follows immediately from Proposition 3.15 which shows that it actually holds already at degree $\operatorname{reg}\mathcal{I} \leq q$.

For the first equality (59a) take a polynomial $f \in \bigl(\langle\mathcal{I}, x_1, \dots, x_{j-1}\rangle : x_j\bigr)_q$. By definition, we have then $x_j f \in \langle\mathcal{I}, x_1, \dots, x_{j-1}\rangle$. If $f \in \langle x_1, \dots, x_{j-1}\rangle$, then there is nothing to prove. Otherwise, a polynomial $g \in \langle x_1, \dots, x_{j-1}\rangle$ exists such that $x_j f - g \in \mathcal{I}$ and obviously $\operatorname{cls}(x_j f - g) = j$. If we introduce the set $\mathcal{H}_{\geq j} = \{h \in \mathcal{H} \mid \operatorname{cls} h \geq j\}$, the involutive standard representation of $x_j f - g$ induces an equation $x_j f = \sum_{h\in\mathcal{H}_{\geq j}} P_h h + \bar{g}$ where $\bar{g} \in x_j\langle x_1, \dots, x_{j-1}\rangle$ and $P_h \in \langle x_j\rangle$ (this is trivial if $\operatorname{cls} h > j$ and follows from $\deg h \leq q$ if $\operatorname{cls} h = j$). Thus we can divide by x_j and find that already $f \in \langle\mathcal{I}, x_1, \dots, x_{j-1}\rangle_q$. □

Bayer and Stillman [10] further proved that in *generic* coordinates it is not possible to find a Gröbner basis of degree less than $\operatorname{reg}\mathcal{I}$ and that this estimate is sharp, as it is realised by bases with respect to the degree reverse lexicographic order. The restriction to the generic case is here essential, as for instance most monomial ideals are trivial counterexamples. Hence their result is only of limited use for the actual computation of the Castelnuovo-Mumford regularity, as one never knows whether one works with generic coordinates.

Example 9.9 Consider the homogeneous ideal

$$\mathcal{I} = \langle z^8 - wxy^6,\ y^7 - x^6 z,\ yz^7 - wx^7\rangle \subset \mathbb{Q}[w,x,y,z]\ . \tag{61}$$

The given basis of degree 8 is already a Gröbner basis for the degree reverse lexicographic term order. If we perform a simple permutation of two variables and consider $\mathcal{I}$ as an ideal in $\mathbb{Q}[w,y,x,z]$, then we obtain for the degree reverse lexicographic term order the following Gröbner basis of degree 50:

$$\begin{aligned}\bigl\{y^7 - x^6 z,\ yz^7 - wx^7,\ z^8 - wxy^6,\ y^8z^6 - wx^{13},& \\ y^{15}z^5 - wx^{19},\ y^{22}z^4 - wx^{25},\ y^{29}z^3 - wx^{31},& \\ y^{36}z^2 - wx^{37},\ y^{43}z - wx^{43},\ y^{50} - wx^{49}\bigr\}\ .&\end{aligned} \tag{62}$$

Unfortunately, neither coordinate system is generic: as $\operatorname{reg}\mathcal{I} = 13$, one yields a basis of too low degree and the other one one of too high degree.

With a Pommaret basis it is no problem to determine the Castelnuovo-Mumford regularity, as the first coordinate system is δ-regular. A Pommaret basis of $\mathcal{I}$ for the degree reverse lexicographic term order is obtained by adding the polynomials $z^k(y^7 - x^6 z)$ for $1 \leq k \leq 6$ and thus the degree of the basis is indeed 13. ◁

Yet another characterisation of q-regularity is due to Eisenbud and Goto [25]. We give a constructive proof of it as an easy corollary of Theorem 9.2.

Theorem 9.10 *The homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ is q-regular, if and only if its truncation $\mathcal{I}_{\geq q}$ admits a linear resolution.*

Proof If $\mathcal{I}$ is q-regular, then by Theorem 9.2 it possesses in suitable coordinates a Pommaret basis $\mathcal{H}$ of degree $\operatorname{reg}\mathcal{I} \leq q$. The set $\mathcal{H}_q$ defined by (1) is a Pommaret basis of the truncated ideal $\mathcal{I}_{\geq q}$ according to Lemma 2.2. Now it follows easily from Theorem 5.10 that $\mathcal{I}_{\geq q}$ possesses a linear free resolution, as all syzygies in the resolution (35) derived from $\mathcal{H}_q$ are necessarily homogeneous of degree 1.

The converse is trivial. The existence of a linear resolution for $\mathcal{I}_{\geq q}$ immediately implies that $\operatorname{reg}\mathcal{I}_{\geq q} = q$. Hence $\mathcal{I}_{\geq q}$ possesses a Pommaret basis of degree q by Theorem 9.2 entailing the existence of a Pommaret basis for $\mathcal{I}$ of degree $q' \leq q$. Hence, again by Theorem 9.2, $\operatorname{reg}\mathcal{I} = q' \leq q$. □

Remark 9.11 We are now finally in a position where we can finish the discussion started in Remark 2.18 on the effective construction of Pommaret bases. There we were not able to prove that after a finite number of coordinate transformations based on our criterion for asymptotic singularity (Theorem 2.13) one always arrives at a δ-regular coordinate system for a given homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$. Recall that our main problem in Remark 2.18 was that we do not have a bound for the degrees of either Pommaret or Janet bases of $\mathcal{I}$. Our results above do not provide us with such a bound, but it still turns out that we can prove the termination of our approach by studying what happens at the finite degree $q = \operatorname{reg}\mathcal{I}$.

We assume from now on that we are working with a class respecting order and with an infinite field $\mathbb{k}$. By the considerations in the proof above, our coordinates $\mathbf{x}$ are δ-regular, if and only if an involutively head autoreduced, $\mathbb{k}$-linear basis of $\mathcal{I}_q$ is also a Pommaret basis of $\mathcal{I}_{\geq q}$. Denote, as in Remark 2.8, by $\beta_q^{(k)}$ the number of elements of class k in such a basis. There we already noted that these numbers are invariants of $\mathcal{I}$, as they are in a one-to-one correspondence with the coefficients of the Hilbert polynomial $H_{\mathcal{I}}$.

Consider now some basis $\mathcal{H}$ arising during the completion process. It induces a subset $\mathcal{H}_q \subset \mathcal{I}_q$ by taking all Pommaret multiplicative multiples of elements up to degree q; let $\tilde{\beta}_q^{(k)}$ be the number of members of class k in it. If $\mathcal{H}$ is not a Pommaret basis, then a comparison of the values $\beta_q^{(k)}$ and $\tilde{\beta}_q^{(k)}$ starting with $k = n$ will sooner or later lead to a smaller value $\tilde{\beta}_q^{(k)}$; more precisely, we have the inequality $\sum_{k=1}^{n} k\tilde{\beta}_q^{(k)} \leq \sum_{k=1}^{n} k\beta_q^{(k)}$ with equality holding, if and only if $\mathcal{H}$ is a Pommaret basis.

Each completion step which adds an element of degree q or less increases the value of the sum $\sum_{k=1}^{n} k\tilde{\beta}_q^{(k)}$. Consider now the effect of a coordinate transformation of the form used in the proof of Theorem 2.13. All new terms arising on the right hand side of (3) are greater than the original one with respect to any class respecting term order. Thus in general we can expect that after such a transformation at least some leading terms of the new set $\mathcal{H}_q$ are greater than before. In fact, by the same argument as in the proof of Theorem 2.13, we even can be sure that after a finite number of transformations this will indeed be the case. But this observation implies that after a finite number of transformations the sum $\sum_{k=1}^{n} k\tilde{\beta}_q^{(k)}$ must increase and eventually we must obtain after a finite number of completion steps and coordinate transformations the right value for this sum implying that we have obtained δ-regular coordinates and a Pommaret basis. $\triangleleft$

As a further corollary to Theorem 9.10, we provide a converse to Theorem 8.2 generalising Theorem 8.9 from the monomial case to polynomial submodules.

Theorem 9.12 *Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a componentwise linear submodule with Pommaret basis $\mathcal{H}$. Then generically the resolution (35) is minimal. It is minimal, if and only if $\mathcal{H}$ is a minimal basis of $\mathcal{M}$.*

Proof As Example 8.3 demonstrated, the problem is that generally the sets $\mathcal{G}_d$ defined by (56) are not Pommaret bases of the modules $\mathcal{M}_{\langle d\rangle}$ for all $d \geq 0$. According to Theorems 9.10 and 9.2, this is trivially the case for all degrees $d \geq q = \deg \mathcal{H}$, since for them $\mathcal{M}_{\langle d\rangle} = \mathcal{M}_{\geq d}$. Thus it suffices to consider the finitely many modules $\mathcal{M}_{\langle 0\rangle}, \dots, \mathcal{M}_{\langle q\rangle}$. By Corollary 2.17, generic coordinate systems are simultaneously δ-regular for all these modules (and then also for the whole submodule $\mathcal{M}$).

Let $\mathcal{H}$ be the Pommaret basis of $\mathcal{M}$ in such a coordinate system and consider the corresponding sets $\mathcal{G}_d$ for $0 \leq d \leq q$. By construction, $\mathcal{M}_{\langle d\rangle} = \langle \mathcal{G}_d \rangle$. According to our assumption, all modules $\mathcal{M}_{\langle d\rangle}$ possess linear resolutions and thus $\operatorname{reg} \mathcal{M}_{\langle d\rangle} = d$. Hence the Pommaret basis of $\mathcal{M}_{\langle d\rangle}$ is of degree d by Theorem 9.2, which is only possible, if it is already given by $\mathcal{G}_d$. If all sets $\mathcal{G}_d$ are involutive, then no first syzygy of the Pommaret basis $\mathcal{H}$ can contain a constant term and it follows from Lemma 8.1 that the resolution (35) is minimal. Furthermore, in this case the basis $\mathcal{H}$ must be a minimal generating set of $\mathcal{M}$. Indeed, it is trivial that the elements of $\mathcal{H}$ of lowest degree are minimal generators and since no element of a higher degree d can be contained in a module $\mathcal{M}_{\langle d'\rangle}$ for any $d' < d$, it must also be a minimal generator. $\square$

Remark 9.13 These considerations in the proof above can be exploited for effectively deciding whether a given submodule $\mathcal{M} \subseteq \mathcal{P}^m$ is componentwise linear. We compute a Pommaret basis $\mathcal{H}$ for $\mathcal{M}$, changing to δ-regular coordinates if necessary. If the resolution (35) determined by $\mathcal{H}$ is minimal, then $\mathcal{M}$ is componentwise linear by Theorem 8.2 (the minimality of the resolution is trivial to check with Lemma 8.1). Otherwise, there are first syzygies in (35) containing a constant term. Let $\mathbf{S}_{\alpha;\ell}$ be one of minimal degree. If $\deg \mathbf{h}_\alpha = d$, then obviously all modules $\mathcal{M}_{\langle d'\rangle}$ for degrees $d' < d$ possess linear resolutions (coming from their Pommaret bases $\mathcal{G}_d$). For analysing the module $\mathcal{M}_{\langle d\rangle}$, we take the corresponding set $\mathcal{G}_d$ and complete it to a Pommaret basis $\mathcal{H}_d$ (potentially performing further coordinate transformation). If $\deg \mathcal{H}_d = d$, then we recompute the Pommaret basis $\mathcal{H}$ of $\mathcal{M}$ in the new coordinates, which trivially are still δ-regular for $\mathcal{M}$ and all modules $\mathcal{M}_{\langle d'\rangle}$ with $d' < d$, and check again for minimality. In the case that obstructions in some degree $d < \bar{d} < \operatorname{reg} \mathcal{M}$ appear, we continue in the same manner. After a finite number of steps we either obtain a minimal resolution and

$\mathcal{M}$ is componentwise linear or we find a degree d such that the module $\mathcal{M}_{\langle d\rangle}$ does not possess a linear resolution. $\triangleleft$

10 Regularity and Saturation

Already in the work of Bayer and Stillman [10] on the Castelnuovo-Mumford regularity the *saturation* $\mathcal{I}^{\text{sat}}$ of a homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ plays an important role. Recall that by definition

$$\mathcal{I}^{\text{sat}} = \mathcal{I} : \mathcal{P}_+^\infty = \left\{ f \in \mathcal{P} \mid \exists k \in \mathbb{N}_0 : f \cdot \mathcal{P}_k \subset \mathcal{I} \right\} . \tag{63}$$

An ideal $\mathcal{I}$ such that $\mathcal{I} = \mathcal{I}^{\text{sat}}$ is called *saturated*. We show now first how $\mathcal{I}^{\text{sat}}$ can be effectively determined from a Pommaret basis of $\mathcal{I}$.[18]

Proposition 10.1 *Let $\mathcal{H}$ be a Pommaret basis of the homogeneous ideal $\mathcal{I}$ for a class respecting term order. We introduce the sets $\mathcal{H}_1 = \{h \in \mathcal{H} \mid \operatorname{cls} h = 1\}$ and $\bar{\mathcal{H}}_1 = \left\{ h/x_1^{\deg_{x_1} \operatorname{lt}_\prec h} \mid h \in \mathcal{H}_1 \right\}$. Then $\bar{\mathcal{H}} = (\mathcal{H} \setminus \mathcal{H}_1) \cup \bar{\mathcal{H}}_1$ is a weak Pommaret basis of the saturation $\mathcal{I}^{\text{sat}}$.*

Proof Recall that for terms of the same degree any class respecting term order coincides with the reverse lexicographic order. Hence of all terms in a generator $h \in \mathcal{H}_1$ the leading term $\operatorname{lt}_\prec h$ has the lowest x_1-degree. This implies in particular that $\bar{\mathcal{H}}_1$ is well-defined and does not contain a generator of class 1 anymore.

