

Deterministic Genericity for Polynomial Ideals

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Abstract

We consider several notions of genericity appearing in algebraic geometry and commutative algebra. Special emphasis is put on various stability notions which are defined in a combinatorial manner and for which a number of equivalent algebraic characterisations are provided. It is shown that in characteristic zero the corresponding generic positions can be obtained with a simple deterministic algorithm. In positive characteristic, only adapted stable positions are reachable except for quasi-stability which is obtainable in any characteristic.

1. Introduction

Genericity appears in many places in algebraic geometry and commutative algebra, as many results considerably simplify, if one assumes that the considered ideal is in a sufficiently generic position. While genericity is well studied theoretically, its algorithmic side has been treated much less. There are two natural questions related to a generic position. To apply the corresponding theoretical results in a concrete computation, one must firstly be able to verify effectively whether a given ideal is in the considered generic position. If this is not the case, one would secondly like to find a (preferably sparse) linear transformation into generic position.

From a theoretical point of view, the second goal is easily achieved by applying a random transformation. In practise, this will destroy all sparsity typically present in problems of interest. Therefore we will study here deterministic algorithms that give us a reasonable chance to render a position generic with a fairly sparse transformation. We make no claims of getting an optimal solution for this problem. In one of the very few articles dealing with such questions, Eisenbud and Sturmfels (1994) argue that different notions of optimality exist. Furthermore, they showed that various related problems are NP-complete.

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Our main emphasis is on generic positions related to Gröbner bases where the leading ideal exhibits certain favourable properties. The most famous generic position here is of course the one where the leading ideal is the generic initial ideal. However, both above mentioned problems are very hard for this position and all computer algebra implementations we know of use random transformations without a check that they have really obtained the generic initial ideal.

One of the main points of this work lies in studying generic positions – mainly of a combinatorial nature and related to stability – that share as many properties with the generic initial ideal as possible but which are still effectively verifiable and constructable. We will present an algorithm to achieve deterministically any stable position via a sequence of elementary moves. The basic idea is due to Hausdorf and Seiler (2002) in the context of differential equations (the original proof contained a gap and was later corrected by Seiler (2009b)). There only the case of quasi-stability was considered and the required moves were selected via a comparison of Pommaret and Janet multiplicative variables. Here we present now a version where the selection criterion is directly based on the combinatorial characterisation of the various stability notions and which is therefore no longer restricted to quasi-stability. While the algorithm itself is very simple, the termination proof is rather long and technical.

In practise, we face here a conflict: the closer we get to the generic initial ideal, the harder it becomes to obtain deterministically the corresponding generic position (meaning the more coordinate transformations are typically needed). This observation explains why we consider so many different kinds of stable positions. They allow us to make a trade-off: we go for a generic position that is just enough for the intended application. One possible application is given by the computation of many fundamental invariants like the depth, the Castelnuovo-Mumford regularity or the reduction number which becomes significantly simpler, if the ideal is in a sufficiently generic position. For lack of space, we cannot discuss here details, but refer e.g. to (Hashemi, 2010, 2012; Hashemi et al., 2012, 2014; Seiler, 2009b, 2012) and references therein. Coordinate transformations to various stable positions also play a crucial role in recent approaches to determine explicit equations for Hilbert and Quot schemes – see Albert (2017) and references therein.

The Castelnuovo-Mumford regularity nicely exemplifies these considerations. Bayer and Stillman (1987a) proved that generically the degree of the Gröbner basis for the degree reverse lexicographic order is the regularity. But no effective criterion is known to verify whether or not a given ideal is in generic position and random transformations are the only way to obtain such a position. (Seiler, 2009b, Ex. 9.9) gives a concrete example where the degree of the Gröbner basis is first smaller than the regularity and then becomes larger after a certain linear coordinate transformation. Thus the result of Bayer and Stillman does not even provide a bound. If the ideal is in quasi-stable position, then it possesses a finite Pommaret basis the degree of which is the regularity. Now the existence of the finite Pommaret basis provides an effective proof of the genericity of the used coordinates and the deterministic algorithm provided in this work effectively constructs such coordinates.

This article is structured as follows. The next section recalls briefly some notions and tools; this concerns in particular Pommaret bases and Gröbner systems. Section 3 discusses the classical combinatorial concept of stability. We will introduce a total of nine different variants of it and provide for all of them equivalent algebraic characterisations. Furthermore, we discuss the role of the characteristic of the base field and componentwise stability. In Section 4, we study four other generic positions. We first show that the classical Noether position coincides with one of our variants of stability. To the best of our knowledge, this represents the first combinatorial characterisation of Noether position. Then we very briefly recall some facts about Borel-fixed ideals and their relation to stability. We also provide a deterministic method to compute the generic initial ideal via Gröbner systems. While we are sure that many people are aware of this method, we could not find it anywhere in the literature. Finally, we introduce the new concept of β -maximal position and show that it corresponds to the genericity notion underlying the generic annihilator numbers.

Section 5 provides a large number of examples demonstrating that the various genericity notions are indeed all distinct. Section 6 contains the main result of this article from a computational point of view: a deterministic algorithm to achieve any variant of stability over fields with characteristic zero. In positive characteristic only the positions related to quasi-stability are effectively reachable (for sufficiently large fields). For all other stability notions only adapted “ p -versions” can be used which, however, lack the algebraic properties of the standard versions. Finally, we present the results of some preliminary experiments with the proposed algorithm.

