Dimension and Depth Dependent Upper Bounds in Polynomial Ideal Theory

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Abstract

We improve certain upper bounds for the degree of Gröbner bases and the Castelnuovo-Mumford regularity of polynomial ideals. For the degree of Gröbner bases, we exclusively work in deterministically verifiable and achievable generic positions of a combinatorial nature, namely either strongly stable position or quasi stable position. Furthermore, we exhibit new dimension and depth depending upper bounds for the Castelnuovo-Mumford regularity and the degrees of the elements of the reduced Gröbner basis (w.r.t. the degree reverse lexicographical ordering) of a homogeneous ideal in these positions. Finally, it is shown that similar upper bounds hold in positive characteristic.

Keywords: Polynomial ideals, Gröbner bases, Pommaret bases, generic positions, stability, degree, dimension, depth, Castelnuovo-Mumford regularity.

1. Introduction

Gröbner bases, introduced by Bruno Buchberger in his Ph.D. thesis (see e.g. (Buchberger, 1965, 2006, 1979)), have become a powerful tool for constructive problems in polynomial ideal theory and related domains. For practical applications, in particular, the implementation in computer algebra systems, it is important to establish upper bounds for the complexity of determining a Gröbner basis for a given homogeneous polynomial ideal. Using Lazard's algorithm (Lazard, 1983), a good measure to estimate such a bound, is an upper bound for the degree of the intermediate polynomials during the Gröbner basis computation. If the input ideal is *not* homogeneous, the maximal degree of the output Gröbner basis is *not* sufficient for this estimation. On the other hand, Möller and Mora (1984) showed that to discuss degree bounds for Gröbner bases, one can restrict to homogeneous ideals. Thus upper bounds for the degrees of the elements of Gröbner bases of homogeneous ideals, allow us to estimate the complexity of computing Gröbner bases in general.

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Let us recall some of the existing results in this direction. Let $\mathcal{P}$ be the polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$ where $\mathbb{k}$ is a field of characteristic zero and $\mathcal{I} \subset \mathcal{P}$ be an ideal generated by *homogeneous* polynomials of degree at most d with $\dim(\mathcal{I}) = D$. The first doubly exponential upper bounds were proven by Bayer, Möller, Mora and Giusti, see (Mora, 2005, Chapter 38) for a comprehensive review of this topic. Based on results due to Bayer (1982) and Galligo (1974, 1979), Möller and Mora (1984) provided the upper bound $(2d)^{(2n+2)^{n+1}}$ for any Gröbner basis of $\mathcal{I}$. They also proved that this doubly exponential behavior cannot be improved. Simultaneously, Giusti (1984) showed the upper bound $(2d)^{2^{n-1}}$ for the degree of the reduced Gröbner basis (w.r.t. the degree reverse lexicographic order) of $\mathcal{I}$ when the ideal is in *generic position*. Then, using a self-contained and constructive combinatorial argument, Dubé (1990) proved the so far sharpest degree bound $2(d^2/2 + d)^{2^{n-1}} \leq 2d^{2^n}$ for $d \geq 2$.

Caviglia and Sbarra (2005) studied upper bounds for the Castelnuovo-Mumford regularity of homogeneous ideals. Analyzing Giusti's proof, they gave a simple proof of the upper bound $(2d)^{2^{n-2}}$ for the degree reverse lexicographic Gröbner basis of an ideal $\mathcal{I}$ in generic position (they also showed that this bound holds independent of the characteristic of $\mathbb{k}$). Finally, Mayr and Ritscher (2013), by following the tracks of Dubé (1990), obtained the dimension-dependent upper bound $2(1/2d^{n-D}+d)^{2^{D-1}}$ for every reduced Gröbner basis of $\mathcal{I}$. It is worth while remarking that there are also *lower* bounds for the worst-case complexity: d^{2^m} with $m = n/10 - O(1)$ from the work of Mayr and Meyer (1982) and d^{2^m} where $m \sim n/2$ due to Yap (1991).

In this article, we will first improve Giusti's bound by showing that if $\mathcal{I}$ is in strongly stable position and $D > 1$, then $2d^{(n-D)2^{D-1}}$ is a simultaneous upper bound for the Castelnuovo-Mumford regularity of $\mathcal{I}$ and for the maximal degree of the elements of the Gröbner basis of $\mathcal{I}$ (with respect to the degree reverse lexicographic order). Furthermore, we will sharpen the bound of Caviglia-Sbarra to $(d^{n-D} + (n-D)(d-1))^{2^{D-1}}$. We will see that the latter improved bound is always lower than all existing bounds. Finally, we will show that, if $\mathcal{I}$ is in quasi stable position and $D \leq 1$, Giusti's bound may be replaced by $nd - n + 1$ (this result was already obtained in (Lazard, 1983) when the ideal is in generic position). In the recent work (Hashemi et al., 2017), we showed how many variants of stable positions – including quasi stable and strongly stable position – can be achieved via linear coordinate transformations constructed with a deterministic algorithm.

This article is a revised and extended version of (Hashemi and Seiler, 2017) presented at ISSAC 2017. The changes to the original version include the addition of a new section (see Sec. 3) on the French course notes (Lejeune-Jalabert, 1984). It contains some results from loc. id. (with simpler proofs) on upper degree bounds for Gröbner bases of certain classes of ideals. In Sec. 4 we exhibit in more detail the properties of one-dimensional ideals in quasi stable position. In addition, in Theorem 50, a new upper degree bound for Gröbner bases is derived and it is shown that the new bound is always sharper than all previously published bounds. We conclude the paper by discussing the case of positive characteristic where we show that similar bounds hold (this study gives a positive answer to the conjecture posed in (Hashemi and Seiler, 2017, Con. 4.5)).

The structure of the paper is as follows: in the next section, we give basic notations and definitions. Section 3 is devoted to some results from Lejeune-Jalabert (1984) concerning upper bounds for the degree of Gröbner bases of particular classes of ideals including zero-dimensional ideals. In Sections 4, 5 and 6 we improve the degree bounds provided by Lazard, Giusti and Caviglia-Sbarra, respectively.

2. Preliminaries

Throughout this article, we keep the following notations. Let $\mathcal{P} = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring (where $\mathbb{k}$ is of characteristic zero). A power product of the variables $x_1, \ldots, x_n$ is called a *term* and $\mathbb{T}$ denotes the monoid of all terms in $\mathcal{P}$. We consider non-zero homogeneous polynomials $f_1, \ldots, f_k \in \mathcal{P}$ and the ideal $\mathcal{I} = \langle f_1, \ldots, f_k \rangle$ generated by them. We assume that f_i is of degree d_i and that the numbering is such that $d_1 \geq d_2 \geq \cdots \geq d_k > 0$. We also set $d = d_1$. Furthermore, we denote by $\mathcal{R} = \mathcal{P}/\mathcal{I}$ the corresponding quotient ring and by D its Krull dimension. Finally, we use throughout the degree reverse lexicographic order with $x_n \prec \cdots \prec x_1$.

The *leading term* of a polynomial $f \in \mathcal{P}$, denoted by $\mathrm{LT}(f)$, is the greatest term (with respect to $\prec$) appearing in f and its coefficient, denoted by $\mathrm{LC}(f)$, is the *leading coefficient* of f. The *leading monomial* of f is the product $\mathrm{LM}(f) = \mathrm{LC}(f)\mathrm{LT}(f)$. The *leading ideal* of $\mathcal{I}$ is defined as $\mathrm{LT}(\mathcal{I}) = \langle \mathrm{LT}(f) \mid f \in \mathcal{I} \rangle$. For the finite set $F = \{f_1, \ldots, f_k\} \subset \mathcal{P}$, $\mathrm{LT}(F)$ denotes the set $\{\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_k)\}$. A finite subset $G \subset \mathcal{I}$ is called a *Gröbner basis* of $\mathcal{I}$ w.r.t. $\prec$, if $\mathrm{LT}(\mathcal{I}) = \langle \mathrm{LT}(G) \rangle$. We refer to (Becker and Weispfenning, 1993) for more details on Gröbner bases.

Given a graded $\mathcal{P}$-module X and a positive integer s, we denote by X_s the set of all homogeneous elements of X of degree s. To define the Hilbert regularity of an ideal, recall that the *Hilbert function* of $\mathcal{I}$ is defined by $\mathrm{HF}_{\mathcal{I}}(t) = \dim_{\mathbb{k}}(\mathcal{R}_t)$; the dimension of $\mathcal{R}_t$ as a $\mathbb{k}$-linear space. From a certain degree on, this function of t is equal to a polynomial in t, called *Hilbert polynomial*, and denoted by $\mathrm{HP}_{\mathcal{I}}$ (see e.g. (Cox et al., 2007) for more details on this topic). The *Hilbert regularity* of $\mathcal{I}$ is $\mathrm{hilb}(\mathcal{I}) = \min\{m \mid \forall t \geq m,\ \mathrm{HF}_{\mathcal{I}}(t) = \mathrm{HP}_{\mathcal{I}}(t)\}$. Finally, recall that the *Hilbert series* of $\mathcal{I}$ is the power series $\mathrm{HS}_{\mathcal{I}}(t) = \sum_{s=0}^{\infty} \mathrm{HF}_{\mathcal{I}}(s)t^s$.

Proposition 1. *There exists a univariate polynomial $p(t)$ with $p(1) \neq 0$ such that $\mathrm{HS}_{\mathcal{I}}(t) = p(t)/(1-t)^D$. Furthermore,* $\mathrm{hilb}(\mathcal{I}) = \max\{0, \deg(p) - D + 1\}$.

For a proof of this result, we refer to (Fröberg, 1997, Theorem 7, page 130). It follows immediately from Macaulay's theorem that the Hilbert function of $\mathcal{I}$ is the same as that of $\mathrm{LT}(\mathcal{I})$ and this provides an effective method to compute it using Gröbner bases, see (Greuel and Pfister, 2007).

Let us state some auxiliary results on regular sequences. Recall that a sequence of polynomials $f_1, \ldots, f_k \in \mathcal{P}$ is called *regular* if it generates a proper ideal and f_i is a non-zero divisor on the ring $\mathcal{P}/\langle f_1, \ldots, f_{i-1} \rangle$ for $i = 2, \ldots, k$. This is equivalent to the condition that f_i does not belong to any associated prime of $\langle f_1, \ldots, f_{i-1} \rangle$. It can be shown that the Hilbert series of a regular sequence $f_1, \ldots, f_k$ is equal to $\prod_{i=1}^{k}(1-t^{d_i})/(1-t^n)$, see e.g. (Lejeune-Jalabert, 1984). The converse of this result is also true, see (Fröberg, 1997, Exercise 7, page 137). As a trivial consequence of this statement we observe that any permutation of a homogeneous regular sequence remains a regular sequence. In addition, these conditions are equivalent to the statement that $D = n - k$, see Lemma 8.