We first show that indeed $\bar{\mathcal{H}}_1 \subset \mathcal{I}^{\text{sat}}$. Let $d_1 = \max_{h \in \mathcal{H}_1} \{\deg_{x_1} \operatorname{lt}_\prec h\}$ and $\Delta = d_1 + \max_{h \in \mathcal{H}_1} \{\deg h\} - \min_{h \in \mathcal{H}_1} \{\deg h\}$. We claim that $\bar{h} \cdot \mathcal{P}_\Delta \subset \mathcal{I}$ for all $\bar{h} \in \bar{\mathcal{H}}_1$. Thus let $x^\mu \in \mathcal{P}_\Delta$ and choose $k \in \mathbb{N}_0$ such that $x_1^k \bar{h} \in \mathcal{H}_1$; obviously, we have $k \leq d_1$. Since the polynomial $x^\mu x_1^k \bar{h}$ lies in $\mathcal{I}$, it possesses an involutive standard representation of the form

$$x^\mu x_1^k \bar{h} = \sum_{h \in \mathcal{H} \setminus \mathcal{H}_1} P_h h + \sum_{h \in \mathcal{H}_1} Q_h h \tag{64}$$

with $P_h \in \mathbb{k}[x_1, \dots, x_{\operatorname{cls} h}]$ and $Q_h \in \mathbb{k}[x_1]$.

The left hand side of this equation is contained in $\langle x_1^k \rangle$ and thus also the right hand side. Analysing an involutive normal form computation leading to the representation (64), one immediately sees that this implies that all coefficients P_h (since here $\operatorname{cls} h > 1$) and all summands $Q_h h$ lie in $\langle x_1^k \rangle$. As a first consequence of this representation we observe that for any monomial x^μ (not necessarily of degree Δ) we may divide (64) by x_1^k and then obtain an involutive standard representation of $x^\mu \bar{h}$ with respect to the set $\bar{\mathcal{H}}$; hence this set is indeed weakly involutive for the Pommaret division and the given term order.

If $x^\mu \in \mathcal{P}_\Delta$, then we find for any $h \in \mathcal{H}_1$ that $|\deg \bar{h} - \deg h| \leq \Delta$ and hence $\deg Q_h = \deg \left(x^\mu x_1^k \bar{h} \right) - \deg h \geq k$. Since $Q_h \in \mathbb{k}[x_1]$, this implies that

[18] It seems to be folklore that for Gröbner bases the construction in Proposition 10.1 yields a Gröbner basis of $\mathcal{I} : x_1^\infty$; in [71, Prop. 5.1.11] this observation is attributed (without reference) to Bayer. In our case we do not only get a Pommaret basis but it also turns out that here $\mathcal{I}^{\text{sat}} = \mathcal{I} : x_1^\infty$ (see the remarks below).

under the made assumption on x^μ already the coefficient Q_h lies in $\langle x_1^k\rangle$ so that the product $x^\mu\bar{h}$ possesses an involutive standard representation with respect to $\mathcal{H}$ and thus is contained in the ideal $\mathcal{I}$ as claimed.

Now we show that every polynomial $f \in \mathcal{I}^{\text{sat}}$ may be decomposed into an element of $\mathcal{I}$ and a linear combination of elements of $\bar{\mathcal{H}}_1$. We may write $f = \tilde{f} + g$ where $\tilde{f}$ is the involutive normal form of f with respect to $\mathcal{H}$ and $g \in \mathcal{I}$. If $\tilde{f} = 0$, then already $f \in \mathcal{I}$ and nothing is to be shown. Hence we assume that $\tilde{f} \neq 0$. By definition of the saturation $\mathcal{I}^{\text{sat}}$, there exists a $k \in \mathbb{N}_0$ such that $\tilde{f} \cdot \mathcal{P}_k \subset \mathcal{I}$, hence in particular $x_1^k\tilde{f} \in \mathcal{I}$. This implies that $\operatorname{lt}_\prec(x_1^k\tilde{f}) \in \langle \operatorname{lt}_\prec\mathcal{H}\rangle_P$. Therefore a unique generator $h \in \mathcal{H}$ exists with $\operatorname{lt}_\prec h \,|_P\, \operatorname{lt}_\prec(x_1^k\tilde{f})$.

So let $\operatorname{lt}_\prec(x_1^k\tilde{f}) = x^\mu \operatorname{lt}_\prec h$ and assume first that $\operatorname{cls} h > 1$. Since the term on the left hand side is contained in $\langle x_1^k\rangle$, we must have $\mu_1 \geq k$ so that we can divide by x_1^k. But this observation implies that already $\operatorname{lt}_\prec\tilde{f} \in \langle \operatorname{lt}_\prec\mathcal{H}\rangle_P$ contradicting our assumption that $\tilde{f}$ is in involutive normal form. Hence we must have $\operatorname{cls} h = 1$ and by the same argument as above $\mu_1 < k$.

Division by x_1^k shows that $\operatorname{lt}_\prec\tilde{f} \in \langle \operatorname{lt}_\prec\bar{\mathcal{H}}_1\rangle_P$. Performing the corresponding involutive reduction leads to a new element $f_1 \in \mathcal{I}^{\text{sat}}$. We compute again its involutive normal form $\tilde{f}_1$ and apply the same argument as above, if $\tilde{f}_1 \neq 0$. After a finite number of such reductions we obtain an involutive standard representation of f with respect to the set $\bar{\mathcal{H}}$ proving our assertion. ☐

By Proposition 5.7 of Part I, an involutive head autoreduction of the set $\bar{\mathcal{H}}$ yields a strong Pommaret basis for the saturation $\mathcal{I}^{\text{sat}}$. As a trivial consequence of the considerations in the proof above, we find that in δ-regular coordinates $\mathcal{I}^{\text{sat}}$ is simply given by the quotient $\mathcal{I} : x_1^\infty$ (in the monomial case this fact also follows immediately from Proposition 4.4 (iv)). This observation in turn implies that for degrees $q \geq \deg\mathcal{H}_1$ we have $\mathcal{I}_q = \mathcal{I}_q^{\text{sat}}$. Hence we recover the well-known fact that all ideals with the same saturation possess also the same Hilbert polynomial and become identical for sufficiently high degrees; $\mathcal{I}^{\text{sat}}$ is the largest among all these ideals. The smallest value q_0 such that $\mathcal{I}_q = \mathcal{I}_q^{\text{sat}}$ for all $q \geq q_0$ is often called the *satiety* $\operatorname{sat}\mathcal{I}$ of the ideal $\mathcal{I}$.

Corollary 10.2 *Let $\mathcal{H}$ be a Pommaret basis of the ideal $\mathcal{I} \subseteq \mathcal{P}$. Then $\mathcal{I}$ is saturated, if and only if $\mathcal{H}_1 = \emptyset$. If $\mathcal{I}$ is not saturated, then* $\operatorname{sat}\mathcal{I} = \deg\mathcal{H}_1$. *Independent of the existence of a Pommaret basis, we have for any homogeneous generating set $\mathcal{F}$ of the socle $\mathcal{I} : \mathcal{P}_+$ the equality*

$$\operatorname{sat}\mathcal{I} = 1 + \max\{\deg f \mid f \in \mathcal{F} \wedge f \notin \mathcal{I}\}\,. \tag{65}$$

Proof Except of the last statement, everything has already been proven in the discussion above. For its proof we may assume without loss of generality that the coordinates are δ-regular so that a Pommaret basis $\mathcal{H}$ of $\mathcal{I}$ exists, as all quantities appearing in (65) are invariant under linear coordinate transformations.

Let $\tilde{h}$ be an arbitrary element of $\mathcal{H}_1$ of maximal degree. We claim that then $\tilde{h}/x_1 \in (\mathcal{I} : \mathcal{P}_+) \setminus \mathcal{I}$. Indeed, since x_1 is always multiplicative for the Pommaret division, we cannot have $\tilde{h}/x_1 \in \mathcal{I}$ (otherwise $\mathcal{H}$ would not be involutively head autoreduced), and if we analyse for any $1 < \ell \leq n$ the involutive standard representation of $x_\ell\tilde{h}$, then all coefficients of generators $h \in \mathcal{H} \setminus \mathcal{H}_1$ are trivially contained in $\langle x_1\rangle$ and for the coefficients of elements $h \in \mathcal{H}_1$ the same holds for degree reasons. Hence we can divide by x_1 and find that $x_\ell\tilde{h}/x_1 \in \mathcal{I}$ for all $1 \leq \ell \leq n$ implying $\tilde{h}/x_1 \in \mathcal{I} : \mathcal{P}_+$.

By our previous results, $\operatorname{sat}\mathcal{I} = \deg\tilde{h}$. By assumption, homogeneous polynomials $P_f \in \mathcal{P}$ exist such that $\tilde{h}/x_1 = \sum_{f\in\mathcal{F}} P_f f$. If $\deg P_f \geq 1$, then $P_f f \in \mathcal{I}$ since $f \in \mathcal{I} : \mathcal{P}_+$. Hence for at least one $\tilde{f} \in \mathcal{F} \setminus \mathcal{I}$, the coefficient $P_{\tilde{f}}$ must be a non-zero constant and thus $\deg\tilde{f} = \operatorname{sat}\mathcal{I} - 1$. □

Remark 10.3 The last statement in Corollary 10.2 is due to Bermejo and Gimenez [12, Prop. 2.1] who proved it in a slightly different way. For *monomial* ideals $\mathcal{I}$, one obtains as further corollary [12, Cor. 2.4] that, if $\mathcal{I} : \mathcal{P}_+ = \mathcal{I} : x_1$, then $\operatorname{sat}\mathcal{I}$ is the maximal degree of a minimal generator of $\mathcal{I}$ divisible by x_1 (this observation generalises a classical result about Borel-fixed ideals [33, Cor. 2.10]). If the considered ideal $\mathcal{I}$ possesses a Pommaret basis $\mathcal{H}$, this statement also follows from the fact that under the made assumptions all elements of $\mathcal{H}_1$ are minimal generators. Indeed, suppose to the contrary that $\mathcal{H}_1$ contains two elements $h_1 \neq h_2$ such that $h_1 \mid h_2$. Obviously, the minimality implies $\deg_{x_1} h_1 = \deg_{x_1} h_2$ and a non-multiplicative index $1 < \ell \leq n$ exists such that $x_\ell h_1 \mid h_2$. Without loss of generality, we may assume that $h_2 = x_\ell h_1$. But this immediately entails that $x_\ell h_1/x_1 = h_2/x_1 \notin \mathcal{I}$ and hence $h_1/x_1 \in (\mathcal{I} : x_1) \setminus (\mathcal{I} : \mathcal{P}_+)$. ◁

A first trivial consequence of our results is the following well-known formula relating Castelnuovo-Mumford regularity and saturation.

Corollary 10.4 *Let* $\mathcal{I} \subseteq \mathcal{P}$ *be an ideal. Then* $\operatorname{reg}\mathcal{I} = \max\{\operatorname{sat}\mathcal{I}, \operatorname{reg}\mathcal{I}^{\mathrm{sat}}\}$.