2. Preliminaries

We begin by fixing our basic notations and assumptions. $\mathcal{P} = \mathbb{k}[\mathbf{x}]$ with $\mathbf{x} = \{x_1, \dots, x_n\}$ will always be the underlying polynomial ring over a base field \mathbb{k} . Some of our results will require \mathbb{k} to be infinite (or at least sufficiently large), some will depend on whether or not the characteristic of \mathbb{k} is positive. The set of all terms in \mathcal{P} is called \mathbb{T} . For a non-constant term $\mathbf{x}^\mu \in \mathbb{T}$, we denote by $m(\mathbf{x}^\mu)$ the maximal index k such that $\mu_k \neq 0$. If $x^\mu = 1$, then we set $m(x^\mu) = 1$. For simplicity, we consider exclusively homogeneous ideals $\mathcal{I} \triangleleft \mathcal{P}$ and thus always assume that all considered polynomials are homogenous, too. The homogeneous maximal ideal in \mathcal{P} is denoted by $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ and the saturation of an ideal $\mathcal{I} \triangleleft \mathcal{P}$ by $\mathcal{I}^{\text{sat}} = \mathcal{I} : \mathfrak{m}^\infty$. Given a finite set F of polynomials, we briefly write $\deg F$ for the maximal degree of an element of F .

A term order \prec , i.e. a total order on \mathbb{T} which is multiplicative and a well-order, defines for any polynomial $0 \neq f \in \mathcal{P}$ its leading term $\text{lt } f$ as the maximal term in the support of f with respect to \prec and we call for any ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ the monomial ideal $\text{lt } \mathcal{I} = \langle \text{lt } f \mid f \in \mathcal{I} \rangle$ its leading ideal. If not explicitly stated otherwise, we will use throughout the degree reverse lexicographic order (with $x_1 \succ x_2 \succ \dots \succ x_n$) for choosing leading terms, as it has a special relation to the stability notions studied here. The use of this order is crucial for the correctness of our algorithm.

A finite polynomial set $G \subset \mathcal{I}$ is a *Gröbner basis* of the ideal $\mathcal{I} \triangleleft \mathcal{P}$, if $\langle \text{lt } G \rangle = \text{lt } \mathcal{I}$. Given a term $x^\mu \in \mathbb{T}$ with $m(\mathbf{x}^\mu) = k$, we call the variables x_k, \dots, x_n (*Pommaret*) *multiplicative* for it and denote them by $\mathbf{x}_P(x^\mu)$. The *non-multiplicative variables* form simply the complement: $\bar{\mathbf{x}}_P(x^\mu) = \mathbf{x} \setminus \mathbf{x}_P(x^\mu)$. A finite set $\mathcal{H} \subset \mathbb{T}$ of terms is a *Pommaret basis*¹ of the monomial ideal $\mathcal{I} = \langle \mathcal{H} \rangle$ they generate, if \mathcal{I} can be written as the direct sum $\mathcal{I} = \bigoplus_{h \in \mathcal{H}} \mathbb{k}[\mathbf{x}_P(h)] \cdot h$. A finite set $\mathcal{H} \subset \mathcal{P}$ of polynomials is a Pommaret basis of $\mathcal{I} = \langle \mathcal{H} \rangle$, if all its elements possess pairwise distinct leading terms and $\text{lt } \mathcal{H}$ is a Pommaret basis of $\text{lt } \mathcal{I}$. Obviously, any Pommaret basis is a (generally not reduced) Gröbner basis but not vice versa.

There is a natural action of $\text{GL}(n, \mathbb{k})$ on the polynomial ring \mathcal{P} via linear coordinate transformations: $x_i \mapsto \sum_{j=1}^n a_{ij} x_j = (A \cdot \mathbf{x})_i$ for a non-singular matrix $A = (a_{ij}) \in \mathbb{k}^{n \times n}$. If we consider the effect of such coordinate transformations, we always assume that term orders are defined via exponent vectors and that we use the same term order before and after the transformation.

For analysing the effect of this $\text{GL}(n, \mathbb{k})$ -action on a given ideal \mathcal{I} , it is useful to recall the notion of a Gröbner system introduced by Weispfenning (1992) as part of his theory of comprehensive Gröbner bases. Let $\tilde{\mathcal{P}} = \mathcal{P}[\mathbf{a}] = \mathbb{k}[\mathbf{a}][\mathbf{x}]$ be a *parametric* polynomial ring with parameters $\mathbf{a} = \{a_1, \dots, a_m\}$. Given term orders $\prec_{\mathbf{a}}$ and $\prec_{\mathbf{x}}$ for terms in the respective variables, we denote by $\prec_{\mathbf{x}, \mathbf{a}}$ the corresponding block elimination order with precedence to the variables \mathbf{x} .