Definition 2. *The* depth *of $\mathcal{I}$ is defined as the maximal integer λ such that there exists a regular sequence of linear forms $y_1, \ldots, y_\lambda$ on $\mathcal{R}$.*

For example, let $\mathcal{J} = \langle x_1^2, x_1x_2 \rangle \subset \mathcal{P} = \mathbb{k}[x_1, x_2]$. Then, no linear form $ax_1 + bx_2$ for $a, b \in \mathbb{k}$ is regular on $\mathcal{R}$. This follows that $\mathrm{depth}(\mathcal{J}) = 0$. It is well-known that the dimension of an ideal is greater than or equal to its depth (a simple proof using Pommaret bases can be found in (Seiler, 2009) after Proposition 3.19). If $\dim(\mathcal{I})$ is equal to the depth of $\mathcal{I}$, then $\mathcal{R}$ is called *Cohen-Macaulay*. For example, one sees that the quotient ring $\mathbb{k}[x_1, x_2]/\langle x_2^2 \rangle$ is Cohen-Macaulay, however, $\mathbb{k}[x_1, x_2]/\langle x_1^2, x_1x_2 \rangle$ is not Cohen-Macaulay. Below, for a given integer e, by $\mathcal{P}(e)$, we shall mean a graded $\mathcal{P}$-module such that $\mathcal{P}(e)_t = \mathcal{P}_{e+t}$ for any t.

Definition 3. *The ideal $\mathcal{I}$ is called m-*regular*, if its minimal graded free resolution is of the form*

$$0 \longrightarrow \bigoplus_{j\in J_r} \mathcal{P}(-e_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j\in J_1} \mathcal{P}(-e_{1j}) \longrightarrow \bigoplus_{j\in J_0} \mathcal{P}(-e_{0j}) \longrightarrow \mathcal{I} \longrightarrow 0$$

with $e_{ij} - i \leq m$ for each i, j where J_i for each i is finite. The Castelnuovo-Mumford regularity *of $\mathcal{I}$ is the smallest m such that $\mathcal{I}$ is m-regular; we denote it by* $\mathrm{reg}(\mathcal{I})$.

For more details on the regularity, we refer to (Mumford, 1966; Eisenbud and Goto, 1984; Bayer and Stillman, 1987; Bermejo and Gimenez, 2006). It is well-known that in generic coordinates $\mathrm{reg}(\mathcal{I})$ is an upper bound for the degree of the Gröbner basis w.r.t. the degree reverse lexicographic order. This upper bound is sharp, if the characteristic of $\Bbbk$ is zero (see (Bayer and Stillman, 1987)). A good measure to estimate the complexity of the computation of the Gröbner basis of $\mathcal{I}$ is the maximal degree of the polynomials which appear in this computation, see (Lazard, 1981, 1983; Giusti, 1984).

Definition 4. *We denote by* $\deg(\mathcal{I}, \prec)$ *the maximal degree of the elements of the reduced Gröbner basis of $\mathcal{I}$ w.r.t. the term order $\prec$.*

We conclude this section with a brief review of the theory of Pommaret bases. Suppose that $f \in \mathcal{P}$ and $\mathrm{LT}(f) = x^\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_n)$. We call $\max\{i \mid \alpha_i \neq 0\}$ the *class* of f, denoted by $\mathrm{cls}(f)$. Then the *multiplicative variables* of f are $\mathcal{X}_P(f) = \{x_{\mathrm{cls}(f)}, \dots, x_n\}$. Furthermore, x^β is a *Pommaret divisor* of x^α, written $x^\beta \mid_P x^\alpha$, if $x^\beta \mid x^\alpha$ and $x^{\alpha-\beta} \in \Bbbk[x_{\mathrm{cls}(f)}, \dots, x_n]$.

Definition 5. *Let $\mathcal{H} \subset \mathcal{I}$ be a finite set such that no leading term of an element of $\mathcal{H}$ is a Pommaret divisor of the leading term of another element. Then $\mathcal{H}$ is called a* Pommaret basis *of $\mathcal{I}$ w.r.t. $\prec$, if*

$$\mathcal{I} = \bigoplus_{h\in\mathcal{H}} \Bbbk[\mathcal{X}_P(h)] \cdot h. \tag{1}$$

One can easily show that any Pommaret basis is a (generally non-reduced) Gröbner basis of the ideal it generates. The main difference between Gröbner and Pommaret bases consists of the fact that by (1) any polynomial $f \in \mathcal{I}$ has a *unique* involutive standard representation. If an ideal $\mathcal{I}$ possesses a Pommaret basis $\mathcal{H}$, then $\mathrm{reg}(\mathcal{I})$ equals the maximal degree of an element of $\mathcal{H}$, cf. (Seiler, 2009, Theorem 9.2). The main drawback of Pommaret bases is however that they do not always exist. Indeed, a given ideal possesses a finite Pommaret basis, if and only if the ideal is in *quasi stable position* – see (Seiler, 2009, Proposition 4.4).

Definition 6. *A monomial ideal $\mathcal{J}$ in $\mathcal{P}$ is called* quasi stable*, if for any term $m \in \mathcal{J}$ and all integers i, j, s with $1 \leq j < i \leq n$ and $s > 0$ such that $x_i^s \mid m$, there exists an exponent $t \geq 0$ such that $x_j^t m / x_i^s \in \mathcal{J}$. A homogeneous ideal $\mathcal{I}$ is in* quasi stable position*, if* $\mathrm{LT}(\mathcal{I})$ *is quasi stable.*

Let us denote by $\deg(m, x_i)$ the degree of m w.r.t. the variable x_i. It follows from (Seiler, 2012, Lemma 3.4) that a monomial ideal $\mathcal{J}$ is quasi stable iff for any monomial $m \in \mathcal{J}$ and all integer $1 \leq j < \mathrm{cls}(m)$ we have $x_j^t m / x_{\mathrm{cls}(m)}^s \in \mathcal{J}$ with $s = \deg(m, x_{\mathrm{cls}(m)})$ and t is the maximum of $\deg(u, x_j)$ where u belongs to a minimal generating set of $\mathcal{J}$.

Remark 7. *Since any linear change of variables is a $\Bbbk$-linear automorphism of $\mathcal{P}$ preserving the degree, it follows trivially that the dimensions over $\Bbbk$ of the homogeneous components of a homogeneous ideal $\mathcal{I}$ or of its factor ring $\mathcal{R}$ remain invariant. Hence the Hilbert function and* *therefore also the Hilbert series, the Hilbert polynomial and the Hilbert regularity of $\mathcal{I}$ do not change. The same is obviously true for the Castelnuovo-Mumford regularity. In addition, due to the special form of the Hilbert series of the ideal generated by a regular sequence, we conclude that any regular sequence remains regular after a linear change of variables and hence the depth is invariant, too. Finally, we note that almost all linear changes of variables transform a given homogeneous ideal into quasi stable position (which is thus a generic position), see (Seiler, 2009). It follows that to study any of the mentioned invariants of $\mathcal{I}$, w.l.o.g. we may assume that $\mathcal{I}$ is in quasi stable position. We write $\mathcal{H} = \{h_1, \dots, h_s\}$ for its Pommaret basis. Finally, for each i we set $m_i = \mathrm{LT}(h_i)$ and it is then easy to see that $\{m_1, \dots, m_s\}$ forms a Pommaret basis of $\mathrm{LT}(\mathcal{I})$.*

3. Lejeune-Jalabert course notes

In her French course notes, Lejeune-Jalabert (1984) studied $\deg(\mathcal{I}, \prec)$ for some particular classes of homogeneous ideals including zero-dimensional ideals. Since these course notes may not be largely available, we discuss briefly some results contained in them related to the subject of this paper and for the sake of completeness we include short proofs for some of them. Before stating the main theorems of this section, we shall give some preliminary results. If an ideal of dimension $n-k$ can be generated by k polynomials, then it is called a complete intersection. The following result is well-known. Proofs for the local case can be found for example in (Bruns and Herzog, 1998, Theorem 2.3.3) or (Matsumura, 1986, Theorem 17.4). The graded version is given in (Lejeune-Jalabert, 1984, Proposition 2.4, page 101).

Lemma 8. *If $D = n - k$, then $f_1, \dots, f_k$ is a regular sequence.*

In this section, we consider the *Nœther position* as a particular generic position for polynomial ideals to derive effective degrees for their Gröbner bases. Recall that a homogeneous ideal $\mathcal{I} \subset \mathcal{P}$ is in *Nœther position* if the injective ring extension $\Bbbk[x_{n-D+1}, \dots, x_n] \hookrightarrow \mathcal{R}$ is integral, i.e. the image in $\mathcal{R}$ of x_i for any $i = 1, \dots, n-D$ is a root of a polynomial $X^s + g_1 X^{s-1} + \cdots + g_s = 0$ where s is an integer and $g_1, \dots, g_s \in \Bbbk[x_{n-D+1}, \dots, x_n]$, see e.g. (Eisenbud, 1995). For example, one can see that the ideal $\langle x_2^2 - x_1 \rangle \subset \Bbbk[x_1, x_2]$ is in Nœther position. Alternatively, Nœther position can be defined combinatorially as a weakened version of quasi stable position, see (Hashemi et al., 2017; **?**, Theorem 4.4).

Lemma 9. *Suppose that $f_1, \dots, f_k$ is a regular sequence and $\mathcal{I}$ is in Nœther position. Then, $f_1, \dots, f_k, x_{n-D+1}, \dots, x_n$ is a regular sequence.*

Proof. Since $\mathcal{I}$ is in Nœther position then $\dim(\mathcal{I} + \langle x_{n-D+1}, \dots, x_n \rangle) = 0$, see (Bermejo and Gimenez, 2001, Lemma 4.1). Thus the assertion follows from Lemma 8. □

Lemma 10. *If z is a linear non-zero divisor on $\mathcal{R}$, then* $\mathrm{hilb}(\mathcal{I}) = \max\{0, \mathrm{hilb}(\mathcal{I} + \langle z \rangle) - 1\}$.

If $f_1, \dots, f_k$ is a regular sequence, from Proposition 1 one deduces that $\mathrm{hilb}(\mathcal{I}) = \max\{0, d_1 + \cdots + d_k - n + 1\}$, see also (Lejeune-Jalabert, 1984, Rem. 3.2.2, page 104). This property is stated in the next proposition.

Proposition 11. *If $f_1, \dots, f_k$ is a regular sequence then* $\mathrm{hilb}(\mathcal{I}) = \max\{0, d_1 + \cdots + d_k - n + 1\}$.

Proposition 12. *(Lejeune-Jalabert, 1984, Proposition 3.4, page 105) Suppose that $1 \le r \le n$ is a positive integer such that the sequence $x_r, \dots, x_n$ is regular on $\mathcal{R}$. Furthermore, let G be the reduced Gröbner basis of $\mathcal{I}$ w.r.t. $\prec$. Then, for each $g \in G$, the leading term of g does not contain any of the variables $x_r, \dots, x_n$.*

Proof. We argue by reductio ad absurdum. Suppose that there exists $g \in G$ such that $x_s \mid \mathrm{LT}(g)$ and $r \leq s \leq n$. Since $\mathcal{I}$ is a homogeneous ideal, G contains only homogeneous polynomials. Assume that x_s is the smallest variable w.r.t. $\prec$ such that $x_s \mid \mathrm{LT}(g)$. Thus, by the definition of $\prec$, we can write g as $x_sA + B$ where $A \in \Bbbk[x_1, \ldots, x_s]$ and $B \in \langle x_{s+1}, \ldots, x_n\rangle \subset \mathcal{P}$. This implies that $x_sA \in \mathcal{I} + \langle x_{s+1}, \ldots, x_n\rangle$, and therefore $A \in \mathcal{I} + \langle x_{s+1}, \ldots, x_n\rangle$ because by the assumption, x_s is a non-zero divisor on the ring $\mathcal{P}/(\mathcal{I} + \langle x_{s+1}, \ldots, x_n\rangle)$. Since $x_r, \ldots, x_n$ is a homogeneous regular sequence, any permutation of this sequence remains regular. So, we can conclude that there exists $C \in \langle x_{s+1}, \ldots, x_n\rangle$ such that $A + C \in \mathcal{I}$. It follows that there exists $g' \in G$ with $\mathrm{LT}(g') \mid \mathrm{LT}(A) = \mathrm{LT}(A + C)$ which contradicts the minimality of G. □

In particular, from Lemma 9 we have the following consequence.