Proof Without loss of generality, we may assume that we use δ-regular coordinates so that $\mathcal{I}$ possesses a Pommaret basis $\mathcal{H}$ with respect to the degree reverse lexicographic order. Now the statement follows immediately from Proposition 10.1 and Corollary 10.2. □

Trung [78] proposed the following approach for computing the regularity of a monomial ideal $\mathcal{I}$ based on evaluations. Let $D = \dim(\mathcal{P}/\mathcal{I})$ and introduce for $j = 0, \dots, D$ the polynomial subrings[19] $\mathcal{P}^{(j)} = \Bbbk[x_{j+1}, \dots, x_n]$ and within them the elimination ideals $\mathcal{I}^{(j)} = \mathcal{I} \cap \mathcal{P}^{(j)}$ and their saturations $\tilde{\mathcal{I}}^{(j)} = \mathcal{I}^{(j)} : x_{j+1}^\infty$. A basis of $\mathcal{I}^{(j)}$ is then obtained by setting $x_1 = \cdots = x_j = 0$ in a basis of $\mathcal{I}$ and for a basis of $\tilde{\mathcal{I}}^{(j)}$ we must additionally set $x_{j+1} = 1$. Now define the numbers

$$c_j = \sup\left\{q \mid (\tilde{\mathcal{I}}^{(j)}/\mathcal{I}^{(j)})_q \neq 0\right\} + 1\,, \qquad 0 \leq j < D \tag{66a}$$

$$c_D = \sup\left\{q \mid (\mathcal{P}^{(D)}/\mathcal{I}^{(D)})_q \neq 0\right\} + 1\,. \tag{66b}$$

Trung [78] proved that whenever none of these numbers is infinite, then their maximum is just $\operatorname{reg}\mathcal{I}$. We show now that this genericity condition is satisfied, if and only if the coordinates are δ-regular and express the numbers c_j as satieties.

[19] Compared with Trung [78], we revert as usual the order of the variables in order to be consistent with our conventions.

Theorem 10.5 *The numbers $c_0, \dots, c_D$ are all finite, if and only if the monomial ideal $\mathcal{I} \subseteq \mathcal{P}$ is quasi-stable. In this case $c_j = \operatorname{sat} \mathcal{I}^{(j)}$ for $0 \leq j \leq D$ and*

$$\max \{c_0, \dots, c_D\} = \operatorname{reg} \mathcal{I} \ . \tag{67}$$

If $d = \operatorname{depth} \mathcal{I}$, then it suffices to consider $c_d, \dots, c_D$.

Proof We assume first that $\mathcal{I}$ is quasi-stable and thus possesses a Pommaret basis which we write $\mathcal{H} = \{h_{k,\ell} \mid 1 \leq k \leq n,\ 1 \leq \ell \leq \ell_k\}$ where $\operatorname{cls} h_{k,\ell} = k$. One easily verifies that the subset $\mathcal{H}^{(j)} = \{h_{k,\ell} \in \mathcal{H} \mid k > j\}$ is the Pommaret basis of the ideal $\mathcal{I}^{(j)}$. If we set $a_{k,\ell} = \deg_{x_k} h_{k,\ell}$, then the Pommaret basis of $\tilde{\mathcal{I}}^{(j)}$ is $\tilde{\mathcal{H}}^{(j)} = \mathcal{H}^{(j+1)} \cup \{h_{j+1,\ell}/x_{j+1}^{a_{j+1,\ell}} \mid 1 \leq \ell \leq \ell_{j+1}\}$. This immediately implies that $c_j = \max \{\deg h_{j+1,\ell} \mid 1 \leq \ell \leq \ell_{j+1}\}$. By construction, $\dim (\mathcal{P}^{(D)}/\mathcal{I}^{(D)}) = 0$ and Proposition 3.15 entails that for $\hat{q} = \deg \mathcal{H}^{(D)}$ the equality $\mathcal{I}^{(D)}_{\hat{q}} = \mathcal{P}^{(D)}_{\hat{q}}$ holds. Hence $c_D = \hat{q}$ (it is not possible that $c_D < \hat{q}$, as otherwise the set $\mathcal{H}$ was not involutively autoreduced).

Thus we find that $\max \{c_0, \dots, c_D\} = \deg \mathcal{H}$ and Theorem 9.2 yields (67). Furthermore, it follows immediately from Corollary 10.2 and Proposition 3.15, respectively, that $c_j = \operatorname{sat} \mathcal{I}^{(j)}$ for $0 \leq j \leq D$. Finally, Proposition 2.20 entails that the values $c_0, \dots, c_{d-1}$ vanish.

Now assume that the ideal $\mathcal{I}$ was not quasi-stable. By Part (ii) of Proposition 4.4, this entails that for some $0 \leq j < D$ the variable x_{j+1} is a zero divisor in the ring $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_j \rangle^{\mathrm{sat}} \cong \mathcal{P}^{(j)}/(\mathcal{I}^{(j)})^{\mathrm{sat}}$. Thus a polynomial $f \notin (\mathcal{I}^{(j)})^{\mathrm{sat}}$ exists for which $x_{j+1} f \in (\mathcal{I}^{(j)})^{\mathrm{sat}}$ which means that we can find for any sufficiently large degree $q \gg 0$ a polynomial $g \in \mathcal{P}^{(j)}$ with $\deg g = q - \deg f$ such that $fg \notin \mathcal{I}^{(j)}$ but $x_{j+1} fg \in \mathcal{I}^{(j)}$. Hence the equivalence class of fg is a non-vanishing element of $(\tilde{\mathcal{I}}^{(j)}/\mathcal{I}^{(j)})_q$ so that for a not quasi-stable ideal $\mathcal{I}$ at least one value c_j is not finite. □

One direction of the proof above uses the same idea as the one of Theorem 9.2: the Castelnuovo-Mumford regularity is determined by the basis members of maximal degree and their classes give us the positions in the minimal resolution where it is attained (recall Remark 9.3; here these are simply the indices j for which c_j is maximal). However, while Theorem 9.2 holds for arbitrary homogeneous ideals, Trung's approach can only be applied to monomial ideals. The formulation using satieties is at the heart of the method of Bermejo and Gimenez [12] to compute the Castelnuovo-Mumford regularity. Similar considerations yield an alternative proof of the following result of Bermejo and Gimenez [12, Cor. 17] for monomial ideals.

Proposition 10.6 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a quasi-stable ideal and $\mathcal{I} = \mathcal{J}_1 \cap \dots \cap \mathcal{J}_r$ its unique irredundant decomposition into irreducible monomial ideals. Then the equality $\operatorname{reg} \mathcal{I} = \max \{\operatorname{reg} \mathcal{J}_1, \dots, \operatorname{reg} \mathcal{J}_r\}$ holds.*

Proof We first note that the Castelnuovo-Mumford regularity of a monomial irreducible ideal $\mathcal{J} = \langle x_{i_1}^{\ell_1}, \dots, x_{i_k}^{\ell_k} \rangle$ is easily determined using the considerations in Remark 2.13 of Part I. There we showed that any such ideal becomes quasi-stable after a simple renumbering of the variables and explicitly gave its Pommaret basis.

Up to the renumbering, the unique element of maximal degree in this Pommaret basis is the term $x_{i_1}^{\ell_1}x_{i_2}^{\ell_2-1}\cdots x_{i_k}^{\ell_k-1}$ and thus it follows from Theorem 9.2 that $\operatorname{reg}\mathcal{J}=\sum_{j=1}^{k}\ell_j-k+1$.

Recall from Proposition 3.10 that an irreducible decomposition can be constructed via standard pairs. As discussed in Section 3, the decomposition (13) is in general redundant; among all standard pairs (ν,N_ν) with $N_\nu=N$ for a given set N only those exponents ν which are maximal with respect to divisibility appear in the irredundant decomposition and thus are relevant.

If we now determine the standard pairs of $\mathcal{I}$ from a Pommaret basis according to Remark 3.12, then we must distinguish two cases. We have first the standard pairs coming from the terms x^μ of degree $q=\deg\mathcal{H}$ not lying in $\mathcal{I}$. They are of the form $\bigl(x^\nu,\{x_1,\dots,x_k\}\bigr)$ where $k=\operatorname{cls}\mu$ and $x^\nu=x^\mu/x_k^{\mu_k}$. By Proposition 3.10, each such standard pair leads to the irreducible ideal $\mathcal{J}=\langle x_\ell^{\nu_\ell+1}\mid k<\ell\leq n\rangle$. By the remarks above, $\operatorname{reg}\mathcal{J}=|\nu|+1\leq|\mu|=q=\operatorname{reg}\mathcal{I}$.

The other standard pairs come from the terms $x^\nu\notin\mathcal{I}$ with $|\nu|<q$. It is easy to see that among these the relevant ones correspond one-to-one to the "end points" of the monomial completion process: we call an element of the Pommaret basis $\mathcal{H}$ of $\mathcal{I}$ an end point, if each non-multiplicative multiple of it has a *proper* involutive divisor in the basis (and thus one branch of the completion process ends with this element[20]). If $x^\mu\in\mathcal{H}$ is such an end point, then the corresponding standard pair consists of the monomial $x^\nu=x^\mu/x_k$ where $k=\operatorname{cls}\mu$ and the empty set and it yields the irreducible ideal $\mathcal{J}=\langle x_\ell^{\nu_\ell+1}\mid 1\leq\ell\leq n\rangle$. Thus we find again $\operatorname{reg}\mathcal{J}=|\nu|+1=|\mu|\leq q=\operatorname{reg}\mathcal{I}$.

These considerations prove the estimate $\operatorname{reg}\mathcal{I}\geq\max\{\operatorname{reg}\mathcal{J}_1,\dots,\operatorname{reg}\mathcal{J}_r\}$. The claimed equality follows from the observation that any element of degree q in $\mathcal{H}$ must trivially be an end point and the corresponding standard pair yields then an irreducible ideal $\mathcal{J}$ with $\operatorname{reg}\mathcal{J}=q$. □

The question of bounding the Castelnuovo-Mumford regularity of a homogeneous ideal $\mathcal{I}$ in terms of the degree q of an arbitrary generating set has attracted quite some interest. Hermann [40] gave already very early a doubly exponential bound; much later Mayr and Meyer [55] showed with explicit examples that this bound is indeed sharp (see [9] for a more detailed discussion).

For monomial ideals $\mathcal{I}$ the situation is much more favourable. It follows immediately from Taylor's explicit resolution of such ideals [75] (see [65] for a derivation via Gröbner bases) that here a *linear* bound

$$\operatorname{reg}\mathcal{I}\leq n(q-1)+1 \tag{68}$$

holds where n is again the number of variables. Indeed, this resolution is supported by the lcm-lattice of the given basis and the degree of its kth term is thus trivially bounded by kq. By Hilbert's Syzygy Theorem, it suffices to consider the first n terms which immediately yields the above bound. If the ideal $\mathcal{I}$ is even quasi-stable, a simple corollary of Proposition 10.6 yields an improved bound using the minimal generators of $\mathcal{I}$.

[20] Note that an end point may very well be a member of the minimal basis of $\mathcal{I}$!

Corollary 10.7 *Let the monomials $m_1, \dots, m_r$ be the minimal generators of the quasi-stable ideal $\mathcal{I} \subseteq \Bbbk[x_1, \dots, x_n]$. If we set $x^\lambda = \operatorname{lcm}(m_1, \dots, m_r)$ and $d = \min\{\operatorname{cls} m_1, \dots, \operatorname{cls} m_r\}$ (i. e. $d = \operatorname{depth} \mathcal{I}$), then the Castelnuovo–Mumford regularity of $\mathcal{I}$ satisfies the estimate*

$$\operatorname{reg} \mathcal{I} \leq |\lambda| + d - n \tag{69}$$

and this bound is sharp.