Definition 2.1. Let $\tilde{\mathcal{I}} \trianglelefteq \tilde{\mathcal{P}}$ be a parametric ideal. A *Gröbner system* for $\tilde{\mathcal{I}}$ for the term order $\prec_{\mathbf{x}, \mathbf{a}}$ is a finite set of triples $\{(\tilde{G}_i, N_i, W_i)\}_{i=1}^\ell$ with finite sets $\tilde{G}_i \subset \tilde{\mathcal{P}}$ and $N_i, W_i \subset \mathbb{k}[\mathbf{a}]$ such that for every index $1 \leq i \leq \ell$ and every specialization homomorphism $\sigma : \mathbb{k}[\mathbf{a}] \rightarrow \mathbb{k}$ with $\sigma(g) = 0$ for every $g \in N_i$ and $\sigma(h) \neq 0$ for every $h \in W_i$ the set $\sigma(\tilde{G}_i)$ is a Gröbner basis of $\sigma(\tilde{\mathcal{I}}) \trianglelefteq \mathcal{P}$ with respect to the order $\prec_{\mathbf{x}}$ and such that for any point $\mathbf{b} \in \mathbb{k}^m$ an index $1 \leq i \leq \ell$ exists with $\mathbf{b} \in \mathcal{V}(N_i) \setminus \mathcal{V}(\prod_{f \in W_i} f)$.

(Weispfenning, 1992, Theorem 2.7) proved that such a Gröbner system exists for every parametric ideal $\tilde{\mathcal{I}} \trianglelefteq \tilde{\mathcal{P}}$ and can be effectively computed. By now, there exists a number of algorithms and implementations for this task (e.g. Kapur et al., 2010; Montes, 2012; Montes and Wibmer, 2010). While it is not part of the definition, every published algorithm for computing Gröbner systems produces systems with an additional property: if two specialisations σ, τ belong to the same triple (\tilde{G}_i, N_i, W_i) , then they yield the same leading terms $\text{lt } \sigma(\tilde{G}_i) = \text{lt } \tau(\tilde{G}_i)$. In the sequel, we will always assume that we are dealing with Gröbner systems possessing this property. We also note that it is always possible to prescribe already at the beginning of the computation some equations or inequations that the parameters must satisfy.

¹See (Seiler, 2009a) or (Seiler, 2010) for a general introduction to involutive bases, a special kind of Gröbner bases with additional combinatorial properties to which Pommaret bases belong. The second reference also contains some historical remarks.

Remark 2.2. Interpreting the entries of a matrix $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{k})$ as parameters, we can compute a Gröbner system $\{(\tilde{G}_i, N_i, W_i)\}_{i=1}^\ell$ of the parametric ideal $\tilde{\mathcal{I}} = A \cdot \mathcal{I} \trianglelefteq \mathbb{k}[a_{ij}][x_1, \dots, x_n]$ imposing at the start the condition that $\det(A) \neq 0$. As such a system is finite by definition, we conclude that under linear coordinate transformations any ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ possesses only finitely many different leading ideals (for a fixed term order).

3. Generic Positions Related To Stability

Stability is a classical combinatorial concept playing an important role in the theory of monomial ideals and depending on the numbering of the variables. There are three basic notions—quasi-stability, stability and strong stability—forming a natural hierarchy. For each of them, we introduce two new weaker versions leading to a total of nine different stability notions following ideas developed in (Hashemi, 2012; Hashemi et al., 2014). The extension to arbitrary ideals is straightforward via a term order.

Definition 3.1. Let $\mathcal{J} \triangleleft \mathcal{P}$ be a monomial ideal, q the maximal degree of a minimal generator of \mathcal{J} and $0 \leq \ell < n$ an integer.

- (i) The ideal \mathcal{J} is *quasi-stable*, if for every term $\mathbf{x}^\mu \in \mathcal{J}$ and every index $j < m = m(\mathbf{x}^\mu)$ the term $x_j^q \mathbf{x}^\mu / x_m^{\mu_m}$ also lies in \mathcal{J} . The ideal is ℓ -*quasi-stable*, if the above condition is satisfied for all terms $\mathbf{x}^\mu \in \mathcal{J}$ with $m(\mathbf{x}^\mu) \geq n - \ell$, and *weakly ℓ -quasi-stable*, if the condition is satisfied with the additional restriction that $j \leq n - \ell$.
- (ii) The ideal \mathcal{J} is *stable*, if for every term $\mathbf{x}^\mu \in \mathcal{J}$ and every index $j < m = m(\mathbf{x}^\mu)$ the term $x_j \mathbf{x}^\mu / x_m$ also lies in \mathcal{J} . The ideal is ℓ -*stable*, if the above condition is satisfied for all terms $\mathbf{x}^\mu \in \mathcal{J}$ with $m(\mathbf{x}^\mu) \geq n - \ell$, and *weakly ℓ -stable*, if the condition is satisfied with the additional restriction that $j \leq n - \ell$.
- (iii) The ideal \mathcal{J} is *strongly stable*, if for every term $\mathbf{x}^\mu \in \mathcal{J}$, and every index pair $i > j$ such that $x_i \mid \mathbf{x}^\mu$ the term $x_j \mathbf{x}^\mu / x_i$ also lies in \mathcal{J} . The ideal is ℓ -*strongly stable*, if the above condition is satisfied for all terms $\mathbf{x}^\mu \in \mathcal{J}$ with $m(\mathbf{x}^\mu) \geq n - \ell$ and all indices $i \geq n - \ell$, and *weakly ℓ -strongly stable*, if the condition is satisfied with the additional restriction that $j \leq n - \ell$.