Corollary 13. *Suppose that $f_1, \ldots, f_k$ is a regular sequence and $\mathcal{I}$ is in Nœther position. Furthermore, let G be the reduced Gröbner basis of $\mathcal{I}$ w.r.t. $\prec$. Then, for each $g \in G$, the leading monomial of g does not contain any of the variables $x_{n-D+1}, \ldots, x_n$.*

We state now the main result of the first part of this section.

Theorem 14. *(Lejeune-Jalabert, 1984, Cor. 3.5, page 107) Suppose that $f_1, \ldots, f_k$ is a regular sequence and $\mathcal{I}$ is in Nœther position. Then* $\deg(\mathcal{I}, \prec) \leq d_1 + \cdots + d_k - k + 1$.

Proof. We know that $f_1, \ldots, f_k, x_{n-D+1}, \ldots, x_n$ is a regular sequence. Let $\mathcal{J}$ be the ideal generated by this sequence. Thus, $\mathrm{hilb}(\mathcal{J}) = d_1 + \cdots + d_k + (n-k) - n + 1 = d_1 + \cdots + d_k - k + 1$. Since $\mathcal{J}$ is a zero-dimensional ideal, $\mathrm{hilb}(\mathcal{J})$ is the maximum degree of the elements of the Gröbner basis of $\mathcal{J}$. On the other hand, from Cor. 13, the maximum degree of the elements of the Gröbner basis of $\mathcal{J}$ is equal to that of $\mathcal{I}$, and this completes the proof. □

Corollary 15. *If $f_1, \ldots, f_n$ is a regular sequence, then* $\deg(\mathcal{I}, \prec) \leq d_1 + \cdots + d_n - n + 1$.

Remark 16. *The bound $d_1 + \cdots + d_n - n + 1$ is known as* Macaulay *bound.*

In the rest of this section, we deal with Cohen-Macaulay rings, and state the results of (Lejeune-Jalabert, 1984) to generalize the above results. To state an analogue of Proposition 11, we need the following lemmata.

Lemma 17. *(Lejeune-Jalabert, 1984, Proposition 4.1, page 108) There exist homogeneous polynomials $g_1, \ldots, g_{n-D} \in \mathcal{P}$ such that*

(1) $\deg(g_i) = d_i$ *for each i,*

(2) $g_i \equiv f_i \mod \langle f_{i+1}, \ldots, f_k\rangle$ *for $i = 1, \ldots, n - D$,*

(3) $g_1, \ldots, g_{n-D}$ *form a regular sequence in $\mathcal{P}$.*

Lemma 18. *(Lejeune-Jalabert, 1984, Proposition 4.4, page 110) Suppose that $\mathcal{R}$ is Cohen-Macaulay and $\mathcal{I}$ is in Nœther position. Then, the sequence $x_{n-D+1}, \ldots, x_n$ is regular on $\mathcal{R}$.*

Based on this lemma, we can state a similar result to Proposition 12.

Corollary 19. *(Lejeune-Jalabert, 1984, Lemma 4.5.1 , page 113) Suppose that the quotient ring $\mathcal{R}$ is Cohen-Macaulay and $\mathcal{I}$ is in Nœther position. Let G be the reduced Gröbner basis of $\mathcal{I}$ w.r.t. $\prec$. Then, for each $g \in G$, the leading term of g does not contain any of the variables $x_{n-D+1}, \ldots, x_n$.*

Proposition 20. *If $\mathcal{R}$ is Cohen-Macaulay, then* $\mathrm{hilb}(\mathcal{I}) \leq \max\{0, d_1 + \cdots + d_{n-D} - n + 1\}$.

Proof. Since $\mathcal{R}$ is Cohen-Macaulay and the Hilbert series of $\mathcal{I}$ is invariant under any invertible linear change of variables, we can assume that $x_{n-D+1}, \ldots, x_n$ is a regular sequence on $\mathcal{R}$. Furthermore, by Lemma 10, we have $\mathrm{hilb}(\mathcal{I}) = \max\{0, \mathrm{hilb}(\mathcal{I} + \langle x_{n-D+1}, \ldots, x_n\rangle) - D\}$. We know from (Cox et al., 2007, Proposition 4, Sec. 3, Chap. 9) that the Hilbert function of $\mathcal{I} + \langle x_{n-D+1}, \ldots, x_n\rangle$ and that of its leading term ideal are equal. By Cor. 19, no leading term of the polynomials in the Gröbner basis of $\mathcal{I}$ is divisible by any of the variables $x_{n-D+1}, \ldots, x_n$. Thus, we conclude that the Hilbert function of this ideal and the one of the ideal $\mathcal{J} = \mathcal{I}|_{x_{n-D+1}=\cdots=x_n=0} \subset \Bbbk[x_1, \ldots, x_{n-D}]$ are the same. Note that $\mathcal{J}$ is generated by the polynomials $h_i = f_i|_{x_{n-D+1}=\cdots=x_n=0}$. For each i, we have either $h_i = 0$ or $\deg(h_i) = d_i$. One observes that $\mathcal{J} = \langle h_1, \ldots, h_k\rangle \subset \Bbbk[x_1, \ldots, x_{n-D}]$ is zero-dimensional. Using Lemma 17 and without loss of generality we may assume that $h_1, \ldots, h_{n-D}$ are non-zero and form a regular sequence. This implies that the inequality $\mathrm{hilb}(\mathcal{J}) \leq \mathrm{hilb}(\langle h_1, \ldots, h_{n-D}\rangle)$ holds. In addition, by Proposition 11, we have $\mathrm{hilb}(\langle h_1, \ldots, h_{n-D}\rangle) \leq \max\{0, d_1 + \cdots + d_{n-D} - (n-D) + 1\}$. These arguments show that $\mathrm{hilb}(\mathcal{I}) = \max\{0, \mathrm{hilb}(\mathcal{I} + \langle x_{n-D+1}, \ldots, x_n\rangle) - D\} = \max\{0, \mathrm{hilb}(\mathcal{J}) - D\} \leq \max\{0, d_1 + \cdots + d_{n-D} - (n-D) + 1 - D\} = \max\{0, d_1 + \cdots + d_{n-D} - n + 1\}$. □

Finally, we state the second main result of this section.

Theorem 21. *(Lejeune-Jalabert, 1984, Proposition 4.8, page 117) If $\mathcal{R}$ is Cohen-Macaulay and $\mathcal{I}$ is in Nœther position, then* $\deg(\mathcal{I}, \prec) \leq d_1 + \cdots + d_{n-D} - (n-D) + 1$.

Proof. It follows from Cor. 19 that the maximal degree of the elements of the Gröbner basis of $\mathcal{I}$ is equal to that of $\mathcal{J} = \mathcal{I} + \langle x_{n-D+1}, \ldots, x_n\rangle$. On the other hand, $\mathcal{J}$ is a zero-dimensional ideal, and therefore, its Hilbert regularity is the maximal degree of the elements of the Gröbner basis of $\mathcal{J}$. By the proof of Proposition 20, $\mathrm{hilb}(\mathcal{J}) \leq d_1 + \cdots + d_{n-D} - (n-D) + 1$. □

In the case that $\mathcal{I}$ is a zero-dimensional ideal, we can derive explicit upper bounds for $\deg(\mathcal{I}, \prec)$ and $\dim_{\Bbbk}(\mathcal{R})$ using the following well-known result. For the reader's convenience, we include an elementary proof for it.

Theorem 22. *Let $\mathcal{I}$ be a zero-dimensional ideal. Then*

(a) $\deg(\mathcal{I}, \prec) \leq d_1 + \cdots + d_n - n + 1$,

(b) $\dim_{\Bbbk}(\mathcal{R}) \leq d_1 \cdots d_n$.

Proof. Since every zero-dimensional ideal is Cohen-Macaulay and in Nœther position then (a) is an immediate consequence of Theorem 21. We present now an elementary proof for (b). The assumption $\dim(\mathcal{I}) = 0$ implies that $\dim_{\Bbbk}(\mathcal{R})$ is equal to the sum of the coefficients of the Hilbert series of $\mathcal{I}$ (which is of course a polynomial here). We may assume w.l.o.g. that the first n generators $f_1, \ldots, f_n$ form a regular sequence (Lemma 17). Thus the Hilbert series of $\mathcal{I}' = \langle f_1, \ldots, f_n\rangle$ is $\mathrm{HS}_{\mathcal{I}'}(t) = \prod_{i=1}^{n}(1 + \cdots + t^{d_i-1})$ and $\dim_{\Bbbk}(\mathcal{P}/\mathcal{I}')$ is at most $\mathrm{HS}_{\mathcal{I}'}(1) = d_1 \cdots d_n$. We obviously have $\dim_{\Bbbk}(\mathcal{R}) \leq \dim_{\Bbbk}(\mathcal{P}/\mathcal{I}') = d_1 \cdots d_n$ and this proves the assertion. □

4. Lazard's upper bound

In this section, we investigate some properties of one-dimensional ideals in generic positions. In particular, we show that the Macaulay bound remains an upper bound for the degree of the reduced Gröbner basis of a one-dimensional ideal in generic position (Lazard, 1983). We provide simpler proofs for some of his results and show that the notion of genericity which is needed to prove this bound is Nœther position (see Theorem 30). Note that for Lazard, dimension was always the one as projective variety, whereas we use throughout this paper the one as affine variety which is one higher. In the sequel, we set $d_i = 1$ for any $i > k$. We first show that for a one-dimensional ideal, being in quasi stable position is equivalent to being in Nœther position.

Lemma 23. *Let* $\dim(\mathcal{I}) = 1$. *Then,* $\mathcal{I}$ *is in Nœther position, if and only if it is in quasi stable position.*

Proof. If $\mathcal{I}$ is in Nœther position then, from (Bermejo and Gimenez, 2001, Lemma 4.1), a pure power of x_i for each $i < n$ belongs to $\mathrm{LT}(\mathcal{I})$. Now, assume that $x_i^s \mid m$ for some $s \in \mathbb{N}$ and $m \in \mathrm{LT}(\mathcal{I})$. Then, we readily deduce that there exists t such that $x_j^t m/x_i^s \in \mathrm{LT}(\mathcal{I})$ for each $j < i \leq n$ and therefore $\mathcal{I}$ is in quasi stable position. Conversely, being in quasi stable position ensures that the ideal $\mathcal{I}$ is in Nœther position, see (Bermejo and Gimenez, 2006, Proposition 3.6). □

Let us first recall the concept of generic initial ideal. Let $A = (a_{ij}) \in \mathrm{GL}(n, \Bbbk)$ be an $n \times n$ invertible matrix. By $A.\mathcal{I}$ we mean the ideal generated by the polynomials $A.f$ with $f \in \mathcal{I}$ where

$$A.f = f(\sum_{i=1}^{n} a_{i1}x_i, \ldots, \sum_{i=1}^{n} a_{in}x_i).$$

The following fundamental theorem is due to Galligo (1974).