Proof Applying repeatedly the rule $\langle \mathcal{F}, t_1 t_2 \rangle = \langle \mathcal{F}, t_1 \rangle \cap \langle \mathcal{F}, t_2 \rangle$ for arbitrary generating sets $\mathcal{F}$ and coprime monomials t_1, t_2, one obtains an irreducible decomposition of $\mathcal{I}$. Obviously, in the worst case one of the irreducible ideals is $\mathcal{J} = \langle x_d^{\lambda_d}, \dots, x_n^{\lambda_n} \rangle$. As we already know that $\operatorname{reg} \mathcal{J} = |\lambda| + d - n$, this value bounds $\operatorname{reg} \mathcal{I}$ by Proposition 10.6. □

Remark 10.8 An alternative direct proof of the corollary goes as follows. Let $\mathcal{H}$ be the Pommaret basis of $\mathcal{I}$. We claim that each generator $x^\mu \in \mathcal{H}$ with $\operatorname{cls} \mu = k$ satisfies $\mu_k \leq \lambda_k$ and $\mu_j < \lambda_j$ for all $j > k$. The estimate for μ_k is obvious, as it follows immediately from our completion algorithm that there is a minimal generator $x^\nu \mid x^\mu$ with $\nu_k = \mu_k$.

Assume for a contradiction that the Pommaret basis $\mathcal{H}$ contains a generator x^μ where $\mu_j > \lambda_j$ for some $j > \operatorname{cls} \mu$. If several such generators exist for the same value j, choose one for which μ_j is maximal. Obviously, j is non-multiplicative for x^μ and hence the multiple $x_j x^\mu$ must contain an involutive divisor $x^\nu \in \mathcal{H}$. Because of our maximality assumption $\nu_j \leq \mu_j$ and hence j must be multiplicative for x^ν so that $\operatorname{cls} \nu \geq j$. But this fact trivially implies that $x^\nu \mid_P x^\mu$ contradicting that $\mathcal{H}$ is by definition involutively autoreduced.

Now the assertion follows immediately: under the made assumptions $\operatorname{cls} \lambda = d$ and in the worst case $\mathcal{H}$ contains the generator $x_d^{\lambda_d} x_{d+1}^{\lambda_{d+1}-1} \cdots x_n^{\lambda_n - 1}$ which is of degree $|\lambda| + d - n$. ◁

Remark 10.9 The same arguments together with Proposition 10.1 also yield immediately a bound for the satiety of a quasi-stable ideal $\mathcal{I}$. As already mentioned above, a quasi-stable ideal is not saturated, if and only if $d = 1$. In this case, we have trivially $\operatorname{sat} \mathcal{I} \leq |\lambda| + 1 - n$. Again the bound is sharp, as shown by exactly the same class of irreducible ideals as considered above.

The estimate (69) also follows immediately from the results in [12]. Yet another derivation is contained in [37]. ◁

If one insists on having an estimate involving only the maximal degree q of the minimal generators and the depth, then the above result yields immediately the following estimate, variations of which appear in [2, 18, 19].

Corollary 10.10 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a quasi-stable ideal minimally generated in degrees less than or equal to q. If $\operatorname{depth} \mathcal{I} = d$, then*

$$q \leq \operatorname{reg} \mathcal{I} \leq (n - d + 1)(q - 1) + 1 \tag{70}$$

and both bounds are sharp.

Proof Under the made assumptions we trivially find that the degree of the least common multiple of the minimal generators is bounded by $|\lambda| \leq (n-d+1)q$. Now (70) follows immediately from (69). The upper bound is realised by the irreducible ideal $\mathcal{I} = \langle x_1^q, \dots, x_n^q \rangle$. The lower bound is attained, if $\mathcal{I}$ is even stable, as then Proposition 8.6 implies that $\operatorname{reg} \mathcal{I} = q$ independent of $\operatorname{depth} \mathcal{I}$. □

Remark 10.11 Eisenbud, Reeves and Totaro [26] presented a variation of the estimate (70). They introduced the notion of *s-stability* as a generalisation of stability: let $s \geq 1$ be an integer; a monomial ideal $\mathcal{I}$ is s-stable, if for every monomial $x^\mu \in \mathcal{I}$ and every index $n \geq j > \operatorname{cls} \mu = k$ an exponent $1 \leq e \leq s$ exists such that $x^{\mu - e_k + e_j} \in \mathcal{I}$. Then it is easy to see that for an s-stable ideal generated in degrees less than or equal to q the estimate

$$\operatorname{reg} \mathcal{I} \leq q + (n-1)(s-1) \tag{71}$$

holds, as $\mathcal{I}_{\geq q+(n-1)(s-1)}$ is stable (thus any s-stable ideal is trivially quasi-stable). However, in general (71) is an overestimate, as it based on the assumption that $\mathcal{I}$ possesses a minimal generator of class 1 and degree q which must be multiplied by $x_2^{s-1} x_3^{s-1} \cdots x_n^{s-1}$ in order to reach a stable set.

Thus for the 8-stable ideal $\langle x^8, y^8, z^8 \rangle$ the estimate is indeed sharp (this is exactly the same worst case as in the proof above for an ideal of depth 1); the Pommaret basis contains as maximal degree element the monomial $x^8 y^7 z^7$. On the other hand, for the also 8-stable ideal $\langle x^6, x^2 y^4, x^2 z^4, y^8, z^8 \rangle$ the regularity is only 16, as now the maximal degree element of the Pommaret basis is $x^2 y^7 z^7$. ◁

Finally, we recall that, given two quasi-stable ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}$ and their respective Pommaret bases, we explicitly constructed in Remarks 2.9 and 6.5, respectively, of Part I weak Pommaret bases for the sum $\mathcal{I} + \mathcal{J}$, the product $\mathcal{I} \cdot \mathcal{J}$ and the intersection $\mathcal{I} \cap \mathcal{J}$. They lead to the following estimates for the regularity of these ideals which were recently also given by Cimpoeaş [21,20].

Proposition 10.12 *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}$ be two quasi-stable ideals. Then the following three estimates hold:*

$$\operatorname{reg}(\mathcal{I} + \mathcal{J}) \leq \max\{\operatorname{reg}\mathcal{I}, \operatorname{reg}\mathcal{J}\}\,, \tag{72a}$$

$$\operatorname{reg}(\mathcal{I} \cdot \mathcal{J}) \leq \operatorname{reg}\mathcal{I} + \operatorname{reg}\mathcal{J}\,, \tag{72b}$$

$$\operatorname{reg}(\mathcal{I} \cap \mathcal{J}) \leq \max\{\operatorname{reg}\mathcal{I}, \operatorname{reg}\mathcal{J}\}\,. \tag{72c}$$

Proof The first two estimates follow immediately from the weak Pommaret bases given in the above mentioned remarks and Theorem 9.2. For the last estimate the weak Pommaret basis constructed in Remark 6.5 of Part I is not good enough; it would also yield $\operatorname{reg}\mathcal{I} + \operatorname{reg}\mathcal{J}$ as upper bound. However, Lemma 2.2 allows us to improve it significantly. Let $\mathcal{G}$ be the Pommaret basis of $\mathcal{I}$ and $\mathcal{H}$ the one of $\mathcal{J}$. If we set $q = \max\{\deg \mathcal{G}, \deg \mathcal{H}\}$, then one easily sees that $\mathcal{G}_q \cap \mathcal{H}_q$ is the Pommaret basis of $(\mathcal{I} \cap \mathcal{J})_{\geq q}$. Hence, the intersection $\mathcal{I} \cap \mathcal{J}$ possesses a Pommaret basis of degree at most q. □

11 Iterated Polynomial Algebras of Solvable Type

In Section 11 of Part I we studied involutive bases in polynomial algebras of solvable type over rings. We had to substitute the notion of an involutively head autoreduced set by the more comprehensive concept of an involutively $\mathcal{R}$-saturated set. In a certain sense this was not completely satisfying, as we had to resort here to classical Gröbner techniques, namely computing normal forms of ideal elements arising from syzygies. Using the syzygy theory developed in Section 5, we provide now an alternative approach for the special case that the coefficient ring $\mathcal{R}$ is again a polynomial algebra of solvable type (over a field). It is obvious that in this case left ideal membership in $\mathcal{R}$ can be decided algorithmically and by Theorem 5.4 it is also possible to construct algorithmically a basis of the syzygy module.

Remark 11.1 In Section 5 we only considered the ordinary commutative polynomial ring, whereas now we return to general polynomial algebras of solvable type (over a field). However, it is easy to see that all the arguments in the proof of the involutive Schreyer Theorem 5.4 depend only on normal form computations and on considerations concerning the leading exponents. The same holds for the classical Schreyer theorem, as one may easily check (see also [52, 53] for a non-commutative version). Thus the in the sequel crucial Theorem 5.4 remains valid in the general case of non-commutative polynomial algebras. $\triangleleft$

We use the following notations in this section: $\mathcal{R} = (\Bbbk[y_1, \dots, y_m], \star, \prec_y)$ and $\mathcal{P} = (\mathcal{R}[x_1, \dots, x_n], \star, \prec_x)$. Furthermore, we are given an involutive division L_y on $\mathbb{N}_0^m$ and a division L_x on $\mathbb{N}_0^n$. For simplicity, we always assume in the sequel that at least L_y is Noetherian. In order to obtain a reasonable theory, we make similar assumptions as in Section 11 of Part I: both $\mathcal{R}$ and $\mathcal{P}$ are solvable algebras with centred commutation relations so that both are (left) Noetherian.

We now propose an alternative algorithm for the involutive $\mathcal{R}$-saturation. Until Line /13/ it is identical with Algorithm 6 of Part I; afterwards we perform an involutive completion and multiply in Line /17/ each polynomial in $\bar{\mathcal{H}}'_{f,L_x}$ by the non-multiplicative variables of its leading coefficient. In the determination of involutive normal forms, we may multiply each polynomial $h' \in \mathcal{H}'$ only by monomials rx^μ such that $x^\mu \in \mathcal{R}[X_{L_x,\mathcal{H}',\prec_x}(h')]$ *and* $r \in \Bbbk\big[Y_{L_y,\mathrm{lc}_{\prec_x}(\bar{\mathcal{H}}'_{h',L_x}),\prec_y}(\mathrm{lc}_{\prec_x} h')\big]$.

Proposition 11.2 *Let L_y be a Noetherian constructive division. Algorithm 1 terminates for any input $\mathcal{F}$ with an involutively $\mathcal{R}$-saturated and head autoreduced set $\mathcal{H}$ such that $\langle \mathcal{H} \rangle = \langle \mathcal{F} \rangle$. Furthermore, the sets $\mathrm{lc}_{\prec_x} \bar{\mathcal{H}}_{h,L_x}$ form weak L_y-involutive bases of the $\mathcal{R}$-ideals generated by them for each $h \in \mathcal{H}$.*

Proof The termination criterion in Line /26/ is equivalent to local involution of all the sets $\mathrm{lc}_{\prec_x} \bar{\mathcal{H}}'_{f,L_x}$. Under the made assumptions on the division L_y and because of the fact that $\mathcal{P}$ is Noetherian, the termination of the algorithm and the assertion about these sets is obvious. In general we only obtain weak involutive bases, as no involutive head autoreductions of these sets are performed. The correctness is a consequence of Theorem 5.4: by analysing all non-multiplicative products we have taken into account a whole basis of the syzygy module $\mathrm{Syz}(\mathrm{lc}_{\prec_x} \bar{\mathcal{H}}'_{f,L_x})$. Thus the output $\mathcal{H}$ is indeed involutively $\mathcal{R}$-saturated. $\square$

Algorithm 1 Involutive $\mathcal{R}$-saturation (and head autoreduction)

Input: finite set $\mathcal{F} \subset \mathcal{P}$, involutive divisions L_y on $\mathbb{N}_0^m$ and L_x on $\mathbb{N}_0^n$
Output: involutively $\mathcal{R}$-saturated and head autoreduced set $\mathcal{H}$ with $\langle\mathcal{H}\rangle = \langle\mathcal{F}\rangle$