If $\mathcal{I} \triangleleft \mathcal{P}$ is an arbitrary polynomial ideal, then we say that \mathcal{I} is in a *stable position* for some term order \prec if its leading ideal $\mathrm{Lt} \mathcal{I}$ is stable. The same terminology is used for any above introduced variant of stability.

It is well-known that the three classical notions of stability are generic (see e.g. (Seiler, 2010, Prop. 4.3.8, Cor. 4.3.16) for the case of quasi-stability). Trivial adaptions of the proofs show that all above considered variants are generic, too. It should be noted that in the literature sometimes strongly stable ideals are simply called stable. Quasi-stable ideals are also called *ideals of nested type* by

Bermejo and Gimenez (2006), *ideals of Borel type* by Herzog et al. (2003) or *weakly stable ideals* by Caviglia and Sbarra (2005). In the above definition, we require that all terms in the monomial ideal \mathcal{J} satisfy certain conditions. It is straightforward to show that it suffices to verify that all minimal generators of \mathcal{J} satisfy these conditions. Furthermore, we note the obvious hierarchy

$$\text{strongly stable} \implies \text{stable} \implies \text{quasi-stable}.$$

3.1. Quasi-Stability

Proposition 3.2. *Let $\mathcal{J} \triangleleft \mathcal{P}$ be a monomial ideal with $\dim(\mathcal{P}/\mathcal{J}) = D$. Then the following statements are equivalent:*

(i) \mathcal{J} is quasi-stable.

(ii) If \mathbf{x}^μ is a term in \mathcal{J} with $\mu_j > 0$ for some $1 < j \leq n$, then for each exponent $0 < r \leq \mu_j$ and each index $1 \leq i < j$ an exponent $s \geq 0$ exists such that $x_i^s \mathbf{x}^\mu / x_j^r$ lies in \mathcal{J} .

(iii) For all $0 \leq j \leq n-1$ we have

$$\mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \dots, x_{n-j} \rangle^\infty. \quad (1)$$

(iv) x_n is not a zero divisor on $\mathcal{P}/\mathcal{J}^{\text{sat}}$ and x_{n-j} is not a zero divisor on $\mathcal{P}/\langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$ for all $0 < j < D$.

(v) $\mathcal{J} : x_n^\infty = \mathcal{J}^{\text{sat}}$ and for all $0 < j < D$ we have

$$\langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : x_{n-j}^\infty = \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}. \quad (2)$$

(vi) We have an ascending chain $\mathcal{J} : x_n^\infty \subseteq \mathcal{J} : x_{n-1}^\infty \subseteq \dots \subseteq \mathcal{J} : x_{n-D+1}^\infty$ and for each $1 \leq j \leq n-D$ there exists a term $x_j^{\ell_j} \in \mathcal{J}$.

(vii) \mathcal{J} has a finite monomial Pommaret basis.

(viii) Let $\mathcal{B} = \{t_1, \dots, t_r\}$ be the minimal basis of \mathcal{J} sorted degree reverse lexicographically with t_1 the largest generator. For each index $1 \leq i \leq r$ set $\mathcal{J}_i = \langle t_1, \dots, t_{i-1} \rangle : t_i$ and $\mathcal{P}_i = \mathbb{k}[x_1, \dots, x_{m(t_i)-1}]$. Then all the ideals $\hat{\mathcal{J}}_i = \mathcal{J}_i \cap \mathcal{P}_i \trianglelefteq \mathcal{P}_i$ are zero-dimensional.

(ix) Every associated prime ideal of \mathcal{P}/\mathcal{J} is of the form $\langle x_1, x_2, \dots, x_j \rangle$ for some index $1 \leq j \leq n-D$.

PROOF. Most equivalences are well known and their proofs can e.g. be found in (Bermejo and Gimenez, 2006, Prop. 3.2), (Herzog et al., 2003, Prop. 2.2), (Seiler, 2009b, Prop. 4.4), (Seiler, 2012, Lem. 3.4). Only the characterisations (v) and (viii) are new with (viii) inspired by ideas of (Caviglia, 2004, Sect. 4.1). We therefore prove now first that (iii) entails (v), then that conversely (v) entails (iv) and finally that (vii) and (viii) are equivalent.

Assume that the monomial $t \in \mathcal{P}$ satisfies $tx_{n-j}^s \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle$ for some integer $s > 0$ and an index $0 \leq j \leq D$. If $\text{m}(t) > n - j$, then we have

$$t \in \langle x_n, \dots, x_{n-j+1} \rangle \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}.$$

Otherwise $tx_{n-j}^s \in \mathcal{J}$ and thus $t \in \mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \dots, x_{n-j} \rangle^\infty$ by (iii). Hence we also find $t \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$. This proves that (v) is a consequence of (iii).

Now assume for the second step that x_{n-j} defines a zero divisor in the ring $\mathcal{P}/\langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$ for some index $1 \leq j \leq D - 1$. This means that a polynomial $f \notin \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$ must exist such that $x_{n-j}f \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$ which in turn entails the existence of an integer s such that $x_{n-j}^{s+1}f \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle$ and thus by (v) that $f \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : x_{n-j}^\infty = \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$ which contradicts the choice of f . Hence (iv) follows from (v).