Theorem 24. *There exists a non-empty Zariski open subset* $\mathcal{U} \subset \mathrm{GL}(n, \Bbbk)$ *such that* $\mathrm{LT}(A.\mathcal{I}) = \mathrm{LT}(A'.\mathcal{I})$ *for all matrices* $A, A' \in \mathcal{U}$.

Definition 25. *The monomial ideal* $\mathrm{LT}(A.\mathcal{I})$ *with* $A \in \mathcal{U}$ *and* $\mathcal{U}$ *as given in Theorem 24 is called* the generic initial ideal *of* $\mathcal{I}$ *(w.r.t.* $\prec$*) and is denoted by* $\mathrm{gin}(\mathcal{I})$.

Theorem 26. *(Lazard, 1983, Theorem 2) Assume that* $\dim(\mathcal{I}) = 1$ *and* λ *is the depth of* $\mathcal{I}$. *Then we have* $\deg(\mathrm{gin}(\mathcal{I}), \prec) \leq d_1 + \cdots + d_r - r + 1$ *where* $r = n - \lambda$.

We showed in (Hashemi et al., 2012) that many properties of $\mathrm{gin}(\mathcal{I})$ also hold for $\mathrm{lt}(\mathcal{I})$ if $\mathcal{I}$ is in quasi stable position. Along these lines, we shall prove that in this theorem we can replace $\mathrm{gin}(\mathcal{I})$ by $\mathcal{I}$, if $\mathcal{I}$ is in Nœther position (or equivalently quasi stable position). For this, we need the next proposition due to Lazard, which was the key point in the proof of the above theorem.

Proposition 27. *(Lazard, 1981, Theorem 3.3) Assume again that* $\dim(\mathcal{I}) = 1$. *Then* $\dim_{\Bbbk}(\mathcal{R}_\ell) = \dim_{\Bbbk}(\mathcal{R}_{\ell+1})$ *for each* $\ell \geq d_1 + \cdots + d_n - n + 1$.

Thus, under the assumptions of this proposition, we can say that $\mathrm{hilb}(\mathcal{I}) \leq d_1 + \cdots + d_n - n + 1$. In addition, in (Lazard, 1981, Theorem 3.3), it was shown that there exists a linear polynomial $y \in \mathcal{P}$ such that the multiplication by y from $\mathcal{R}_{\ell-1}$ to R_ℓ for each $\ell \geq d_1 + \cdots + d_n - n + 1$ is bijective. We show that if $\mathcal{I}$ is in Nœther position, then we can choose $y = x_n$.

Proposition 28. *Assume that* $\dim(\mathcal{I}) = 1$ *and* $\mathcal{I}$ *is in Nœther position. Then, the multiplication* $x_n : \mathcal{R}_{\ell-1} \longrightarrow R_\ell$ *for each* $\ell \geq d_1 + \cdots + d_n - n + 1$ *is bijective.*

Proof. Since by (Cox et al., 2007, Proposition 4, page 458), $\mathcal{I}$ and $\mathrm{LT}(\mathcal{I})$ share the same Hilbert function then to prove the assertion one can replace $\mathcal{I}$ by $\mathrm{LT}(\mathcal{I})$. By reductio ad absurdum, suppose that there exists an integer ℓ such that $x_n m \in \mathrm{LT}(\mathcal{I})$ where m is a term with $m \notin \mathrm{LT}(\mathcal{I})$. Since $\mathcal{I}$ is in Nœther position, from (Bermejo and Gimenez, 2001, Lemma 4.1) it yields that $x_i^\ell \in \mathrm{LT}(\mathcal{I})$ for each $i < n$. Moreover, each term of degree ℓ in $x_1, \ldots, x_{n-1}$ belongs to $\mathrm{LT}(\mathcal{I})$. It follows that there exists an integer s so that for each term $u \in \mathcal{P}$ of degree s we have $um \in \mathrm{LT}(\mathcal{I})$. Assume that s is minimal for this property. Thus, there exists a term v of degree $s-1$ so that $vm \notin \mathrm{LT}(\mathcal{I})$. We conclude that $x_i vm \in \mathrm{LT}(\mathcal{I})$ for each i which is in contradiction with the existence of the linear bijection $y : \mathcal{R}_{\deg(m)+s-1} \longrightarrow R_{\deg(m)+s}$. ☐

As a consequence of this proof, we derive the next useful corollary.

Corollary 29. *Let* $\dim(\mathcal{I}) = 1$. *Then,* $\dim_{\Bbbk}(\mathcal{R}_\ell) \leq d_1 \cdots d_{n-1}$ *for each* $\ell \geq d_1 + \cdots + d_n - n + 1$.

Proof. Since the Hilbert function of an ideal remains invariant under any invertible linear change of variables, we may assume that $\mathcal{I}$ is in Nœther position. Furthermore, it is enough to prove the assertion for $\mathrm{LT}(\mathcal{I})$. From Macaulay's theorem (Cox et al., 2007, page 232), for each ℓ the set of terms $\{x_n^s m \mid s \in \mathbb{N}, \deg(m) + s = \ell, m \notin \mathrm{LT}(\mathcal{I}), m \in \Bbbk[x_1, \ldots, x_{n-1}]\}$ contains a basis for $\mathcal{R}_\ell$. On the other hand, from the assumption we know that $\mathcal{I}|_{x_n=0}$ is a zero-dimensional ideal and hence by Theorem 22 the set of terms $m \in \Bbbk[x_1, \ldots x_{n-1}]\}$ with $m \notin \mathrm{LT}(\mathcal{I})$ has at most $d_1 \cdots d_{n-1}$ elements. ☐

The Hilbert polynomial of a one-dimensional ideal is a constant polynomial. On the other hand, $(D-1)!$ times the leading coefficient of the Hilbert polynomial of an ideal is called the *degree* of the ideal, see (Cox et al., 2007, page 476). Thus, from this corollary it follows that the degree of a one-dimensional ideal $\mathcal{I}$ is bounded above by $d_1 \cdots d_{n-1}$.

Theorem 30. *Assume that* $\dim(\mathcal{I}) = 1$ *and* $\mathcal{I}$ *is in Nœther position. Then,* $\deg(\mathcal{I}, \prec) \leq d_1 + \cdots + d_r - r + 1$ *where* $r = n - \lambda$.

Proof. It suffices to show that $\deg(\mathcal{I}, \prec) \leq d_1 + \cdots + d_n - n + 1$, since then the desired inequality follows immediately from Proposition 35. As $\mathcal{I}$ is in quasi stable position, from (Hashemi, 2010, Theorem 4.17), (Seiler, 2012, Theorem 4.7) we have $\deg(\mathcal{I}, \prec) \leq \max\{\mathrm{hilb}(\mathcal{I}'), \mathrm{hilb}(\mathcal{I})\}$ where $\mathcal{I}' = (\mathcal{I} + \langle x_n \rangle) \cap \Bbbk[x_1, \ldots, x_{n-1}]$ is an ideal in the ring $\Bbbk[x_1, \ldots, x_{n-1}]$. Obviously, $\mathcal{I}'$ is generated by the polynomials $f_1|_{x_n=0}, \ldots, f_k|_{x_n=0}$. In addition, using the fact that $\mathcal{I}$ is in Nœther position we have $\dim(\mathcal{I}') = 0$. These arguments show that, by Props. 20 and 27, $\mathrm{hilb}(\mathcal{I}') \leq d_1 + \cdots + d_{n-1} - (n-1) + 1$ and $\mathrm{hilb}(\mathcal{I}) \leq d_1 + \cdots + d_n - n + 1$ which proves the assertion. ☐

Example 31. *Lazard (1983, Conj. 3) conjectured that the conclusion of Theorem 26 remained true, if one replaces* $\mathrm{gin}\,\mathcal{I}$ *by* $\mathcal{I}$. *Mora claimed that the following ideal (see the Appendix of (Lazard, 1983)) provided a counter-example. Consider the homogeneous ideal* $\mathcal{I} = \langle x_1 x_2^{t-1} - x_3^t, x_1^{t+1} - x_2 x_3^{t-1} x_4, x_1^t x_3 - x_2^t x_4 \rangle$ *in the polynomial ring* $\mathcal{P} = \Bbbk[x_1, \ldots, x_4]$. *Thus we have* $d_1 = t, d_2 = d_3 = t + 1$. *One can show that the polynomial* $x_3^{t^2+1} - x_2^{t^2} x_4$ *appears in the Gröbner basis of* $\mathcal{I}$ *and hence* $\deg(\mathcal{I}, \prec) \geq t^2 + 1$. *For simplicity we restrict to the case* $t = 4$ *where we obtain*

$$\mathrm{LT}(\mathcal{I}) = \langle x_1 x_2^3, x_1^4 x_3, x_1^5, x_1^3 x_3^5, x_1^2 x_3^9, x_1 x_3^{13}, x_3^{17} \rangle .$$

Thus we find here $\deg(\mathcal{I}, \prec) = 17 > d_1 + d_2 + d_3 - 3 = 11$. *But as* $\dim(\mathcal{I}) = 2$, $\mathcal{I}$ *does not yield a counter-example to Lazard's conjecture. However, if we consider* $\mathcal{I}' = \mathcal{I}|_{x_4=0} \subset \Bbbk[x_1, x_2, x_3]$, *then we find that $\mathcal{I}'$ has dimension 1 and that $\mathrm{LT}(\mathcal{I}')$ is generated by the same terms as $\mathrm{LT}(\mathcal{I})$. $\mathcal{I}'$ is not in Nœther position, as no pure power of x_2 belongs to $\mathrm{LT}(\mathcal{I}')$. Hence $\mathcal{I}'$ represents a counter-example to Lazard's conjecture. This example shows furthermore that in Theorem 30 it is not possible to drop the assumption of Nœther position.*

Remark 32. *We gave above a direct proof for Theorem 30. However, we can provide a more concise proof using Theorem 26 and Pommaret bases. Indeed, from Theorem 26 it follows that* $\mathrm{reg}(\mathcal{I}) \le d_1 + \cdots + d_r - r + 1$ *where $r = n - \lambda$, as by (Bayer and Stillman, 1987, Proposition 2.9),* $\mathrm{reg}(\mathcal{I})$ *is equal to the maximum degree of the minimal generating set of* $\mathrm{gin}(\mathcal{I})$*. Since the ideal $\mathcal{I}$ is in quasi stable position, it possesses a finite Pommaret basis $\mathcal{H}$ where* $\mathrm{reg}(\mathcal{I})$ *is the maximal degree of the elements of $\mathcal{H}$ and therefore* $\deg(\mathcal{I}, \prec) \le d_1 + \cdots + d_r - r + 1$*. These considerations also yield immediately the following corollary.*

Corollary 33. *If* $\dim(\mathcal{I}) = 1$*, then* $\mathrm{reg}(\mathcal{I}) \le d_1 + \cdots + d_r - r + 1$ *where* $r = n - \lambda$*.*

We conclude this section by presenting an affine version of Theorem 30. We drop now the assumption that the polynomials $f_1, \ldots, f_k$ generating $\mathcal{I}$ are homogeneous. Let x_{n+1} be an extra variable and $\tilde{f}$ the homogenization of f using x_{n+1}. We further denote by $\tilde{\mathcal{I}}$ the ideal generated by $\tilde{f}_1, \ldots, \tilde{f}_k$ (note that in general this is *not* equal to the homogenization of $\mathcal{I}$). Following (Fröberg, 1997, Def. 31, page 112), we consider a *good extension* $\prec_h$ of $\prec$ defined by the degree reverse lexicographic order with $x_{n+1} \prec x_n \prec \cdots \prec x_1$. The next proposition may be considered as a generalization of (Lazard, 1983, Theorem 2) to ideals in Nœther position.