/1/ $\mathcal{H} \leftarrow \mathcal{F}$; $\mathcal{S} \leftarrow \mathcal{F}$
/2/ **while** $S \neq \emptyset$ **do**
/3/ $\quad\nu \leftarrow \max_{\prec_x} \mathrm{le}_{\prec_x}\mathcal{S}$; $\mathcal{S}_\nu \leftarrow \{f \in \mathcal{H} \mid \mathrm{le}_{\prec_x} f = \nu\}$
/4/ $\quad\mathcal{S} \leftarrow \mathcal{S} \setminus \mathcal{S}_\nu$; $\mathcal{H}' \leftarrow \mathcal{H}$
/5/ $\quad$**for all** $f \in \mathcal{S}_\nu$ **do**
/6/ $\qquad h \leftarrow \mathtt{HeadReduce}_{L_x,\prec_x}(f, \mathcal{H})$
/7/ $\qquad$**if** $f \neq h$ **then**
/8/ $\qquad\quad\mathcal{S}_\nu \leftarrow \mathcal{S}_\nu \setminus \{f\}$; $\mathcal{H}' \leftarrow \mathcal{H}' \setminus \{f\}$
/9/ $\qquad\quad$**if** $h \neq 0$ **then**
/10/ $\qquad\qquad\mathcal{H}' \leftarrow \mathcal{H}' \cup \{h\}$
/11/ $\qquad\quad$**end_if**
/12/ $\qquad$**end_if**
/13/ $\quad$**end_for**
/14/ $\quad$**if** $\mathcal{S}_\nu \neq \emptyset$ **then**
/15/ $\qquad$choose $f \in \mathcal{S}_\nu$ and determine the set $\bar{\mathcal{H}}'_{f,L_x}$
/16/ $\qquad$**repeat**
/17/ $\qquad\quad\mathcal{T} \leftarrow \bigl\{y_j \star \bar{f} \mid \bar{f} \in \bar{\mathcal{H}}'_{f,L_x},\ y_j \in \bar{Y}_{L_y,\mathrm{lc}_{\prec_x}(\bar{\mathcal{H}}'_{f,L_x}),\prec_y}(\mathrm{lc}_{\prec_x}\bar{f})\bigr\}$
/18/ $\qquad\quad$**repeat**
/19/ $\qquad\qquad$choose $h' \in \mathcal{T}$ such that $\mathrm{le}_{\prec_y}(\mathrm{lc}_{\prec_x} h')$ is minimal
/20/ $\qquad\qquad\mathcal{T} \leftarrow \mathcal{T} \setminus \{h'\}$
/21/ $\qquad\qquad h \leftarrow \mathtt{NormalForm}_{L_x,\prec_x,L_y,\prec_y}(h', \mathcal{H}')$
/22/ $\qquad\qquad$**if** $h \neq 0$ **then**
/23/ $\qquad\qquad\quad\mathcal{H}' \leftarrow \mathcal{H}' \cup \{h\}$
/24/ $\qquad\qquad$**end_if**
/25/ $\qquad\quad$**until** $\mathcal{T} = \emptyset \vee h \neq 0$
/26/ $\qquad$**until** $\mathcal{T} = \emptyset \wedge h = 0$
/27/ $\quad$**end_if**
/28/ $\quad$**if** $\mathcal{H}' \neq \mathcal{H}$ **then**
/29/ $\qquad\mathcal{H} \leftarrow \mathcal{H}'$; $\mathcal{S} \leftarrow \mathcal{H}$
/30/ $\quad$**end_if**
/31/ **end_while**
/32/ **return** $\mathcal{H}$

Theorem 11.3 *Let the polynomial ring $\mathcal{P}$ satisfy the made assumptions and L_x be a Noetherian constructive division. If in Algorithm 3 of Part I the subalgorithm $\mathtt{InvHeadAutoReduce}_{L_x,\prec_x}$ is substituted by Algorithm 1, then the completion will terminate with a weak involutive basis of $\mathcal{I} = \langle\mathcal{F}\rangle$ for any finite input set $\mathcal{F} \subset \mathcal{P}$. Furthermore, the sets $\mathrm{lc}_{\prec_x}\bar{\mathcal{H}}_{h,L_x}$ form strong L_y-involutive bases of the $\mathcal{R}$-ideals generated by them for each $h \in \mathcal{H}$.*

Proof The proof of the termination and of the correctness of the algorithm is as in Part I. The only new claim is that the sets $\mathrm{lc}_{\prec_x}\bar{\mathcal{H}}_{h,L_x}$ are strongly L_y-involutive. This is a simple consequence of the fact that under the made assumption on the product in $\mathcal{P}$ the loop in Lines /5-13/ of Algorithm 1 leads to an involutive head autoreduction of these sets. Hence we indeed obtain strong involutive bases. □

Corollary 11.4 *If L_x is the Janet division, then each polynomial $f \in \mathcal{I}$ possesses a unique involutive standard representation $f = \sum_{h\in\mathcal{H}} P_h \star h$ where $P_h \in \Bbbk[Y_{L_y,\mathrm{lc}_{\prec_x}(\bar{\mathcal{H}}_{h,L_x}),\prec_y}(\mathrm{lc}_{\prec_x} h)][X_{L_x,\mathcal{H},\prec_x}(h)]$.*

Proof For the Janet division the only obstruction for $\mathcal{H}$ being a strong involutive basis is that some elements of it may have the same leading exponents. More precisely, for any $h \in \mathcal{H}$ we have $\mathcal{H}_{h,L_x} = \{h' \in \mathcal{H} \mid \mathrm{le}_{\prec_x} h' = \mathrm{le}_{\prec_x} h\}$. This immediately implies furthermore $\bar{\mathcal{H}}_{h,L_x} = \mathcal{H}_{h,L_x}$. By Theorem 11.3 the sets $\mathrm{lc}_{\prec_x} \bar{\mathcal{H}}_{h,L_x}$ form a strong L_y-involutive basis of the ideals they generate. Hence the claimed representation must be unique. □

12 Conclusions

One can find in the literature algorithms for the effective determination of all the invariants considered in this article. However, typically one has for each invariant a separate algorithm requiring often a number of Gröbner bases computations. In fact, the classical approach for many of them would be to construct the minimal free resolution—an obviously quite expensive approach. By contrast, with the determination of a single Pommaret basis (with respect to the degree reverse lexicographic order) we obtain simultaneously *without further computations* the following information on our module: its Hilbert function (and thus the Krull dimension and the multiplicity) together with a maximal strongly independent set of variables, its depth and a very simple maximal regular sequence, a Noether normalisation together with a sparse homogeneous system of parameters and finally its projective dimension together with all extremal Betti numbers (and thus the Castelnuovo-Mumford regularity) plus bounds on the remaining Betti numbers. Taking a converse point of view, we may say that compared with ordinary Gröbner bases Pommaret bases are much less arbitrary but to a considerable extent determined by structural properties of the module they generate. This fact makes them a natural choice for many computational problems in algebraic geometry.

The price to pay for this power of Pommaret bases is the problem of δ-regularity which makes their effective construction somewhat more cumbersome. We proposed a simple deterministic method for solving this problem which even gives us a fighting chance of finding a sparse coordinate transformation. Generally, it should be much more efficient than the usually proposed probabilistic approach which inevitably destroys all sparsity present in the original generating set and thus makes all subsequent computations very expensive.

δ-regularity is often considered as a purely technical nuisance. Our results show a different picture. Asymptotic regularity is indeed a technical concept used by our method for the construction of δ-regular coordinates and relevant for the termination of the completion algorithm presented in Part I. By contrast, δ-regularity has an intrinsic meaning. This can already be seen from the simple fact that in the case of linear differential operators there is a close relation to characteristics (see any textbook on partial differential equations, e. g. [49,61]): a necessary condition for a coordinate system to be δ-regular is that the hypersurface $x_n = 0$ is non-characteristic. Indeed, the standard definition of a characteristic hypersurface may be rephrased that on it one cannot solve for all derivatives of class n. We also mentioned in Remark A.5 that δ-regularity is essentially equivalent to quasi-regularity in the sense of Serre which in turn is related to the associated prime ideals of the module and thus again to intrinsic properties (see [66] for details).

We have also seen that δ-regularity is related to many genericity concepts in commutative algebra and algebraic geometry. Many statements that are only generically true hold in δ-regular coordinates. In particular, in δ-regular coordinates many properties of an affine algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$ may already be read off the monomial algebra $\mathcal{A}' = \mathcal{P}/\mathrm{lt}_{\prec}\mathcal{I}$ where $\prec$ is the degree reverse lexicographic order.

For example, it follows immediately from Proposition 3.19 that $\operatorname{depth}\mathcal{A} = \operatorname{depth}\mathcal{A}'$ and that $(x_1, \dots, x_d)$ is a maximal regular sequence for both algebras. As in the homogeneous case it is also trivial that $\dim\mathcal{A} = \dim\mathcal{A}'$, we see that the algebra $\mathcal{A}$ is Cohen-Macaulay, if and only if $\mathcal{A}'$ is so. Similarly, it is an easy consequence of Theorem 8.11 that $\operatorname{proj}\dim\mathcal{A} = \operatorname{proj}\dim\mathcal{A}'$ and of Theorem 9.2 that $\operatorname{reg}\mathcal{A} = \operatorname{reg}\mathcal{A}'$; in fact, all extremal Betti numbers are the same as mentioned in Remark 9.7. An exception are the remaining Betti numbers where Example 8.10 shows that even in δ-regular coordinates $\mathcal{A}$ and $\mathcal{A}'$ may have different ones.

These equalities are of course not new; they can already be found in [10] (some even in earlier references). However, one should note an important difference: Bayer and Stillman [10] work with the *generic initial ideal*, whereas we assume δ-regularity of the coordinates. These are two different genericity concepts, as even in δ-regular coordinates $\mathrm{lt}_{\prec}\mathcal{I}$ is not necessarily the generic initial ideal (in contrast to the former, the latter is always Borel-fixed).

When we proved in Corollary 8.13 and 9.5, respectively, the two inequalities $\operatorname{proj}\dim\mathcal{A} \leq \operatorname{proj}\dim\mathcal{A}'$ and $\operatorname{reg}\mathcal{I} \leq \operatorname{reg}(\mathrm{lt}_{\prec}\mathcal{I})$ for arbitrary term orders $\prec$, we had to assume the existence of a Pommaret basis of $\mathcal{I}$ for $\prec$. It is well-known that these inequalities remain true, if we drop this assumption (see for example the discussions in [9,10,15]). We included here our alternative proofs because of their great simplicity and they cover at least the generic case.

Many of the results in Sections 6–8 on monomial ideals are generalisations of the work of Eliahou and Kervaire [27]. They considered exclusively the case of stable modules where we obtain a minimal resolution. If one analyses closely their proofs, it is not difficult to see that implicitly they introduce Pommaret bases and exploit some of their basic properties. Our proof of Theorem 8.9 appears so much simpler only because we have already shown all these properties in Part I. Furthermore, Eliahou and Kervaire did not realise that they constructed a syzygy resolution in Schreyer form. Hence they had to give a lengthy and rather messy proof that the complex $(\mathcal{S}_*, \delta)$ is exact, whereas in our approach this is immediate.

We rediscover all their complicated calculations in the proof of Theorem 7.2. But note that this explicit formula for the differential is needed neither for proving the minimality of the resolution nor for its construction, although the latter is of course simplified by it. Furthermore, the theory of involutive standard representations gives us a clear guideline how to proceed.

Our results strongly suggest a homological background of the Pommaret division. Most of the quantities like the depth or the Castelnuovo-Mumford regularity determined by the Pommaret basis of an ideal $\mathcal{I}$ are of a homological nature; more precisely, they correspond to certain extremal points in the Betti diagram and thus come from the Koszul homology. For the special case of monomial ideals, Sahbi showed in his diploma thesis [63] how the Koszul homology of a quasi-stable ideal can be explicitly computed from the P-graph of its Pommaret basis.

The combination of Corollary 3.8 and Proposition 3.19 allows us to make some statements about the so-called *Stanley conjecture*. It concerns the minimal number of multiplicative variables for a generator in a Stanley decomposition. Following Apel [5, Def. 1] and Herzog et al. [43] we call this number the *Stanley depth* of the decomposition and for an ideal $\mathcal{I} \subseteq \mathcal{P}$ the Stanley depth of the algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$, written $\operatorname{sdepth} \mathcal{A}$, is defined as the maximal Stanley depth of a complementary decomposition for $\mathcal{I}$. In its simplest form the Stanley conjecture claims that we always have the inequality $\operatorname{sdepth} \mathcal{A} \geq \operatorname{depth} \mathcal{A}$. Obviously, Corollary 3.8 together with Proposition 3.19 (plus the existence Theorem 2.16 for Pommaret bases) shows that this inequality holds for arbitrary ideals.

The rigorous formulation of the Stanley conjecture [69, Conj. 5.1] concerns monomial ideals and requires that all generators in the decomposition are again monomials. Furthermore, no variables transformation is allowed. Then our results only allow us to conclude that the Stanley conjecture is true for all quasi-stable ideals. Some further results on this question have been achieved by Apel [5,6] with the help of a slightly different notion of involutive bases.