For the proof of the equivalence of (vii) and (viii), we write \mathcal{C}_i for the set of all terms in \mathcal{P}_i which are not contained in \mathcal{J}_i and k_i for $\text{m}(t_i)$. Thus (viii) is equivalent to the fact that all these sets are finite. We will now prove that if this is the case, then the Pommaret basis of \mathcal{J} is given by the finite set

$$\mathcal{H} = \mathcal{B} \cup \bigcup_{i=1}^r \{st_i \mid s \in \mathcal{C}_i\}. \quad (3)$$

Obviously, \mathcal{H} generates \mathcal{I} and thus we only have to prove that it is involutive for the Pommaret division. Consider a term $r \in \bar{\mathcal{C}}_i = \{t_i\} \cup \{st_i \mid s \in \mathcal{C}_i\}$; obviously, $\text{m}(r) = k_i$. We choose an index $1 \leq j < k_i$ which is thus non-multiplicative for r . If $x_j r \in \bar{\mathcal{C}}_i$, then there is nothing to prove. Otherwise write $r = st_i$ with $s = 1$ or $s \in \mathcal{C}_i$. Then $x_j r \notin \bar{\mathcal{C}}_i$ is equivalent to $x_j s \notin \mathcal{C}_i$ which in turn implies that $x_j r \in \langle t_1, \dots, t_{i-1} \rangle$. Let $1 \leq \ell < i$ be the smallest index such that $t_\ell \mid x_j s t_i$ and write $x_j r = r_m r_{nm} t_\ell$ with terms $r_m \in \mathbb{k}[x_{k_\ell}, \dots, x_n]$ and $r_{nm} \in \mathbb{k}[x_1, \dots, x_{k_\ell-1}]$. Because of the minimality of the index ℓ , we must have that $r_{nm} \in \mathcal{C}_\ell$. Hence $r_{nm} t_\ell$ is an element of \mathcal{H} and an involutive divisor of $x_j r$ so that we are done. \square

The following two results generalise some of the characterisations in Proposition 3.2 to the above introduced weaker forms of quasi-stability and thus provide also for these algebraic interpretations.

Proposition 3.3. *Let $\mathcal{J} \triangleleft \mathcal{P}$ be a monomial ideal and ℓ an integer. Then the following statements are equivalent.*

- (i) \mathcal{J} is ℓ -quasi-stable.
- (ii) If $\mathbf{x}^\mu \in \mathcal{J}$ satisfies $\text{m}(\mathbf{x}^\mu) \geq n - \ell$ and $\mu_j > 0$ for some $n - \ell \leq j \leq n$, then for each $0 < r \leq \mu_j$ and $1 \leq i < j$ an integer $s \geq 0$ exists such that $x_i^s \mathbf{x}^\mu / x_j^r \in \mathcal{J}$.

(iii) For all $0 \leq j \leq \ell$ we have

$$\mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \dots, x_{n-j} \rangle^\infty. \quad (4)$$

PROOF. Assume first that \mathcal{J} is ℓ -quasi-stable and denote by B its minimal basis. Let $\mathbf{x}^\mu \in \mathcal{J}$ be a term with $\mu_j > 0$ for some $n - \ell \leq j \leq n$ and r an integer with $0 < r \leq \mu_j$. Hence $k = m(\mathbf{x}^\mu) \geq j$. We want to prove (ii) by showing that $x_i^{\deg B} \mathbf{x}^\mu / x_j^r$ lies in \mathcal{J} for all integers $i < j$. By the definition of ℓ -quasi-stability, $x_i^{\deg B} \mathbf{x}^\mu / x_k^{\mu_k} \in \mathcal{J}$ for $i < k$. Therefore there exists a term $\mathbf{x}^{\nu^{(1)}} \in B$ with

$$\mathbf{x}^{\nu^{(1)}} \mid x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \quad (5)$$

and $k_1 = m(\mathbf{x}^{\nu^{(1)}}) \leq m(x_i^{\deg B} \mathbf{x}^\mu / x_k^{\mu_k}) < k$. Obviously, $\nu_\alpha^{(1)} \leq \mu_\alpha$ for all $i \neq \alpha < k$ and $\nu_i^{(1)} \leq \mu_i + \deg B$. Again it follows from the assumed ℓ -quasi-stability that $x_i^{\deg B} \mathbf{x}^{\nu^{(1)}} / x_{k_1}^{\nu_{k_1}^{(1)}} \in \mathcal{J}$ and thus there exists a term $\mathbf{x}^{\nu^{(2)}} \in B$ with $\mathbf{x}^{\nu^{(2)}} \mid x_i^{\deg B} \mathbf{x}^{\nu^{(1)}} / x_{k_1}^{\nu_{k_1}^{(1)}}$ and $m(\mathbf{x}^{\nu^{(2)}}) = k_2 < k_1$. Furthermore by (5), $\mathbf{x}^{\nu^{(2)}} \mid x_i^{2 \cdot \deg B} \mathbf{x}^\mu / x_{k_1}^{\nu_{k_1}^{(1)}} x_k^{\mu_k}$ and—since $\deg(\mathbf{x}^{\nu^{(2)}}) \leq \deg B$ and $\nu_{k_1}^{(1)} \leq \mu_{k_1}$ —this entails

$$\mathbf{x}^{\nu^{(2)}} \mid x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_{k_1}^{\mu_{k_1}} x_k^{\mu_k}}. \quad (6)$$