Proposition 34. *Assume that $\tilde{\mathcal{I}}$ is in Nœther position, that* $\dim(\tilde{\mathcal{I}}) \le 1$ *and that* $\mathrm{depth}(\tilde{\mathcal{I}}) = \lambda$*. Then,* $\deg(\mathcal{I}, \prec) \le d_1 + \cdots + d_r - r + 1$ *where* $r = n + 1 - \lambda$*.*

Proof. Suppose that $\{g_1, \ldots, g_t\}$ is a Gröbner basis of $\tilde{\mathcal{I}}$ w.r.t. $\prec_h$. Then, from Theorem 30, we have $\deg(g_i) \le d_1 + \cdots + d_r - r + 1$ for each i. By (Fröberg, 1997, Proposition 34, page 113), we know that $\{g_1|_{x_{n+1}=1}, \ldots, g_t|_{x_{n+1}=1}\}$ forms a Gröbner basis for $\mathcal{I}$ w.r.t. $\prec$ which ends the proof. □

5. Improving Giusti's upper bound

Giusti (1984) established the upper bound $(2d)^{2^{n-1}}$ for $\deg(\mathcal{I}, \prec)$ in the case that the ideal is in generic position. The key point of Giusti's proof is the use of the combinatorial structure of the generic initial ideal in characteristic zero. Later on, Mora (2005, Ch. 38), by a deeper analysis of Giusti's proof, improved this bound to $(d+1)^{(n-D)2^{D-\lambda}}$ where λ is the depth of $\mathcal{I}$. In this section, we improve Mora's bound by following his general approach and correcting some flaws in his method. Our presentation seems to be simpler than the ones by Mora and Giusti.

We first note that for a given ideal in quasi stable position, we are able to reduce the number of variables by the depth of the ideal to obtain a sharper bound for $\deg(\mathcal{I}, \prec)$. A novel proof *à la Pommaret* of this result is given below.

Proposition 35. *Let $U(n, d, D)$ be a function depending on n, d and D such that* $\deg(\mathcal{I}, \prec) \le U(n, d, D)$ *for any ideal $\mathcal{I}$ which is in quasi stable position and is generated by homogeneous polynomials of degree at most d. Then,* $\deg(\mathcal{I}, \prec) \le U(n - \lambda, d, D - \lambda)$ *where* $\mathrm{depth}(\mathcal{I}) = \lambda$*.*

Proof. Let t be the maximal class of the elements in $\mathcal{H}$. It is shown in (Seiler, 2009, Prop 2.20) that in quasi-stable position the variables $x_{t+1}, \ldots, x_n$ define a regular sequence on $\mathcal{R}$ and that thus $\lambda = n - t$ (note that this reference distinguishes between $\mathrm{depth}(\mathcal{I})$ and $\mathrm{depth}(\mathcal{R})$ with the two related by $\text{depth}(\mathcal{R}) = \text{depth}(\mathcal{I}) - 1$; what we call here $\text{depth}(\mathcal{I})$ corresponds to $\text{depth}(\mathcal{R})$ in Seiler (2009)). By definition of t, no leading term of an element of $\mathcal{H}$ is divisible by any of these variables. Thus $\tilde{\mathcal{H}} = \mathcal{H}|_{x_{t+1}=\cdots=x_n=0}$ is the Pommaret basis of the ideal $\tilde{\mathcal{I}} = \mathcal{I}|_{x_{t+1}=\cdots=x_n=0}$ in $\mathbb{k}[x_1,\ldots,x_t]$ and hence $\deg(\mathcal{I},\prec) = \deg(\tilde{\mathcal{I}},\prec)$. This entails our claim. $\square$

As consequences of the above proof we obtain the following corollaries.

Corollary 36. *As a similar statement to Proposition 35, suppose that $R(n,d,D)$ is a function depending on n,d and D such that* $\text{reg}(\mathcal{I}) \leq R(n,d,D)$. *Then,* $\text{reg}(\mathcal{I}) \leq R(n-\lambda,d,D-\lambda)$.

Proof. The claim follows by the same argument as in the proof of Proposition 35 and using the facts that for each $f \in \mathcal{H}$, the corresponding element $\bar{f} \in \bar{\mathcal{H}}$ has the same degree as f and in quasi stable position $\text{reg}(\mathcal{I}) = \text{reg}(\bar{\mathcal{I}})$ is given by the maximal degree of the elements of $\mathcal{H}$ and $\bar{\mathcal{H}}$. $\square$

Corollary 37. *(Hashemi et al., 2012, Theorem 16) Let $\mathcal{I}$ be in quasi stable position. Then we have* $\text{depth}(\mathcal{I}) = \text{depth}(\text{LT}(\mathcal{I}))$.

To state the refined version of Giusti's bound, we need to recall the *crystallisation principle*. Suppose that $\mathcal{I} = \langle f_1,\ldots,f_k\rangle$ and that for some $s \in \mathbb{N}$ we have $\deg(f_i) \leq s$ for all i and $\text{gin}(\mathcal{I})$ has no minimal generator in degree $s+1$. Then, the crystallisation principle (CP) states that for each m in the generating set of $\text{gin}(\mathcal{I})$ we have $\deg(m) \leq s$, see (Green, 1998, Prop 2.28). Note that this principle holds only in characteristic zero and it has been proven only for generic initial ideals and for lexicographic ideals, see (Green, 1998, Theorem 3.8).

Giusti's approach to prove his degree upper bound consists of applying CP along with an induction on the number of variables. One crucial fact in his analysis is that CP also holds for a generic initial ideal modulo the last variable. Below, we will show that both these properties remain true for an arbitrary *strongly stable* ideal.

Definition 38. *A monomial ideal $\mathcal{J}$ is called* strongly stable*, if for any term $m \in \mathcal{J}$ we have $x_jm/x_i \in \mathcal{J}$ for all i and j such that $j < i$ and x_i divides m. A homogeneous ideal $\mathcal{I}$ is in* strongly stable position*, if* $\text{LT}(\mathcal{I})$ *is strongly stable.*

Proposition 39. *Let $\mathcal{I}$ be in strongly stable position. Then, CP holds for* $\text{LT}(\mathcal{I})$.

Proof. The following arguments are inspired by (Mora, 2005, page 728). Let us consider an integer $s \geq d$. Suppose that we are computing a Gröbner basis of $\mathcal{I}$ using Buchberger's algorithm and by applying the normal strategy. In addition, assume that we have already computed the set $G = \{g_1,\ldots,g_t\}$ up to degree s (this set will be enlarged to a Gröbner basis of $\mathcal{I}$), and there is no new polynomial of degree $s+1$ to be added into G. Note that we have chosen $s \geq d$ to be sure that G generates $\mathcal{I}$. To prove the assertion, it suffices to show that G is a Gröbner basis of $\mathcal{I}$.

We introduce the set $M_s = \langle \text{LT}(G)\rangle_s \cap \mathbb{T}$ where $\mathbb{T}$ is the set of all terms of $\mathcal{P}$. We now claim that for each pair of terms $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \neq x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$ in M_s either $\deg(\text{lcm}(x^\alpha,x^\beta)) = s+1$ or there exists a further term $x^\gamma \in M_s \setminus \{x^\alpha,x^\beta\}$ such that

- $x^\gamma \mid \text{lcm}(x^\alpha,x^\beta)$,
- $\deg(\text{lcm}(x^\gamma,x^\alpha)) < \deg(\text{lcm}(x^\alpha,x^\beta))$ and $\deg(\text{lcm}(x^\gamma,x^\beta)) < \deg(\text{lcm}(x^\alpha,x^\beta))$.

If this claim is true, then Buchberger's second criterion implies that it suffices to consider those pairs $\{g_i, g_j\}$ with the property $\deg(\mathrm{lcm}(\mathrm{LT}(g_i), \mathrm{LT}(g_j))) = s + 1$. If for each such pair the corresponding S-polynomial reduces to zero, then G is a Gröbner basis and we are done. Otherwise, there exists a new generator of degree $s + 1$ contradicting the made assumptions.

For proving the made claim, it suffices to show that, if we have $\deg(\mathrm{lcm}(x^\alpha, x^\beta)) > s + 1$, then there exists a term $x^\gamma \in M_s \setminus \{x^\alpha, x^\beta\}$ satisfying the above conditions. Let j be an integer such that $\alpha_j \neq \beta_j$ and $\alpha_{j+1} = \beta_{j+1}, \ldots, \alpha_n = \beta_n$. W.l.o.g., we may assume that $\alpha_j > \beta_j$. Since x^α and x^β have the same degree, there is an index $i < j$ such that $\beta_i > \alpha_i$. The strongly stable position of $\mathcal{I}$ implies that M_s is a strongly stable set. Therefore the term $x^\gamma = x_i x^\alpha / x_j$ satisfies $x^\gamma \in M_s \setminus \{x^\alpha, x^\beta\}$ and $x^\gamma \mid \mathrm{lcm}(x^\alpha, x^\beta)$. Furthermore, $\deg(\mathrm{lcm}(x^\gamma, x^\alpha)) = s+1 < \deg(\mathrm{lcm}(x^\alpha, x^\beta))$ and $\deg(\mathrm{lcm}(x^\gamma, x^\beta)) = \deg(\mathrm{lcm}(x^\alpha, x^\beta)) - 1$, proving the claim. □

Example 40. *We show that there exists an ideal $\mathcal{I}$ (due to Green (1998)) in strongly stable position such that* $\mathrm{LT}(\mathcal{I}) \neq \mathrm{gin}(\mathcal{I})$. *Let* $\mathcal{I} = \langle x_1x_3, x_1x_2 + x_2^2, x_1^2\rangle \subset \Bbbk[x_1, x_2, x_3]$. *Its leading term ideal* $\mathrm{LT}(\mathcal{I}) = \langle x_1x_3, x_1x_2, x_1^2, x_2^2x_3, x_2^3\rangle$ *is strongly stable, but we find* $\mathrm{gin}(\mathcal{I}) = \langle x_2^2, x_1x_2, x_1^2, x_1x_3^2\rangle \neq \mathrm{LT}(\mathcal{I})$. *Nevertheless, one observes that both* $\mathrm{LT}(\mathcal{I})$ *and* $\mathrm{gin}(\mathcal{I})$ *satisfy CP.*

As a consequence of the proof of this proposition, we can infer a generalization of CP.