Many of the results mentioned above are quite well-known for Borel-fixed ideals and thus for generic initial ideals. However, it appears that for many purposes it is not necessary to move to this highly special class of ideals; quasi-stable ideals which are easier to produce algorithmically share many of their properties. Thus it is not surprising that quasi-stable ideals have appeared under different names in quite a number of recent works in commutative algebra (e. g. [12,18,42]).

The results presented in this article offer two heuristic explanations for the efficiency of the involutive completion algorithm already mentioned in Part I. The first one is that according to our proof of Theorem 5.4 the involutive algorithm automatically takes into account many instances of Buchberger's second criterion for redundant S-polynomials. Whereas a naive implementation of Buchberger's algorithm without such criteria fails already for rather small examples, a naive implementation of the involutive completion algorithm works reasonably for not too large examples.

The second explanation concerns Proposition 3.2. It is well-known that the so-called “Hilbert driven” Buchberger algorithm [77] is often very fast, but it requires a priori knowledge of the Hilbert polynomial. The involutive completion algorithm may also be interpreted as “Hilbert driven”. The assignment of multiplicative variables to the elements of the current basis $\mathcal{H}$ defines at each iteration a trial Hilbert function $h_{\mathcal{H},L,\prec}$ measuring the size of the involutive span $\langle\mathcal{H}\rangle_{L,\prec}$. This trial function is the true Hilbert function, if and only if we have already reached an involutive basis; otherwise it yields too small values. For continuous divisions the analysis of the products of the generators with their non-multiplicative variables represents a simple check for the trial Hilbert function to be the true one.

While for many ideals the involutive approach is an interesting alternative for the construction of Gröbner bases, there exist some obvious cases where this is not the case. For monomial ideals any basis is already a Gröbner basis, whereas an involutive basis still has to be constructed. Blinkov and Gerdt [14] showed that for toric ideals involutive bases are typically much larger than Gröbner bases. In both cases, the reason is that these ideals are rarely in general position and often possess Gröbner bases of a much lower degree than $\operatorname{reg}\mathcal{I}$.

An interesting question is whether the results of this second part can be extended to the polynomial algebras of solvable type introduced in the first part. Such a generalisation is trivial only for the determination of Stanley decompositions, as these are defined as vector space isomorphisms and therefore do not feel the non-commutativity (note that we always have the commutative product $\cdot$ on the right hand side of the defining equation (7) of a Stanley decomposition). Thus involutive bases are a valuable tool for computing Hilbert functions even in the non-commutative case. They give immediately the *Gelfand-Kirillov dimension* [56, Sect. 8.1.11] as the degree of the Hilbert polynomial (only in the commutative case it always coincides with the Krull dimension). Some examples for such computations in the context of quantum groups (however, using Gröbner instead of involutive bases) may be found in [17].

By contrast, our results on the depth and on the Castelnuovo-Mumford regularity rely on the fact that for commutative polynomials $f \in \langle x_j \rangle$ implies that any term in f is divisible by x_j. In a non-commutative algebra of solvable type we have the relations $x_i \star x_j = c_{ij}x_ix_j + h_{ij}$ and in general the polynomial h_{ij} is not divisible by x_j.

For syzygies the situation is complicated, too. The proof of Theorem 5.4 is independent of the precise form of the multiplication and thus we may conclude that we can always construct at least a Gröbner basis of the syzygy module. Our proof of Theorem 5.10 relies mainly on normal form arguments that generalise. A minimal requirement is that the term order $\prec_{\mathcal{H}}$ respects the multiplication $\star$, as otherwise the theorem does not even make sense. Furthermore, we must be careful with all arguments involving multiplicative variables. We need that if x_i and x_j are both multiplicative for a generator, then $x_i \star x_j = c_{ij}x_ix_j + h_{ij}$ must also contain only multiplicative variables which will surely happen, if h_{ij} depends only on variables x_k with $k \leq \max\{i, j\}$. This is for example the case for linear differential operators, so that we may conclude that Theorem 5.10 (and its consequences) remain true for the Weyl algebra and other rings of differential operators.

Example 12.1 Recall from Example 3.9 of Part I that the universal enveloping algebra of the Lie algebra $\mathfrak{so}(3)$ is isomorphic to the ring $(\Bbbk[x_1, x_2, x_3], \star)$ with the product $\star$ induced by the relations

$$\begin{aligned} x_1 \star x_2 &= x_1x_2\ , & x_2 \star x_1 &= x_1x_2 - x_3\ ,\\ x_1 \star x_3 &= x_1x_3\ , & x_3 \star x_1 &= x_1x_3 + x_2\ ,\\ x_2 \star x_3 &= x_2x_3\ , & x_3 \star x_2 &= x_2x_3 - x_1\ . \end{aligned} \tag{73}$$

Obviously $x_1x_2 - x_3 \in \langle x_1 \rangle$, but the term x_3 is not divisible by x_1. It follows from the same relation that $x_2 \star x_1$ depends on x_3 and thus the arguments on multiplicative variables required by our proof of Theorem 5.10 break down. $\lhd$

A Rees Decompositions à la Sturmfels-White

Sturmfels and White [74, Algo. 4.2] presented an algorithm for the effective construction of Rees decompositions (based on works by Baclawski and Garsia [7]). We show now that generically it yields a Pommaret basis. However, we believe that the involutive approach is much more efficient. It does not only allow us to avoid completely computations in factor algebras, using our results in Section 2 we obtain more easily and deterministically good coordinates whereas Sturmfels and White must rely on a probabilistic approach.

We introduce some additional notations. Let again $\mathcal{M}$ be a finitely generated module over the ring $\mathcal{P} = \mathbb{k}[x_1, \dots, x_n]$. The *annihilator* of an element $\mathbf{m} \in \mathcal{M}$ is $\operatorname{Ann}(\mathbf{m}) = \{f \in \mathcal{P} \mid f\mathbf{m} = 0\}$. The $\mathbb{k}$-vector space $Z_{\mathcal{M}} \subseteq \mathcal{M}$ is defined as the set $Z_{\mathcal{M}} = \{\mathbf{m} \in \mathcal{M} \mid \operatorname{Ann}(\mathbf{m}) = \mathcal{P}_+\}$ (of course, $Z_{\mathcal{M}}$ is nothing but the nth Koszul homology group $H_n(\mathcal{M})$ of $\mathcal{M}$). The approach of Sturmfels and White is based on the following fact which may be interpreted as their version of the concept of δ-regularity.

Lemma A.1 *If $Z_{\mathcal{M}} = 0$, then there exists a non zero divisor $y \in \mathcal{P}_1$, i. e. $y\mathbf{m} = 0$ implies $\mathbf{m} = 0$ for all $\mathbf{m} \in \mathcal{M}$. Identifying $\mathcal{P}_1$ with $\mathbb{k}^n$, the set of all non zero divisors contains a Zariski open subset.*

The Sturmfels-White Algorithm 2 computes a basis $\mathcal{Y} = \{y_1, \dots, y_n\}$ of $\mathcal{P}_1$ and a set $\mathcal{H} \subset \mathcal{M}$ of generators such that (as graded $\mathbb{k}$-vector spaces)

$$\mathcal{M} \cong \bigoplus_{\mathbf{h} \in \mathcal{H}} \mathbb{k}[y_1, \dots, y_{\operatorname{cls} \mathbf{h}}] \cdot \mathbf{h}\,. \tag{74}$$

Here the class $\operatorname{cls} \mathbf{h}$ is automatically assigned in the course of the algorithm and not necessarily equal to the notion of class we introduced in the definition of the Pommaret division. Within this appendix, the latter one will be referred to as *coordinate class* $\operatorname{ccls}_{\mathbf{x}} \mathbf{h}$, since its definition depends on the chosen coordinates $\mathbf{x}$.

Whether the individual steps of Algorithm 2 can be made effective depends on how $\mathcal{M}$ is given. If it is presented by generators and relations, as we always assume, one may use Gröbner bases; Sturmfels and White formulated their algorithm directly for this case. Note that they need repeated Gröbner bases calculations in order to perform algorithmically all the computations in factor modules. A further problem is to find the non zero divisors, as Lemma A.1 only guarantees their existence but says nothing about their determination. Sturmfels and White proposed a probabilistic approach. As the non zero divisors contain a Zariski open subset of $\mathcal{P}_1$, the random choice of a one-form yields one with probability 1.

Theorem A.2 *The Algorithm 2 terminates for any finitely generated polynomial module $\mathcal{M}$ with a Rees decomposition.*

For a proof we refer to [7,74] where Lemma A.1 is proved, too (note also Remark A.5 below). We will now show that generically the Sturmfels-White Algorithm 2 returns a Pommaret basis when it is applied to a submodule $\mathcal{M} \subseteq \mathcal{P}^r$ of a free module. We begin by studying the relation between the classes and the

Algorithm 2 Construction of a Rees Decomposition à la Sturmfels-White

Input: polynomial module $\mathcal{M}$ over $\mathcal{P} = \Bbbk[x_1, \dots, x_n]$
Output: basis $\mathcal{Y}$ of $\mathcal{P}_1$, set $\mathcal{H}$ of generators defining Rees decomposition (74)

$k \leftarrow 0$; $p \leftarrow 0$; $\mathcal{M}' \leftarrow \mathcal{M}$
while $\mathcal{M}' \neq 0$ **do**
  compute $Z_{\mathcal{M}'}$
  if $Z_{\mathcal{M}'} = 0$ **then**
    $k \leftarrow k + 1$
    choose a non zero divisor $y_k \in \mathcal{P}_1$ linearly independent of $\{y_1, \dots, y_{k-1}\}$
    $\mathcal{M}' \leftarrow \mathcal{M}'/y_k\mathcal{M}'$
  else
    compute $\mathbf{h}_{p+1}, \dots, \mathbf{h}_{p+\ell} \in \mathcal{M}$ such that $\bigl\{[\mathbf{h}_i] \mid p < i \leq p + \ell\bigr\}$ is a basis of $Z_{\mathcal{M}'}$
    for i **from** $p + 1$ **to** $p + \ell$ **do**
      $\operatorname{cls} \mathbf{h}_i \leftarrow k$
    end_for
    $p \leftarrow p + \ell$
    $\mathcal{M}' \leftarrow \mathcal{M}'/Z_{\mathcal{M}'}$
  end_if
end_while
if $k < n$ **then**
  complete $\{y_1, \dots, y_k\}$ to a basis $\mathcal{Y}$ of $\mathcal{P}_1$
end_if
return $\bigl(\mathcal{Y}, \mathcal{H} = \bigl\{(\mathbf{h}_1, \operatorname{cls} \mathbf{h}_1), \dots, (\mathbf{h}_p, \operatorname{cls} \mathbf{h}_p)\bigr\}\bigr)$

coordinate classes of the generators of a Rees decomposition determined with the help of this algorithm (the restriction to submodules concerns only the very first step of the next proof).

Proposition A.3 *Let $\mathcal{H}$ define a Rees decomposition of the form (74) for the submodule $\mathcal{M} \subseteq \mathcal{P}^r$ with respect to the basis $\mathcal{Y}$ of $\mathcal{P}_1$ and let $\mathcal{H}$ and $\mathcal{Y}$ be determined with the Sturmfels-White Algorithm 2. With respect to the basis $\mathcal{Y}$ we have the inequalities* $\operatorname{cls} \mathbf{h} \geq \operatorname{ccls}_{\mathbf{y}} \mathbf{h}$ *for all generators* $\mathbf{h} \in \mathcal{H}$.

Proof $Z_{\mathcal{M}} = 0$ for a submodule $\mathcal{M} \subset \mathcal{P}^r$. Thus Algorithm 2 produces no generators of class 0. The coordinate class is always greater than 0.

We follow step by step Algorithm 2. In the first iteration some non zero divisor $y_1 \in \mathcal{P}_1$ is chosen and in the second iteration we must treat the factor module $\mathcal{M}^{(1)} = \mathcal{M}/y_1\mathcal{M}$. Now $\mathbf{m} \in \mathcal{M}$ represents an element of $Z_{\mathcal{M}^{(1)}}$, if and only if $y_k\mathbf{m} \in y_1\mathcal{M}$ for all $k > 1$. Thus $\operatorname{ccls}_{\mathbf{y}}(y_k\mathbf{m}) = 1$ for all $k > 1$ which is only possible if $\operatorname{ccls}_{\mathbf{y}} \mathbf{m} = 1$.