We go on like this until we end up with a term $\mathbf{x}^{\nu^{(\omega)}} \in B$ such that $\mathbf{x}^{\nu^{(\omega)}} \mid x_i^{\deg B} \mathbf{x}^{\nu^{(\omega-1)}} / x_{k_{\omega-1}}^{\nu_{k_{\omega-1}}^{(\omega-1)}}$ and $m(\mathbf{x}^{\nu^{(\omega)}}) = k_\omega < \dots < k_1 < k$ such that $k_{\omega-1} = j$. Hence the following holds:

- $\nu_\alpha^{(\omega)} = 0$ for all $\alpha \geq j > k_\omega$.
- $\nu_\alpha^{(\omega)} \leq \nu_\alpha^{(\omega-1)} \leq \dots \leq \nu_\alpha^{(1)} \leq \mu_\alpha$ for all $i \neq \alpha < j$.
- $\nu_i^{(\omega)} \leq \nu_i^{(\omega-1)} \leq \dots \leq \nu_i^{(1)} \leq \mu_i + \deg B$.

Analogously to (5) and (6), we have

$$\mathbf{x}^{\nu^{(\omega)}} \mid x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_j^{\mu_j} x_{k_{\omega-2}}^{\mu_{k_{\omega-2}}} \dots x_{k_1}^{\mu_{k_1}} x_k^{\mu_k}}$$

which entails that $\mathbf{x}^{\nu^{(\omega)}}$ divides $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_j^r}$ and we are done.

Now assume (ii) and let t be a term such that $x_{n-j}^r t \in \mathcal{J}$ for some exponent r and index $0 \leq j \leq \ell$. Since $m(x_{n-j}^r t) \geq n - j \geq n - \ell$, it follows from (ii) that for all $i < n - j \leq m(x_{n-j}^r t)$ there is an integer s_i such that the term $x_i^{s_i} x_{n-j}^r t / x_{n-j}^r = x_i^{s_i} t$ lies in \mathcal{J} . Hence we have the inclusion $t \langle x_1, \dots, x_{n-j} \rangle^{(s_1 + \dots + s_{n-j-1} + r)(n-j)} \subseteq \mathcal{J}$ entailing $t \in \mathcal{J} : \langle x_1, \dots, x_{n-j} \rangle^\infty$ which shows (iii).

Finally assume that the equality (4) holds and consider a term $\mathbf{x}^\mu \in \mathcal{J}$ such that $m(\mathbf{x}^\mu) = n - j$ with $j \leq \ell$. Because of (4), we have $\mathbf{x}^\mu/x_{n-j}^{\mu_{n-j}} \in \mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \dots, x_{n-j} \rangle^\infty$. Hence there is an integer s such that $(\mathbf{x}^\mu/x_{n-j}^{\mu_{n-j}})\langle x_1, \dots, x_{n-j} \rangle^s \subseteq \mathcal{J}$. But this inclusion means that for every index $1 \leq i < n - j$ a minimal generator t_i of \mathcal{J} exists which divides $x_i^s \mathbf{x}^\mu/x_{n-j}^{\mu_{n-j}} \in \mathcal{J}$. Because of $\deg_{x_i} t_i \leq \deg B$, it is clear that we may choose $s \leq \deg B$ which finally shows that \mathcal{J} is ℓ -quasi-stable. \square

Corollary 3.4. *Let $\mathcal{J} \triangleleft \mathcal{P}$ be a monomial and ℓ -quasi-stable ideal. If $\ell \geq D - 1$ where $D = \dim(\mathcal{P}/\mathcal{J})$, then \mathcal{J} is even quasi-stable.*

PROOF. Since the equality $\mathcal{I} : x_{n-j}^\infty = \mathcal{I} : \langle x_1, \dots, x_{n-j} \rangle^\infty$ for all $0 < j < D$ implies that also $\langle \mathcal{I}, x_n, \dots, x_{n-j+1} \rangle : x_{n-j}^\infty = \langle \mathcal{I}, x_n, \dots, x_{n-j+1} \rangle^{\text{sat}}$ for all $0 < j < D$, the assertion follows from Propositions 3.2 and 3.3.

For low-dimensional ideals, this observation significantly reduces the computational costs of checking quasi-stability. However, its straightforward application requires the knowledge of the dimension of the ideal. The following simple Algorithm 1 verifies whether a given monomial ideal is D -quasi-stable without a priori knowledge of D . It is an adaption of a similar algorithm for checking D -stability presented in (Hashemi et al., 2014, Alg. 1). We will prove its correctness later (Proposition 4.5).