Corollary 41. *Suppose we know in advance that $\mathcal{I}$ is in strongly stable position. Let us fix an integer t (not necessarily greater than d). Suppose that we are computing a Gröbner basis for $\mathcal{I}$ using Buchberger's algorithm and by applying the normal strategy. Assume that we have treated all S-polynomials of degree at most t and G_t is the set of all polynomials computed so far. If all S-polynomials of degree $t + 1$ reduce to zero, then any critical pair $\{f, g\}$ with* $\max\{\deg(f), \deg(g)\} \leq t$ *is superfluous. In particular, G_t remains a Gröbner basis for $\langle \mathcal{I}_{\leq t}\rangle$.*

In the sequel, for an index i we shall denote by $\mathcal{I}_i$ the ideal $\mathcal{I}|_{x_i = \cdots = x_n = 0} \subset \Bbbk[x_1, \ldots, x_{i-1}]$. Since we assume that $\prec$ is the degree reverse lexicographic term order, strongly stable position of $\mathcal{I}$ entails that for any index i, $\mathcal{I}_i$ is in strongly stable position, too. The essence of Giusti's approach consists in finding, by repeated evaluation, relations between $\deg(\mathcal{I}, \prec)$ and $\deg(\mathcal{I}_i, \prec)$ for $i = n, \ldots, n - D + 1$. For this purpose, we introduce some further notations for an ideal $\mathcal{I}$ in strongly stable position. We denote by $N(\mathcal{I})$ the set of all terms $m \notin \mathrm{LT}(\mathcal{I})$. If $\dim(\mathcal{I}) = 0$, then we define $F(\mathcal{I}) = N(\mathcal{I})$. Otherwise we set $F(\mathcal{I}) = \{\tau x_n^a \in N(\mathcal{I}) \mid \tau \in F(\mathcal{I}_n) \text{ and } \deg(\tau x_n^a) < \deg(\mathcal{I}, \prec)\}$. Since $\mathcal{I}$ is in strongly stable position, $N(\mathcal{I})$ is strongly stable for the reverse ordering of the variables. More precisely, if $x^\alpha \in N(\mathcal{I})$ with $\alpha_i > 0$, then we claim that $x_j x^\alpha / x_i \in N(\mathcal{I})$ for any $j > i$. Indeed, otherwise it belonged to $\mathrm{LT}(\mathcal{I})$ and thus – since $\mathrm{LT}(\mathcal{I})$ is strongly stable – $x^\alpha \in \mathrm{LT}(\mathcal{I})$ which is a contradiction. Below, $\#X$ denotes the cardinality of a finite set X.

Lemma 42. *Suppose that $\mathcal{I}$ is in strongly stable position. Then the following statements hold.*

(*a*) $\deg(\mathcal{I}, \prec) \leq \max\{d, \deg(\mathcal{I}_n, \prec)\} + \#F(\mathcal{I}_n)$,

(*b*) $\#F(\mathcal{I}) \leq \left(\max\{d, \#F(\mathcal{I}_n)\}\right)^2$.

Proof. (*a*) Let G be the reduced Gröbner basis of $\mathcal{I}$ for $\prec$. Because of our use of the degree reverse lexicographic term order, we easily see that $G|_{x_n=0}$ is the reduced Gröbner basis of $\mathcal{I}_n$ for $\prec$. Let $G' \subset G$ be the subset of all polynomials in G of maximal degree. We distinguish two cases. If $\mathrm{LT}(G') \cap \Bbbk[x_1, \ldots, x_{n-1}] \neq \emptyset$, then obviously $\deg(\mathcal{I}, \prec) = \deg(\mathcal{I}_n, \prec)$ and the assertion is proved. Otherwise, CP (applicable by Proposition 39) implies that for each degree $\max\{d, \deg(\mathcal{I}_n, \prec)\} < i \leq \deg(\mathcal{I}, \prec)$ there exists a polynomial $g_i \in G$ with $\deg(g_i) = i$ (note that if $\deg(\mathcal{I}, \prec) = d$ then (a) holds and we are done). Thus, we can write $\mathrm{LT}(g_i)$ in the form $x_n^{a_i}\tau_i$ with $a_i > 0$ and $\tau_i \in \Bbbk[x_1, \dots, x_{n-1}]$. We claim that $\tau_i \in F(\mathcal{I}_n)$. Writing $\tau_i = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}}$ where $i_1 < \cdots < i_k$, we may conclude by the assumed reducedness of G that $\tau_i \notin \mathrm{LT}(\mathcal{I})$ and by the strong stability of $\mathrm{LT}(\mathcal{I})$ that $x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}+a_i} \in \mathrm{LT}(\mathcal{I})$. Hence there exists an integer $a > 0$ such that $x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}+a-1} \notin \mathrm{LT}(\mathcal{I})$ and $x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}+a} \in \mathrm{LT}(\mathcal{I})$. It follows that there exists a generator $g \in G \cap \mathcal{I}_n$ such that its leading term $\mathrm{LT}(g) = x_{i_1}^{\beta_{i_1}} \cdots x_{i_k}^{\beta_{i_k}}$ divides the latter term. We must have $\beta_\ell \leq \alpha_\ell$ for each $\ell < i_k$ and $\beta_{i_k} = \alpha_{i_k} + a$ by the definition of a. Furthermore, the strong stability of $\mathrm{LT}(\mathcal{I})$ implies that $\deg(g) > \deg(\tau_i)$, as otherwise another generator $g' \in G$ would exist with $\mathrm{LT}(g') \mid \tau_i$. Thus $\deg(\tau_i) < \deg(\mathcal{I}_n, \prec)$. If we write $\tau_i = \bar{\tau}_i x_{i_k}^{\alpha_{i_k}}$, then there only remains to show that $\bar{\tau}_i \in F(\mathcal{I}_{i_k})$. Note that the membership $\tau_i \in N(\mathcal{I}_n)$ is a trivial consequence of $\tau_i \in N(\mathcal{I})$. If $\dim(\mathcal{I}_{i_k}) = 0$, the claim follows immediately from $F(\mathcal{I}_{i_k}) = N(\mathcal{I}_{i_k})$. Otherwise we repeat the same arguments as above.

Thus for each i with $\max\{d, \deg(\mathcal{I}_n, \prec)\} < i \leq \deg(\mathcal{I}, \prec)$ there exists a generator $g_i \in G$ such that $\mathrm{LT}(g_i) = x_n^{a_i}\tau_i$ and $\tau_i \in F(\mathcal{I}_n)$. Since G is reduced, the terms τ_i are pairwise different. Hence $\deg(\mathcal{I}, \prec) - \max\{d, \deg(\mathcal{I}_n, \prec)\} \leq \#F(\mathcal{I}_n)$ and this proves (a).

To show (b), we use the proof of (a) in which, to apply CP, we should consider the polynomials of degree at least d. Thus in the sequel we shall replace $\#F(\mathcal{I}_n)$ by $\max\{d, \#F(\mathcal{I}_n)\}$. Let us introduce for each degree $\delta \in \mathbb{N}$ the subset $F_\delta(\mathcal{I}) = \{x_n^\delta\tau \mid x_n^\delta\tau \in F(\mathcal{I})\} \subset F(\mathcal{I})$. By definition, $x_n^\delta\tau \in F_\delta(\mathcal{I})$ implies $\tau \in F(\mathcal{I}_n)$ and thus $\#F_\delta(\mathcal{I}) \leq \#F(\mathcal{I}_n)$. On the other hand, from $\deg(\mathcal{I}, \prec) - \max\{d, \deg(\mathcal{I}_n, \prec)\} \leq \max\{d, \#F(\mathcal{I}_n)\}$ we deduce that the maximal δ such that $x_n^\delta\tau \in F(\mathcal{I})$ is $\max\{d, \#F(\mathcal{I}_n)\}$ and thus

$$\#F(\mathcal{I}) \leq \sum_{\delta=0}^{\max\{d,\#F(\mathcal{I}_n)\}-1} \max\{d, \#F(\mathcal{I}_n)\}$$

which immediately yields the inequality in (b). $\square$

Remark 43. *Mora (2005, Theorem 38.2.7) presented another version of this lemma. Instead of our set $F(\mathcal{I})$, he defined $\tilde{F}(\mathcal{I}) = \{\tau x_n^a \in N(\mathcal{I}) \mid \tau \in N(\mathcal{I}_n),\ \deg(\tau x_n^a) < \deg(\mathcal{I}, \prec)\}$ which differs only in the condition on τ. Assuming the equality $\tilde{F}_0(\mathcal{I}) = \tilde{F}(\mathcal{I}_n)$ where $\tilde{F}_0(\mathcal{I})$ contains the elements of $\tilde{F}(\mathcal{I})$ with $a = 0$, he proved the following two properties:*

(a) $\deg(\mathcal{I}, \prec) \leq \deg(\mathcal{I}_n, \prec) + \#\tilde{F}(\mathcal{I}_n)$,

(b) $\#\tilde{F}(\mathcal{I}) \leq (\#\tilde{F}(\mathcal{I}_n))^2$.

However, in general these assertions are not correct – not even for an ideal in generic position. Indeed, in general we have only $\tilde{F}(\mathcal{I}_n) \subsetneq \tilde{F}_0(\mathcal{I})$ and if $\dim(\mathcal{I}) > 0$ and $\deg(\mathcal{I}_1 \prec) < \deg(\mathcal{I}, \prec)$ then equality does not hold. As a concrete example consider $\mathcal{I} = \langle x_1^2, x_2^{11}x_1\rangle \subset \Bbbk[x_1, x_2]$. We perform a generic linear change $x_1 = ay_1 + by_2$ and $x_2 = cy_1 + dy_2$ with parameters $a, b, c, d \in \Bbbk$. The leading term ideal of the new ideal is then $\langle y_1^2, y_2^{11}y_1\rangle$. This show that $\mathcal{I} = \mathrm{gin}(\mathcal{I})$ and therefore the original coordinates for $\mathcal{I}$ are already generic. We have $\mathcal{I}_2 = \langle x_1^2\rangle$, $\tilde{F}(\mathcal{I}_2) = \{1, x_1\}$ and $\deg(\mathcal{I}_2, \prec) = 2$. Furthermore, we have $\tilde{F}(\mathcal{I}) = \{x_2^{11}\} \cup \{x_2^i, x_2^i x_1 \mid i = 0, \dots, 10\}$ and $\#\tilde{F}(\mathcal{I}) = 23$. Thus, $12 = \deg(\mathcal{I}, \prec) \nleq \deg(\mathcal{I}_n, \prec) + \#\tilde{F}(\mathcal{I}_n) = 2 + 2 = 4$ and $23 = \#\tilde{F}(\mathcal{I}) \nleq (\#\tilde{F}(\mathcal{I}_n))^2 = 4$.

We state now the main result of this section.