If $Z_{\mathcal{M}^{(1)}} \neq 0$, then Algorithm 2 proceeds with $\mathcal{M}^{(2)} = \mathcal{M}^{(1)}/Z_{\mathcal{M}^{(1)}}$. Now $\mathbf{m} \in \mathcal{M}$ represents an element of $Z_{\mathcal{M}^{(2)}}$, if and only if for all $k > 1$ the product $y_k\mathbf{m}$ either is an element of $y_1\mathcal{M}$ or represents an element of $Z_{\mathcal{M}^{(1)}}$. In both cases this is only possible, if $\operatorname{ccls}_{\mathbf{y}} \mathbf{m} = 1$. The same argument holds until $Z_{\mathcal{M}^{(\ell)}} = 0$ for some ℓ. Thus all generators $\mathbf{h}$ to which Algorithm 2 assigns the class 1 are divisible by y_1 and hence all their terms possess the coordinate class 1.

Now $\mathcal{M}^{(\ell+1)} = \mathcal{M}^{(\ell)}/y_2\mathcal{M}^{(\ell)}$ must be considered. Proceeding as above, we see that $\mathbf{m} \in \mathcal{M}$ represents an element of $Z_{\mathcal{M}^{(\ell+1)}}$, if and only if $y_k\mathbf{m} \in y_2\mathcal{M}^{(\ell)}$ for all $k > 2$ implying that $\operatorname{ccls}_{\mathbf{y}}\mathbf{m} \leq 2$. Using the same argument as above, we conclude that all generators of class 2 according to the Sturmfels-White algorithm consist of terms with a coordinate class less than or equal to 2. Following Algorithm 2 until the end we obtain the assertion, namely that $\operatorname{cls}\mathbf{h} \geq \operatorname{ccls}_{\mathbf{y}}\mathbf{h}$ for all generators $\mathbf{h} \in \mathcal{H}$. □

As it may happen that $\operatorname{cls}\mathbf{h} > \operatorname{ccls}_{\mathbf{y}}\mathbf{h}$ for some generator $\mathbf{h}$, the set $\mathcal{H}$ is not necessarily a Pommaret basis. More precisely, the coordinates $\mathbf{y}$ are not necessarily δ-regular for $\mathcal{M}$. We show now similarly to the proof of Theorem 2.13 that we may always transform $\mathbf{y}$ into a δ-regular coordinate system $\mathbf{z}$.

For simplicity, let us assume that only one generator $\mathbf{h}$ with $\operatorname{ccls}_{\mathbf{y}}\mathbf{h} < \operatorname{cls}\mathbf{h}$ exists and that $\operatorname{cls}\mathbf{h} = 2$. Consider a coordinate transformation $z_k = y_k$ for $k > 1$ and $z_1 = y_1 + cy_2$ where $c \in \mathbb{k}$ is chosen such that with respect to the new coordinates $\operatorname{ccls}_{\mathbf{z}}\mathbf{h} = 2$. The possible values of c form a Zariski open set in $\mathbb{k}$. By Lemma A.1, the non zero divisors among which y_1 was chosen in Algorithm 2 contain a Zariski open subset of $\mathcal{P}_1$. Thus there exist values of c such that both $\operatorname{ccls}_{\mathbf{z}}\mathbf{h} = 2$ *and* z_1 is a non zero divisor.

As in the proof of Theorem 2.13, it is not difficult to show that this transformation increases the Hilbert function of the involutive span of $\mathcal{H}$. Applying a finite number of similar changes of coordinates leads to a new basis $\mathcal{Z}$ of $\mathcal{P}_1$ in which $\operatorname{ccls}_{\mathbf{z}}\mathbf{h} = \operatorname{cls}\mathbf{h}$. As we still have a Rees decomposition, the set $\mathcal{H}$ is a Pommaret basis of the submodule $\mathcal{M}$ and the coordinates $\mathbf{z}$ are δ-regular for $\mathcal{M}$. Obviously, the one-forms $z_1, \dots, z_n$ would have been valid choices for the non zero divisors in Algorithm 2. Thus we conclude that this algorithm may be used for the construction of Pommaret bases. The following proposition shows that in fact any Pommaret basis may be constructed this way.

Proposition A.4 *Let $\mathcal{H}$ be a Pommaret basis of the submodule $\mathcal{M} \subseteq \mathcal{P}^r$ with respect to the δ-regular coordinates $\mathbf{y}$ and a class respecting term order $\prec$. The one-forms $y_1, \dots, y_n$ may be used as non zero divisors in Algorithm 2 and the then obtained generators $\bar{\mathbf{h}}$ satisfy $\operatorname{cls}\bar{\mathbf{h}} = \operatorname{ccls}_{\mathbf{y}}\bar{\mathbf{h}}$. They are $\mathbb{k}$-linear combinations of the elements of $\mathcal{H}$; one may even simply take use the elements of $\mathcal{H}$.*

Proof The Pommaret basis $\mathcal{H}$ defines a Rees decomposition

$$\mathcal{M} = \bigoplus_{\mathbf{h}\in\mathcal{H}} \mathbb{k}[y_1, \dots, y_{\operatorname{ccls}_{\mathbf{y}}\mathbf{h}}] \cdot \mathbf{h}\,. \tag{75}$$

As in the previous proof, we follow step by step the Sturmfels-White Algorithm 2. Let $\mathcal{M}^{(1)} = \mathcal{M}/y_1\mathcal{M}$ and $\mathcal{H}_1 = \{\mathbf{h} \in \mathcal{H} \mid \operatorname{ccls}_{\mathbf{y}}\mathbf{h} = 1\}$. The vector space $Z_{\mathcal{M}^{(1)}}$ is isomorphic to a subspace of the $\mathbb{k}$-linear space freely generated by $\mathcal{H}_1$, as $Z_{\mathcal{M}^{(1)}}$ contains only elements with coordinate class 1 by Proposition A.3 and the only elements of $\mathcal{M}$ of coordinate class 1 which are not in $y_1\mathcal{M}$ are $\mathbb{k}$-linear combinations of the elements of $\mathcal{H}_1$.

Let $\mathbf{h} \in \mathcal{H}_1$ and $k > 1$. We determine the involutive normal form of $y_k\mathbf{h}$ induced by (75). Every term in $y_k\mathbf{h}$ has coordinate class 1, thus $\mathbf{t} = \operatorname{lt}_{\prec}(y_k\mathbf{h})$

satisfies $\operatorname{ccls}_{\mathbf{y}}\mathbf{t} = 1$. Since $\mathcal{H}$ is a strong basis, there exists precisely one generator $\mathbf{h}' \in \mathcal{H}$ such that $\operatorname{lt}_{\prec}\mathbf{h}' \,|_P\, \mathbf{t}$. If $\operatorname{lt}_{\prec}\mathbf{h}' = \mathbf{t}$, then $\mathbf{h}' \in \mathcal{H}_1$, as $\prec$ is class respecting. After the corresponding reduction step, the initial term of the result is still of coordinate class 1. So the normal form of $y_k\mathbf{h}$ has the following structure

$$y_k\mathbf{h} = \sum_{\tilde{\mathbf{h}}\in\mathcal{H}_1} c_{\tilde{\mathbf{h}}}\tilde{\mathbf{h}} + y_1\mathbf{m} \tag{76}$$

for some coefficients $c_{\tilde{\mathbf{h}}} \in \Bbbk$ and an element $\mathbf{m} \in \mathcal{M}$. The vector space $Z_{\mathcal{M}^{(1)}}$ is generated by those $\tilde{\mathbf{h}} \in \mathcal{H}_1$ where the first summand in (76) is zero; this includes in particular all elements of $\mathcal{H}_1$ of maximal degree.

Thus if $\mathcal{H}_1 \neq \emptyset$, then $Z_{\mathcal{M}^{(1)}} \neq \emptyset$. Algorithm 2 proceeds in this case with $\mathcal{M}^{(2)} = \mathcal{M}^{(1)}/Z_{\mathcal{M}^{(1)}}$. If $\dim Z_{\mathcal{M}^{(1)}} < |\mathcal{H}_1|$, then $Z_{\mathcal{M}^{(2)}} \neq \emptyset$. Algorithm 2 will iterate Line /14/, until all elements of $\mathcal{H}_1$ have been used up. When this stage is reached, $Z_{\mathcal{M}^{(\ell)}} = 0$. It follows from the direct sum in (75) that y_2 is a non zero divisor for $\mathcal{M}^{(\ell)}$ and we may proceed with $\mathcal{M}^{(\ell+1)} = \mathcal{M}^{(\ell)}/y_2\mathcal{M}^{(\ell)}$.

Let $\mathcal{H}_2 = \{\mathbf{h} \in \mathcal{H} \mid \operatorname{ccls}_{\mathbf{y}}\mathbf{h} = 2\}$. As above, $Z_{\mathcal{M}^{(\ell+1)}}$ is isomorphic to a subspace of the vector space freely generated by $\mathcal{H}_2$. There are some minor modifications in (76): the first sum is over all $\tilde{\mathbf{h}} \in \mathcal{H}_2$ and there are additional summands which vanish either modulo $y_1\mathcal{M}$ or modulo $y_2\mathcal{M}^{(\ell)}$ or modulo some $Z_{\mathcal{M}^{(i)}}$ for $1 \leq i \leq \ell$. Again Algorithm 2 will iterate line /14/, until all elements of $\mathcal{H}_2$ have been used up. The same argument may be repeated for $y_3, \dots, y_n$.

Thus, Algorithm 2 terminates with a Rees decomposition (with respect to the basis $\mathcal{Y} \subset \mathcal{P}_1$) generated by $\bar{\mathcal{H}}$ where $|\bar{\mathcal{H}}| = |\mathcal{H}|$ and where the elements $\bar{\mathbf{h}} \in \bar{\mathcal{H}}$ with $\operatorname{cls}\bar{\mathbf{h}} = k$ freely generate the same vector space as the elements $\mathbf{h} \in \mathcal{H}$ with $\operatorname{ccls}_{\mathbf{y}}\mathbf{h} = k$. We may even choose $\bar{\mathcal{H}} = \mathcal{H}$. In any case, $\operatorname{cls}\bar{\mathbf{h}} = \operatorname{ccls}_{\mathbf{y}}\bar{\mathbf{h}}$ and $\bar{\mathcal{H}}$ is a Pommaret basis of $\mathcal{M}$. □

Remark A.5 Note the strong similarity between this proof and the proof of Proposition 2.20. This fact is not surprising, as the minimal class assigned by Algorithm 2 is equal to the depth of $\mathcal{M}$ [74] and the basis $\mathcal{Y}$ determined by it is quasi-regular for the module $\mathcal{M}$ in the sense of Serre (see either the letter of Serre appended to [36] or [66]). In fact, Lemma A.1 follows immediately from the results of Serre. They imply furthermore that $\mathcal{Z}_{\mathcal{M}}$ is always finite-dimensional and thus it is not really necessary to factor by $\mathcal{Z}_{\mathcal{M}}$. In [66] it is shown that coordinates are δ-regular for the submodule $\mathcal{M} \subseteq \mathcal{P}^r$, if and only if they are quasi-regular for the factor module $\mathcal{P}^r/\mathcal{M}$. Thus in principle, one should always compare Algorithm 2 applied to $\mathcal{P}^r/\mathcal{M}$ with the Pommaret basis of $\mathcal{M}$ (recall that the latter also leads immediately to a Rees decomposition of $\mathcal{P}^r/\mathcal{M}$ via Corollary 3.8). ◁

Acknowledgements The author would like to thank V.P. Gerdt for a number of interesting discussions on involutive bases. M. Hausdorf and R. Steinwandt participated in an informal seminar at Karlsruhe University where many ideas of this article were presented and gave many valuable comments. The constructive remarks of the anonymous referees were also very helpful. This work received partial financial support by Deutsche Forschungsgemeinschaft, INTAS grant 99-1222 and NEST-Adventure contract 5006 (*GIFT*).