Algorithm 1 DQS-TEST: Test for D -quasi-stability

Input: minimal basis $G = \{t_1, \dots, t_r\}$ of monomial ideal $\mathcal{J} \triangleleft \mathcal{P}$

Output: The answer to: is \mathcal{J} D -quasi-stable?

```

1:  $\ell \leftarrow$  smallest  $j$  such that  $x_\alpha^{\deg G} \in \mathcal{I}$  for  $\alpha = 1, \dots, n - j$ 
2: for all  $\mathbf{x}^\mu \in G$  with  $k = m(\mathbf{x}^\mu) \geq n - \ell$  do
3:   for  $i = 1, \dots, k - 1$  do
4:     if  $x_i^{\deg G} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \langle G \rangle$  then
5:       return false
6:     end if
7:   end for
8: end for
9: return true

```

With minor adaptions of the proof given above, one obtains the following version of Proposition 3.3 for the weakly ℓ -quasi-stable case. In Section 4.1, we will relate this notion of stability to Noether position.

Proposition 3.5. *Let $\mathcal{J} \triangleleft \mathcal{P}$ be a monomial ideal and ℓ an integer. Then the following statements are equivalent.*

1. \mathcal{J} is weakly ℓ -quasi-stable
2. If \mathbf{x}^μ in \mathcal{J} with $m(\mathbf{x}^\mu) \geq n - \ell$ and $\mu_j > 0$ for some $n - \ell \leq j \leq n$, then for each $0 < r \leq \mu_j$ and $1 \leq i \leq n - \ell$ an integer $s \geq 0$ exists such that $x_i^s \frac{\mathbf{x}^\mu}{x_j^r}$ lies in \mathcal{J} .

3. For all $0 \leq j \leq \ell$ holds

$$\mathcal{J} : x_{n-j}^\infty \subseteq \mathcal{J} : \langle x_1, \dots, x_{n-\ell} \rangle^\infty. \quad (7)$$

3.2. Stability

A study of the problem of characterising algebraically the various variants of stability has already been started by Hashemi et al. (2014) because of its relevance for computing reduction numbers. For completeness, we first recall without proof the following result about ℓ -stability.

Proposition 3.6 ((Hashemi et al., 2014, Prop. 3.5)). *The monomial ideal $\mathcal{J} \trianglelefteq \mathcal{P}$ is ℓ -stable, if and only if it satisfies for all $0 \leq j \leq \ell$*

$$\langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : x_{n-j} = \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : \mathfrak{m}. \quad (8)$$

Hashemi et al. (2014) showed furthermore that D -stability implies quasi-stability whereas this is not the case for weak D -stability. The following novel result provides an analogous characterisation of weak ℓ -stability.

Proposition 3.7. *Let $\mathcal{J} \trianglelefteq \mathcal{P}$ be a monomial ideal. If \mathcal{J} is weakly ℓ -stable, then it satisfies for all $0 \leq j \leq \ell$ the equality*

$$\langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle : x_{n-j} = \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle : \mathfrak{m}. \quad (9)$$

PROOF. Assume first that \mathcal{J} is weakly ℓ -stable and let t be a term such that $x_{n-j}t \in \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle$ for some $j \leq \ell$. If $m(t) > n - \ell$, then $t \in \langle x_n, \dots, x_{n-\ell+1} \rangle$ and nothing is to be proven. Otherwise, we have $x_{n-j}t \in \mathcal{J}$ and $m(x_{n-j}t) = n - j \geq n - \ell$. The weak ℓ -stability now entails that $xt = x_i \frac{x_{n-j}t}{x_{n-j}} \in \mathcal{J}$ for all $i \leq n - \ell$. Hence $t\langle x_1, \dots, x_{n-\ell} \rangle \subseteq \mathcal{J}$ implying $t\mathfrak{m} \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle$. Thus we have shown the inclusion “ \subseteq ” and the converse one is trivial. \square

3.3. Componentwise Stability

Herzog and Hibi (1999) introduced the notion of a componentwise linear ideal as a generalisation of the notion of a stable monomial ideal to polynomial ideals. Such ideals have many special properties, in particular concerning their Betti numbers. If $\mathcal{I} \triangleleft \mathcal{P}$ is a homogeneous ideal, then we denote the ideal generated by the homogeneous component \mathcal{I}_d by $\mathcal{I}_{\langle d \rangle} = \langle \mathcal{I}_d \rangle$. One can now extend every stable position defined above to a *componentwise stable position* by requiring that all ideals $\mathcal{I}_{\langle d \rangle}$ with $d \geq 0$ are simultaneously in the corresponding stable position. For monomial ideals, componentwise (strong) stability is equivalent to ordinary (strong) stability, as the defining criterion involves only terms of the same degree. By contrast, componentwise quasi-stability is a stronger condition than the ordinary version. As for polynomial ideals we do not simply consider their leading ideals but the (polynomial) component ideals $\mathcal{I}_{\langle d \rangle}$, for them componentwise (strongly) stable position is generally also a stronger condition than its ordinary counterpart.

We will concentrate in the sequel on componentwise quasi-stability, as it appears to be the most important notion for applications. For example, if a componentwise linear ideal is in componentwise quasi-stable position, then all its Betti numbers can be directly read off from its Pommaret basis, as this basis induces the minimal resolution of the ideal (Seiler, 2009b, Thm. 9.12). Another quite remarkable fact about this position is that it is of all the generic positions considered in this work the only one which is not automatically implied by the GIN position (see Definition 4.8 below). Example 5.23 provides a concrete counter example.