Theorem 44. *If the ideal $\mathcal{I}$ is in strongly stable position, then $\#F(\mathcal{I}) \leq d^{(n-D)2^D}$ and we have*

$$\deg(\mathcal{I}, \prec) \leq \max\left\{(n - D + 1)(d - 1) + 1, 2d^{(n-D)2^{D-1}}\right\}.$$

Proof. We proceed by induction over $D = \dim(\mathcal{I})$. In this proof without loss of generality, we may assume that $d \geq 2$. If $D = 0$, the assertions follow immediately from Theorem 22. For $D > 0$, we exploit that $\dim(\mathcal{I}) = \dim(\mathcal{I}_n) + 1$ and we may consider $\mathcal{I}_n$ as an ideal in $\Bbbk[x_1, \ldots, x_{n-1}]$. Lemma 42 now entails that

$$\begin{aligned}\#F(\mathcal{I}) &\leq \max\{d, \#F(\mathcal{I}_n)\}^2 \\ &\leq (d^{(n-1-(D-1))2^{D-1}})^2 = d^{(n-D)2^D}\end{aligned}$$

and thus the first inequality holds. For the second inequality, Thms. 22 and 30 immediately imply our claim for $D \leq 1$. For $D \geq 2$, we obviously have $(n-1-(D-1)+1)(d-1)+1 \leq d^{(n-D)2^{D-1}}$ and $2d^{(n-1-(D-1))2^{D-2}} \leq d^{(n-1-(D-1))2^{D-1}}$. We can thus rewrite the induction hypothesis as

$$\begin{aligned}\deg(\mathcal{I}_n, \prec) \leq \max\{&(n-1-(D-1)+1)(d-1)+1, \\ &2d^{(n-1-(D-1))2^{D-2}}\} \leq d^{(n-D)2^{D-1}}.\end{aligned}$$

Again by Lemma 42, we can also estimate

$$\begin{aligned}\deg(\mathcal{I}, \prec) &\leq \max\{d, \deg(\mathcal{I}_n, \prec)\} + \#F(\mathcal{I}_n) \\ &\leq d^{(n-D)2^{D-1}} + d^{(n-1-(D-1))2^{D-1}} \\ &= 2d^{(n-D)2^{D-1}}\end{aligned}$$

proving the second assertion. □

Example 45. *Let us consider the values $n = 2, d = 2$ and $D = 0$. The above theorem states* $\deg(\mathcal{I}, \prec) \leq 4$. *Consider the ideal $\mathcal{I} = \langle x_1^2, x_1x_2 + x_2^2\rangle$. By performing a generic linear change of coordinates, we get* $\operatorname{gin}(\mathcal{I}) = \langle x_2x_1, x_1^2, x_2^3\rangle$. *Therefore $\#F(\mathcal{I}) = 4 \leq 4$ and* $\deg(\mathcal{I}, \prec) = 3 \leq 4$ *confirming the accuracy of the presented upper bounds. It should be noted that for such a zero-dimensional ideal Theorem 22 provides the best upper bound for* $\deg(\mathcal{I}, \prec)$*, namely $d_1 + \cdots + d_n - n + 1$ which is equal to the exact value 3 for this example.*

Using Proposition 35, we obtain even sharper bounds depending on both the dimension and the depth of $\mathcal{I}$. We continue to write $\dim(\mathcal{I}) = D$ and $\operatorname{depth}(\mathcal{I}) = \lambda$. For the rest of this paper, we assume that $\mathcal{R}$ is not Cohen-Macaulay, i. e. that $D > \lambda$. Note that if $\mathcal{R}$ is Cohen-Macaulay, then Theorem 21 presents a sharp upper bound for $\deg(\mathcal{I}, \prec)$ in generic position.

Corollary 46. *If $\mathcal{I}$ is in strongly stable position and $D > 1$, then $\#F(\mathcal{I}) \leq d^{(n-D)2^{D-\lambda-1}}$ and* $\deg(\mathcal{I}, \prec) \leq 2d^{(n-D)2^{D-\lambda-1}}$.

The maximal degree of an element of the Pommaret basis of an ideal in quasi stable position equals the Castelnuovo-Mumford regularity (Seiler, 2009, Theorem 9.2). If the ideal is even in stable position, then the Pommaret basis coincides with the reduced Gröbner basis (Mall, 1998, Theorem 2.15). These considerations imply now immediately the following two results.

Corollary 47. *If the ideal $\mathcal{I}$ is in strongly stable position and $D > 1$, then* $\operatorname{reg}(\mathcal{I}) \leq 2d^{(n-D)2^{D-\lambda-1}}$.

Corollary 48. *Let the ideal $\mathcal{I}$ be in quasi stable position, $\mathcal{H}$ its Pommaret basis and $D > 1$. If we write* $\deg(\mathcal{H})$ *for the maximal degree of an element of $\mathcal{H}$, then* $\deg(\mathcal{I}, \prec) \leq \deg(\mathcal{H}) \leq 2d^{(n-D)2^{D-\lambda-1}}$.

6. Improving the upper bound of Caviglia-Sbarra

Caviglia and Sbarra (2005) gave a simple proof for the upper bound $(2d)^{2^{n-2}}$ for $\deg(\mathcal{I}, \prec)$ when the coordinates are in generic position by analyzing Giusti's proof and exploiting some properties of quasi stable ideals. We will now improve this bound to a dimension and depth dependent bound. As a by-product, we will show that the notion of genericity that one needs here is strongly stable position. We end this section by showing that a similar upper bound holds in positive characteristic, too.

We begin with a quick review of the approach presented in (Caviglia and Sbarra, 2005). For any monomial ideal $\mathcal{J} \subset \mathcal{P}$ let $G(\mathcal{J})$ be its unique minimal generating set. We write $\deg_i(\mathcal{J}) = \max\{\deg_i(u) \mid u \in G(\mathcal{J})\}$ where $\deg_i(u)$ denotes the degree in the variable x_i of u. Slightly changing our previous notation, we now denote by $\mathcal{J}_i$ the ideal $\mathcal{J}|_{x_{i+1}=\cdots=x_n=0} \subset \Bbbk[x_1, \ldots, x_i]$. It follows immediately that for any quasi stable monomial ideal $\mathcal{J}$ we have $\deg_i(\mathcal{J}_i) = \deg_i(\mathcal{J})$, see (Caviglia and Sbarra, 2005, Lemma 1.5). We note that two distinct terms in $G(\mathcal{J})$ must differ already in the first $n-1$ variables because of the minimality of $G(\mathcal{J})$. Hence $\#G(\mathcal{J}) \leq \prod_{i=1}^{n-1}(\deg_i(\mathcal{J}) + 1)$.

Assume that $\mathcal{I}$ is in quasi stable position and satisfies CP w.r.t. d. CP implies that $\deg(\mathcal{I}, \prec) - d + 1 \leq \#G(\mathrm{LT}(\mathcal{I}))$ and hence $\deg(\mathcal{I}, \prec) \leq d - 1 + \prod_{i=1}^{n-1}(\deg_i(\mathrm{LT}(\mathcal{I})) + 1)$. Quasi stability of $\mathrm{LT}(\mathcal{I})$ implies that $\deg_i(\mathrm{LT}(\mathcal{I})) = \deg(\mathcal{I}_i, \prec)$ and thereby $\deg(\mathcal{I}, \prec) \leq d - 1 + \prod_{i=1}^{n-1}(\deg(\mathcal{I}_i, \prec) + 1)$.

Set $B_1 = d$ and for $i \geq 2$ recursively $B_i = d - 1 + \prod_{j=1}^{i-1}(B_j + 1)$. If we assume that for each index $1 \leq i < n$ the reduced ideal $\mathcal{I}_i$ satisfies CP w.r.t. d, then by the considerations above $\deg(\mathcal{I}_i, \prec) \leq B_i$. In particular, $B_2 = 2d$ and $\deg(\mathcal{I}, \prec) \leq B_n$. One easily sees that the B_i satisfy the recursion relation $B_i = d - 1 + (B_{i-1} + 1)(B_{i-1} - d + 1) = B_{i-1}^2 - (d-2)B_{i-1}$ for all $i \geq 2$. Since we may suppose that $d \geq 2$, we have $B_i \leq B_{i-1}^2$. Thus, for all $i \geq 2$ we have $B_i \leq (2d)^{2^{i-2}}$ and therefore $B_n = \deg(\mathcal{I}, \prec) \leq (2d)^{2^{n-2}}$. We summarize the above discussion in the next theorem.

Theorem 49. *(Caviglia and Sbarra, 2005) Suppose that $\mathcal{I}$ is in quasi stable position and that the ideals $\mathcal{I}_1, \ldots, \mathcal{I}_{n-1}, \mathcal{I}$ satisfy CP w.r.t. d. Then* $\deg(\mathcal{I}, \prec) \leq \mathrm{reg}(\mathcal{I}) \leq (2d)^{2^{n-2}}$.

Proof. Since $\mathrm{reg}(\mathcal{I})$ equals the maximal degree of an element of the Pommaret basis of $\mathcal{I}$, $\deg(\mathcal{I}, \prec) \leq \mathrm{reg}(\mathcal{I})$. So, we must only show the second inequality. As the regularity remains invariant under linear coordinate transformations, we may w.l.o.g. assume that $\mathcal{I}$ is even in strongly stable position. Then, by (Mall, 1998, Theorem 2.15), we have $\deg(\mathcal{I}, \prec) = \mathrm{reg}(\mathcal{I})$. In addition, Proposition 39 entails that also $\mathcal{I}_1, \ldots, \mathcal{I}_{n-1}, \mathcal{I}$ satisfy CP w.r.t. d. Now the assertion follows from the consideration above. □

In the following, we derive a dimension dependent upper bound for $\deg(\mathcal{I}, \prec)$.

Theorem 50. *Suppose that $\mathcal{I}$ is in strongly stable position and $D = \dim(\mathcal{I}) \geq 2$. Then*

$$\deg(\mathcal{I}, \prec) = \mathrm{reg}(\mathcal{I}) \leq \left(d^{n-D}((n-D+1)(d-1)+2) + (n-D)(d-1)\right)^{2^{D-2}}.$$

Proof. The equality follows from (Mall, 1998, Theorem 2.15). Since $\mathcal{I}$ is in strongly stable position, $\mathcal{I}_{n-D} \subset \Bbbk[x_1, \ldots, x_{n-D}]$ is zero-dimensional (Seiler, 2009, Prop 3.15). According to Theorem 22, $\deg(\mathcal{I}_{n-D}, \prec) \leq (n-D)(d-1) + 1$. Hence the maximal degree of a term in $G(\mathrm{LT}(\mathcal{I}))$ which depends only on $x_1, \ldots, x_{n-D}$ is at most this bound. We shall now construct an upper bound for the degree of the terms in $G(\mathrm{LT}(\mathcal{I}))$ containing at least one of the remaining variables $x_{n-D+1}, \ldots, x_n$. Following the approach by Caviglia and Sbarra, we first look for an upper bound for the number of these terms.

Consider a term $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in G(\mathrm{LT}(\mathcal{I}))$ with $\alpha_i > 0$ for some $i \geq n-D+1$. It is clear that $x_1^{\alpha_1} \cdots x_{n-D}^{\alpha_{n-D}}$ belongs to the complement of $\mathrm{LT}(\mathcal{I}_{n-D})$. Since the ideal $\mathcal{I}_{n-D} \subset \Bbbk[x_1,\dots,x_{n-D}]$ is zero-dimensional, Theorem 22 entails that

$$\dim_{\Bbbk}(\Bbbk[x_1,\dots,x_{n-D}]/\mathcal{I}_{n-D}) \leq d^{n-D}.$$

Hence the number of terms $x_1^{\alpha_1} \cdots x_{n-D}^{\alpha_{n-D}}$ is at most d^{n-D}. On the other hand, for any index $n-D+1 \leq i \leq n$ we have $\alpha_i \leq \deg_i(\mathrm{LT}(\mathcal{I})) \leq \deg(\mathcal{I}_i, \prec)$. Furthermore, we know that two distinct term in $G(\mathrm{LT}(\mathcal{I}))$ differ already in their first $n-1$ variables. These arguments imply that the number of terms in $G(\mathrm{LT}(\mathcal{I}))$ containing at least one of the variables $x_{n-D+1},\dots,x_n$ is at most $d^{n-D}\prod_{i=n-D+1}^{n-1}(\deg(\mathcal{I}_i,\prec)+1)$.