References

1. W.W. Adams and P. Loustaunau. *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics 3. American Mathematical Society, Providence, 1994.
2. S. Ahmad and I. Anwar. An upper bound for the regularity of ideals of Borel type. *Commun. Alg.*, 36:670–673, 2008.
3. M. Amasaki. Application of the generalized Weierstrass preparation theorem to the study of homogeneous ideals. *Trans. Amer. Math. Soc.*, 317:1–43, 1990.
4. M. Amasaki. Generic Gröbner bases and Weierstrass bases of homogeneous submodules of graded free modules. *J. Pure Appl. Algebra*, 152:3–16, 2000.
5. J. Apel. On a conjecture of R.P. Stanley. Part I – monomial ideals. *J. Algebr. Comb.*, 17:39–56, 2003.
6. J. Apel. On a conjecture of R.P. Stanley. Part II – quotients modulo monomial ideals. *J. Algebr. Comb.*, 17:57–74, 2003.
7. K. Baclawski and A.M. Garsia. Combinatorial decompositions of rings. *Adv. Math.*, 39:155–184, 1981.
8. D. Bayer, H. Charalambous, and S. Popescu. Extremal Betti numbers and applications to monomial ideals. *J. Alg.*, 221:497–512, 1999.
9. D. Bayer and D. Mumford. What can be computed in algebraic geometry? In D. Eisenbud and L. Robbiano, editors, *Computational Algebraic Geometry and Commutative Algebra*, Symposia Mathematica 34, pages 1–48. Cambridge University Press, Cambridge, 1993.
10. D. Bayer and M. Stillman. A criterion for detecting m-regularity. *Invent. Math.*, 87:1–11, 1987.
11. Th. Becker and V. Weispfenning. *Gröbner Bases*. Graduate Texts in Mathematics 141. Springer-Verlag, New York, 1993.
12. I. Bermejo and P. Gimenez. Saturation and Castelnuovo-Mumford regularity. *J. Alg.*, 303:592–617, 2006.
13. L.J. Billera, R. Cushman, and J.A. Sanders. The Stanley decomposition of the harmonic oscillator. *Indagat. Math.*, 50:375–394, 1988.
14. Yu.A. Blinkov and V.P. Gerdt. Janet bases of toric ideals. In H. Kredel and W.K. Seiler, editors, *Proc. Rhine Workshop on Computer Algebra*, pages 125–135. Universität Mannheim, 2002.
15. W. Bruns and A. Conca. Gröbner bases, initial ideals and initial algebras. Preprint math.AC/0308102, 2003.
16. B. Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*. PhD thesis, Universität Innsbruck, 1965. (Engl. translation: J. Symb. Comput. 41 (2006) 475–511).
17. J.L. Bueso, J. Gómez-Torrecillas, F.J. Lobillo, and F.J. Castro-Jiménez. An introduction to effective calculus in quantum groups. In S. Caenepeel and A. Verschoren, editors, *Rings, Hopf Algebras, and Brauer Groups*, Lecture Notes in Pure and Applied Mathematics 197, pages 55–83. Marcel Dekker, New York, 1998.
18. G. Caviglia and E. Sbarra. Characteristic-free bounds for the Castelnuovo-Mumford regularity. *Compos. Math.*, 141:1365–1373, 2005.
19. M. Cimpoeaş. Monomial ideals with linear upper bound regularity. Preprint math.AC/0611064, 2006.
20. M. Cimpoeaş. A stable property of Borel type ideals. *Commun. Alg.*, 36:674–677, 2008.
21. M. Cimpoeaş. Some remarks on Borel type ideals. *Commun. Alg.*, 37:724–727, 2009.
22. D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992.

23. D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*. Graduate Texts in Mathematics 185. Springer-Verlag, New York, 1998.
24. D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics 150. Springer-Verlag, New York, 1995.
25. D. Eisenbud and S. Goto. Linear free resolutions and minimal multiplicity. *J. Alg.*, 88:89–133, 1984.
26. D. Eisenbud, A. Reeves, and B. Totaro. Initial ideals, Veronese subrings and rates of algebras. *Adv. Math.*, 109:168–187, 1994.
27. S. Eliahou and M. Kervaire. Minimal resolutions of some monomial ideals. *J. Alg.*, 129:1–25, 1990.
28. A. Galligo. A propos du théorème de préparation de Weierstrass. In F. Norguet, editor, *Fonctions de Plusieurs Variables Complexes*, Lecture Notes in Mathematics 409, pages 543–579. Springer-Verlag, Berlin, 1974.
29. V.P. Gerdt. On the relation between Pommaret and Janet bases. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing — CASC 2000*, pages 167–182. Springer-Verlag, Berlin, 2000.
30. V.P. Gerdt and Yu.A. Blinkov. Involutive bases of polynomial ideals. *Math. Comp. Simul.*, 45:519–542, 1998.
31. V.P. Gerdt and Yu.A. Blinkov. Minimal involutive bases. *Math. Comp. Simul.*, 45:543–560, 1998.
32. H. Grauert. Über die Deformation isolierter Singularitäten analytischer Mengen. *Invent. Math.*, 15:171–198, 1972.
33. M.L. Green. Generic initial ideals. In J. Elias, J.M. Giral, R.M. Miró-Roig, and S. Zarzuela, editors, *Six Lectures on Commutative Algebra*, Progress in Mathematics 166, pages 119–186. Birkhäuser, Basel, 1998.
34. G.-M. Greuel and G. Pfister. *A* SINGULAR *Introduction to Commutative Algebra*. Springer-Verlag, Berlin, 2002.
35. W. Gröbner. *Moderne Algebraische Geometrie*. Springer-Verlag, Wien, 1949.
36. V.W. Guillemin and S. Sternberg. An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.*, 70:16–47, 1964.
37. A. Hashemi. Polynomial-time algorithm for Hilbert series of Borel type ideals. *Albanian J. Math.*, 1:145–155, 2007.
38. M. Hausdorf, M. Sahbi, and W.M. Seiler. δ- and quasi-regularity for polynomial ideals. In J. Calmet, W.M. Seiler, and R.W. Tucker, editors, *Global Integrability of Field Theories*, pages 179–200. Universitätsverlag Karlsruhe, Karlsruhe, 2006.
39. M. Hausdorf and W.M. Seiler. An efficient algebraic algorithm for the geometric completion to involution. *Appl. Alg. Eng. Comm. Comp.*, 13:163–207, 2002.
40. G. Hermann. Die Frage der endlich vielen Schritte in der Theorie der Polynomideale. *Math. Ann.*, 95:736–788, 1926.
41. J. Herzog and T. Hibi. Componentwise linear ideals. *Nagoya Math. J.*, 153:141–153, 1999.
42. J. Herzog, D. Popescu, and M. Vladoiu. On the Ext-modules of ideals of Borel type. In *Commutative Algebra*, Contemp. Math. 331, pages 171–186. Amer. Math. Soc., Providence, 2003.
43. J. Herzog, A. Soleyman Jahan, and S. Yassemi. Stanley decompositions and partionable simplicial complexes. *J. Algebr. Comb.*, 27:113–125, 2008.
44. H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. Math.*, 79:109–326, 1964.
45. H. Hironaka. Bimeromorphic smoothing of complex analytic spaces. *Acta Math. Vietnam.*, 2:103–168, 1977.

46. H. Hironaka. Idealistic exponents of singularity. In J.-I. Igusa, editor, *Algebraic Geometry – The Johns Hopkins Centennial Lectures*, pages 52–125. Johns Hopkins University Press, 1977.
47. S. Hoşten and G.G. Smith. Monomial ideals. In D. Eisenbud, D.R. Grayson, M. Stillman, and B. Sturmfels, editors, *Computations in Algebraic Geometry with Macaulay 2*, Algorithms and Computation in Mathematics 8, pages 73–100. Springer-Verlag, Berlin, 2002.
48. M. Janet. *Leçons sur les Systèmes d'Équations aux Dérivées Partielles*. Cahiers Scientifiques, Fascicule IV. Gauthier-Villars, Paris, 1929.
49. F. John. *Partial Differential Equations*. Applied Mathematical Sciences 1. Springer-Verlag, New York, 1982.
50. H. Kredel and V. Weispfenning. Computing dimension and independent sets for polynomial ideals. *J. Symb. Comp.*, 6:231–247, 1988.
51. M. Kreuzer and L. Robbiano. *Computational Commutative Algebra 1*. Springer-Verlag, Berlin, 2000.
52. V. Levandovskyy. On Gröbner bases for non-commutative G-algebras. In J. Calmet, M. Hausdorf, and W.M. Seiler, editors, *Proc. Under- and Overdetermined Systems of Algebraic or Differential Equations*, pages 99–118. Fakultät für Informatik, Universität Karlsruhe, 2002.
53. V. Levandovskyy. *Non-commutative Computer Algebra for Polynomial Algebras: Gröbner Bases, Applications and Implementation*. PhD thesis, Fachbereich Mathematik, Universität Kaiserslautern, 2005.
54. D. Mall. On the relation between Gröbner and Pommaret bases. *Appl. Alg. Eng. Comm. Comp.*, 9:117–123, 1998.
55. E.W. Mayr and A.R. Meyer. The complexity of the word problems for commutative semigroups and polynomial ideals. *Adv. Math.*, 46:305–329, 1982.
56. J.C. McConnell and J.C. Robson. *Non-commutative Noetherian Rings*. Wiley, 1987.
57. E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra*. Graduate Texts in Mathematics 227. Springer-Verlag, New York, 2005.
58. W. Plesken and D. Robertz. Janet's approach to presentations and resolutions for polynomials and linear PDEs. *Arch. Math.*, 84:22–37, 2005.
59. D. Rees. A basis theorem for polynomial modules. *Proc. Cambridge Phil. Soc.*, 52:12–16, 1956.
60. G.J. Reid. Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution. *Eur. J. Appl. Math.*, 2:293–318, 1991.
61. M. Renardy and R.C. Rogers. *An Introduction to Partial Differential Equations*. Texts in Applied Mathematics 13. Springer-Verlag, New York, 1993.
62. D. Robertz. Noether normalization guided by monomial cone decompositions. Preprint, Dept. Mathematics, RWTH Aachen, 2008.
63. M. Sahbi. Pommaret bases and the computation of the Koszul homology in the monomial case. Diploma Thesis, Fakultät für Informatik, Universität Karlsruhe, 2007.
64. F.O. Schreyer. Die Berechnung von Syzygien mit dem verallgemeinerten Weierstraßschen Divisionssatz. Master's thesis, Fakultät für Mathematik, Universität Hamburg, 1980.
65. W.M. Seiler. Taylor and Lyubeznik resolutions via Gröbner bases. *J. Symb. Comp.*, 34:597–608, 2002.
66. W.M. Seiler. Spencer cohomology, differential equations, and Pommaret bases. In M. Rosenkranz and D. Wang, editors, *Gröbner Bases in Symbolic Analysis*, Radon Series on Computation and Applied Mathematics 2, pages 171–219. Walter de Gruyter, Berlin, 2007.

67. W.M. Seiler. *Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra*. Algorithms and Computation in Mathematics 24. Springer-Verlag, Berlin, 2009 (to appear).
68. R.P. Stanley. Hilbert functions of graded algebras. *Adv. Math.*, 28:57–83, 1978.
69. R.P. Stanley. Linear diophantine equations and local cohomology. *Invent. Math.*, 68:175–193, 1982.
70. R.P. Stanley. *Combinatorics and Commutative Algebra*. Birkhäuser, 1983.
71. M. Stillman. Tools for computing primary decompositions and applications to ideals associated to Bayesian networks. In A. Dickenstein and I.Z. Emiris, editors, *Solving Polynomial Equations*, Algorithms and Computation in Mathematics 14, pages 203–239. Springer-Verlag, 2005.
72. B. Sturmfels. *Algorithms in Invariant Theory*. Texts and Monographs in Symbolic Computation. Springer-Verlag, Wien, 1993.
73. B. Sturmfels, N.V. Trung, and W. Vogel. Bounds on degrees of projective schemes. *Math. Ann.*, 302:417–432, 1995.
74. B. Sturmfels and N. White. Computing combinatorial decompositions of rings. *Combinatorica*, 11:275–293, 1991.
75. D. Taylor. *Ideals Generated by Monomials in an R-Sequence*. PhD thesis, University of Chicago, 1960.
76. J.M. Thomas. *Differential Systems*. Colloquium Publications XXI. American Mathematical Society, New York, 1937.
77. C. Traverso. Hilbert functions and the Buchberger algorithm. *J. Symb. Comput*, 22:355–376, 1996.
78. N.G. Trung. Evaluations of initial ideals and Castelnuovo–Mumford regularity. *Proc. Amer. Math. Soc.*, 130:1265–1274, 2002.
79. W.V. Vasconcelos. *Computational Methods in Commutative Algebra and Algebraic Geometry*. Algorithms and Computations in Mathematics 2. Springer-Verlag, Berlin, 1997.