The strongly stability of $\mathcal{I}$ implies that CP holds for $\mathrm{LT}(\mathcal{I})$ w.r.t. $(n-D)(d-1)+1 \geq d$ by Proposition 39. Hence $\deg(\mathcal{I},\prec) - ((n-D)(d-1)+1)+1$ must be less than or equal to the number of terms in $G(\mathrm{LT}(\mathcal{I}))$ containing at least one of the variables $x_{n-D+1},\dots,x_n$ leading to the estimate

$$\deg(\mathcal{I},\prec) \leq d^{n-D}\prod_{i=n-D+1}^{n-1}(\deg(\mathcal{I}_i,\prec)+1) + (n-D)(d-1)\,.$$

Since $\mathcal{I}_{n-D+1}$ is a one-dimensional ideal then from Theorem 30, it follows that $\deg(\mathcal{I}_{n-D+1},\prec) \leq (n-D+1)(d-1)+1$. Thus, we can write

$$\deg(\mathcal{I},\prec) \leq d^{n-D}((n-D+1)(d-1)+2)\prod_{i=n-D+2}^{n-1}(\deg(\mathcal{I}_i,\prec)+1) + (n-D)(d-1)\,.$$

Set $B_{n-D+1} = d^{n-D}((n-D+1)(d-1)+2) + (n-D)(d-1)$ and recursively

$$B_j = d^{n-D}((n-D+1)(d-1)+2)\prod_{i=n-D+2}^{j-1}(B_i+1) + (n-D)(d-1)$$

for $n-D+3 \leq j \leq n$. One easily verifies that these numbers satisfy the recursion relation $B_j = (B_{j-1}-(n-D)(d-1))(B_{j-1}+1)+(n-D)(d-1) = B_{j-1}^2 - ((n-D)(d-1)-1)B_{j-1}$. We may again assume that $d \geq 2$, and therefore $B_j \leq B_{j-1}^2$ for $n-D+3 \leq j \leq n$. This implies that $B_j \leq (d^{n-D}((n-D+1)(d-1)+2)+(n-D)(d-1))^{2^{j-n+D-2}}$ and in particular we have $B_n \leq (d^{n-D}((n-D+1)(d-1)+2)+(n-D)(d-1))^{2^{D-2}}$. □

Remark 51. *It should be noticed that in the case that $D \leq 1$, Thms. 22 and 30 present sharp upper bounds for $\deg(\mathcal{I},\prec)$ and therefore in all of our improved bound we can ignore these two cases. Now, let us compare the dimension dependent bounds $A(n,d,D) = 2d^{(n-D)2^{D-1}}$ derived in Theorem 44 and $B(n,d,D) = 2(1/2d^{n-D}+d)^{2^{D-1}}$ due to Mayr and Ritscher (2013) with $C(n,d,D) = (d^{n-D}((n-D+1)(d-1)+2)+(n-D)(d-1))^{2^{D-2}}$ obtained now. Obviously, all three bounds describe essentially the same qualitative behavior, although they are derived with fairly different approaches. However, the bound $C(n,d,D)$ for $D \geq 2$ has always the best constants. We claim that for infinite values of n,d,D we have $C(n,d,D) < B(n,d,D) < A(n,d,D)$. To illustrate these relations, let us take $n-D=2$ and $D=2$. Then, $A(n,d,D)=2d^4$, $B(n,d,D)=2(d^2/2+d)^2$ and $C(n,d,D) = 3d^3-d^2+2d-2$. One observes readily that the dominant term of $C(n,d,D)$ has smallest degree and so it is the sharpest upper bound. In the case of a hypersurface, i.e. for $D=n-1$, even the bound $A(n,d,D)$ is always better than $B(n,d,D)$, as its constant is always 2 versus $2(3/2)^{2^{n-2}}$ for the Mayr-Ritscher bound.*

Again an application of Proposition 35 yields immediately the following improved bound depending on both the depth and the dimension of $\mathcal{I}$.

Corollary 52. *Under the assumptions of Theorem 50, one has*

$$\deg(\mathcal{I}, \prec) = \operatorname{reg}(\mathcal{I}) \leq (d^{n-D}((n-D+1)(d-1)+2) + (n-D)(d-1))^{2^{D-\lambda-2}}.$$

We conclude the paper by presenting an upper bound for $\operatorname{reg}(\mathcal{I})$ when the characteristic of $\Bbbk$ is positive. It should be noted that in this case, it is not always possible to achieve strongly stable position by linear coordinate transformations (see (Hashemi et al., 2017) for a more detailed discussion). Below, by using Lazard's upper bound, we improve the upper bound stated in (Caviglia and Sbarra, 2005, Cor. 2.6). For this purpose, we adopt some notations and definitions from loc. id. and give a simpler proof to Theorem 2.4 in that paper.

In the rest of the paper, we assume that the characteristic of $\Bbbk$ is positive. According to the flat base change property of local cohomology, we may assume that the size of $\Bbbk$ is infinite. Then, given an ideal $\mathcal{I}$, by (Hashemi et al., 2017, Alg. 3), there exists a linear coordinate transformation Ψ so that $\Psi(\mathcal{I})$ is in quasi stable position. Therefore, to obtain an upper bound for $\operatorname{reg}(\mathcal{I})$, w.l.o.g. we may assume in the sequel that $\mathcal{I}$ is in quasi stable position, see Rem. 7.

Recall that a sequence of non-constant homogeneous polynomials $g_1, \ldots, g_t \in \mathcal{P}$ is called *almost regular* for $\mathcal{R}$ if g_i for $i = 2, \ldots, t$ is a non-zero divisor on the ring $\mathcal{P}/(\mathcal{I} + \langle g_1, \ldots, g_{i-1} \rangle)_{\geq \ell}$ for some ℓ sufficiently large. This is equivalent to the condition that g_i does not belong to any associated prime of $I + \langle g_1, \ldots, g_{i-1} \rangle$ except the maximal homogeneous ideal of $\mathcal{P}$. Since $\mathcal{I}$ is in quasi stable position then by (Seiler, 2009, Proposition 4.4) we conclude that the sequence $x_{n-D+1}, \ldots, x_n$ is almost regular on $\mathcal{R}$. This observations allows us that in the next lemma we apply this sequence instead of a sequence of generic linear forms and this leads to a simpler proof than the one given by Caviglia and Sbarra. Below, we let $\lambda(M)$ be the *length* of an $\mathcal{P}$-module M; i.e. the largest integer m so that there exists a chain $N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_m$ of sub-modules of M. If no such largest length exists, then M has infinite length. In the case that $\lambda(M)$ is finite, one can use Hilbert series to compute it. For the special case that we use below, when $\mathcal{J}$ is a monomial ideal, $\lambda((\mathcal{J} : x_n)/\mathcal{J})$ is finite and it is equal to the value of $\mathrm{HS}_{\mathcal{J}}(t) - \mathrm{HS}_{\mathcal{J}:x_n}(t)$ at $t = 1$. For example, let $\mathcal{J} = \langle x_1^4, x_1 x_2^{10} \rangle \subset \Bbbk[x_1, x_2]$. Then, $\mathrm{HS}_{\mathcal{J}}(t) - \mathrm{HS}_{\mathcal{J}:x_n}(t) = t^{10} + t^{11} + t^{12}$ and therefore $\lambda((\mathcal{J} : x_n)/\mathcal{J}) = 3$. One observes that for this special case, we have $\lambda((\mathcal{J} : x_n)/\mathcal{J}) \leq \prod_{i=1}^{n-1}(\deg(\mathcal{J}_i)) = 4$.

Lemma 53. *(Caviglia and Sbarra, 2005, Theorem 2.4) Under the above assumptions, if $D > 0$ then* $\operatorname{reg}(\mathcal{I}) \leq \max\{d, \operatorname{reg}(\mathcal{I}_{n-1})\} + d^{n-D} \prod_{i=n-D+1}^{n-1} \operatorname{reg}(\mathcal{I}_i)$.

Proof. In their proof, Caviglia and Sbarra proved $\operatorname{reg}(\mathcal{I}) \leq \max\{d, \operatorname{reg}(\mathcal{I}_{n-1})\} + \lambda((\mathcal{I} : x_n)/\mathcal{I})$. Thus, it suffices to show that $\lambda((\mathcal{I} : x_n)/\mathcal{I}) \leq d^{n-D} \prod_{i=n-D+1}^{n-1} \operatorname{reg}(\mathcal{I}_i)$. Since the ideals $\mathcal{I}$ and $\mathcal{I}_i$ for each i are in quasi stable position, using (Seiler, 2009, Cor. 9.5), we can replace $\mathcal{I}$ by $\mathcal{J} = \mathrm{LT}(\mathcal{I})$. It is clear that $\lambda((\mathcal{J} : x_n)/\mathcal{J})$ is equal to the number of terms $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \notin \mathcal{J}$ so that $m x_n \in \mathcal{J}$. It follows that $x_1^{\alpha_1} \cdots x_{n-D}^{\alpha_{n-D}}$ belongs to the complement of $\mathcal{J}$. Since the ideal $\mathcal{I}_{n-D} \subset \Bbbk[x_1, \ldots, x_{n-D}]$ is zero-dimensional, we get from Theorem 22

$$\dim_{\Bbbk}(\Bbbk[x_1, \ldots, x_{n-D}]/\mathcal{I}_{n-D}) \leq d^{n-D}$$

and the number of choices for $x_1^{\alpha_1} \cdots x_{n-D}^{\alpha_{n-D}}$ is at most d^{n-D}. Hence, a repetition of the arguments in the proof of Theorem 50 yields $\lambda((\mathcal{J} : x_n)/\mathcal{J}) \leq d^{n-D} \prod_{i=n-D+1}^{n-1} \deg(\mathcal{I}_i, \prec) \leq d^{n-D} \prod_{i=n-D+1}^{n-1} \operatorname{reg}(\mathcal{I}_i)$ and this completes the proof of the lemma. $\square$

In the case $D \leq 1$, Theorem 22 and Cor. 33 provide sharp upper bounds for $\operatorname{reg}(\mathcal{I})$ and therefore we will ignore this case in the sequel. The next theorem improves the upper bound $\operatorname{reg}(\mathcal{I}) \leq (d^{n-D} + (n-D)(d-1)+1)^{2^{D-1}}$ presented in (Caviglia and Sbarra, 2005, Cor. 2.6).

Theorem 54. *Under the above assumptions, if* $D > 1$ *then*

$$\operatorname{reg}(\mathcal{I}) \leq (d^{n-D}((n-D+1)(d-1)+1) + (n-D)(d-1)+1)^{2^{D-2}}.$$

Proof. Our proof follows similar ideas as the proof of (Caviglia and Sbarra, 2005, Cor. 2.6). We first observe that $I_{n-D+1} \subset \Bbbk[x_1, \ldots, x_{n-D+1}]$ is one-dimensional and therefore by Cor. 33, $\operatorname{reg}(\mathcal{I}_{n-D+1}) \leq (n-D+1)(d-1)+1$ which is indeed equal to B_0. Now, set $B_1 = d^{n-D}((n-D+1)(d-1)+1) + (n-D)(d-1)+1)$ and recursively

$$B_j = B_{j-1} + \prod_{i=1}^{j-1} B_i$$

for $2 \leq j \leq D-1$. It is obvious that $B_j = B_{j-1} + (B_{j-1} - B_{j-2})B_{j-1} \leq B_{j-1}^2$ and $B_j \leq B_1^{2^{j-1}}$. On the other hand, by virtue of Lemma 53, we have $\operatorname{reg}(\mathcal{I}) \leq B_{D-1}$ and therefore $\operatorname{reg}(\mathcal{I}) \leq B_1^{2^{D-2}}$, as desired. □

Applying Proposition 35 yields the following improved bound depending on both the depth and the dimension of $\mathcal{I}$, which also settles the conjecture (Hashemi and Seiler, 2017, Con. 4.5).

Corollary 55. *Under the assumptions of Theorem 54,*

$$\operatorname{reg}(\mathcal{I}) \leq (d^{n-D}((n-D+1)(d-1)+1) + (n-D)(d-1)+1)^{2^{D-\lambda-2}}.$$

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