The following elementary result implies that it is not really necessary to work componentwise which requires to treat many and rather large bases and thus is computationally very inefficient. In the case of (strongly) stable position only the only-if-part remains true. However, for the subsequent results only this direction is needed so that appropriately adapted versions can be provided.

Lemma 3.8. *Let $\mathcal{I} \triangleleft \mathcal{P}$ be a homogeneous polynomial ideal. The ideal $\mathcal{I}_{\langle d \rangle} = \langle \mathcal{I}_d \rangle$ is in quasi-stable position, if and only if the ideal $\mathcal{I}_{[d]} = \langle \bigcup_{r \leq d} \mathcal{I}_r \rangle$ is in quasi-stable position.*

PROOF. Obviously, $\mathcal{I}_{\langle d \rangle} = (\mathcal{I}_{[d]})_{\geq d}$. Now the claim follows immediately from (Seiler, 2009b, Lemma 2.2). \square

We now develop a sufficient criterion for an ideal \mathcal{I} to be in componentwise quasi-stable position which does not require the consideration of the component ideals $\mathcal{I}_{\langle d \rangle}$ (or equivalently $\mathcal{I}_{[d]}$). Such a criterion is important for deciding componentwise linearity. Assuming that the ideal \mathcal{I} is already in quasi-stable position (so that it possesses a Pommaret basis), we can derive one based on the first syzygies of \mathcal{I} .

If the set $\mathcal{H} = \{h_1, \dots, h_s\}$ is a Pommaret basis of \mathcal{I} and x_k is a non-multiplicative variable for the generator $h_\alpha \in \mathcal{H}$, then the product $x_k h_\alpha$ possesses a unique involutive standard representation

$$x_k h_\alpha = \sum_{\beta=1}^s P_\beta^{(\alpha;k)} h_\beta \quad (10)$$

where each non-vanishing coefficient $P_\beta^{(\alpha;k)}$ depends only on variables which are multiplicative for h_β and satisfies $\text{lt}(P_\beta^{(\alpha;k)} h_\beta) \preceq \text{lt}(x_k h_\alpha)$. Seiler (2009b) showed that the corresponding syzygies form a Pommaret basis of the first syzygy module of \mathcal{I} (for the Schreyer order induced by \mathcal{H}). Given a degree $d \geq 0$ such that $\mathcal{I}_d \neq 0$, we introduce two subsets of the Pommaret basis \mathcal{H} : the set $\mathcal{H}_d = \{h \in \mathcal{H} \mid \deg h \leq d\}$ collects all generators up to degree d and the set $\widehat{\mathcal{H}}_d = \{\hat{h} \in \mathcal{H} \mid \exists h \in \mathcal{H}_d : \text{lt } h \mid \text{lt } \hat{h}\}$ contains in addition all higher order generators which have a leading term divisible by the leading term of an element of \mathcal{H}_d .

Proposition 3.9. *Let $\mathcal{I} \triangleleft \mathcal{P}$ be a homogeneous ideal in quasi-stable position and $d \geq 0$ a degree such that $\mathcal{I}_d \neq 0$. The ideal $\mathcal{I}_{[d]}$ is in quasi-stable position, if in every involutive standard representation (10) with $h_\alpha \in \widehat{\mathcal{H}}_d$ all generators h_β with $P_\beta^{(\alpha;k)} \neq 0$ also lie in $\widehat{\mathcal{H}}_d$. In this case, $\widehat{\mathcal{H}}_d$ is the Pommaret basis of $\mathcal{I}_{[d]}$.*

PROOF. We first note that obviously $\mathcal{I}_{[d]} = \langle \mathcal{H}_d \rangle$. Then we denote by $\widehat{\mathcal{I}}$ the ideal generated by $\widehat{\mathcal{H}}_d$. If the condition on the involutive standard representations is satisfied, then $\widehat{\mathcal{H}}_d$ is the Pommaret basis of $\widehat{\mathcal{I}}$. As obviously, $\mathcal{I}_{[d]} \subseteq \widehat{\mathcal{I}}$, it suffices to show that $\mathcal{I}_{[d]}$ cannot be a proper subset of $\widehat{\mathcal{I}}$. Assume that this was the case. Then there must exist a generator $\hat{h} \in \widehat{\mathcal{H}}_d$ which is not contained in $\mathcal{I}_{[d]}$. Let \hat{h} be among all such generators the one with the smallest leading term with respect to the used term order. By construction, there exists $h \in \mathcal{H}_d$ such that $\text{lt } \hat{h} = x^\nu \text{lt } h$ for some term x^ν . We consider the polynomial $g = \text{lc}(h)\hat{h} - \text{lc}(\hat{h})x^\nu h \in \widehat{\mathcal{I}}$. It possesses an involutive standard representation with respect to the Pommaret basis $\widehat{\mathcal{H}}_d$ of the form $g = \sum_{\hat{f} \in \widehat{\mathcal{H}}_d} P_{\hat{f}} \hat{f}$. Every generator \hat{f} with $P_{\hat{f}} \neq 0$ must have a leading term smaller than \hat{h} , as by construction $\text{lt } g \prec \text{lt } \hat{h}$, and thus must lie in $\mathcal{I}_{[d]}$ according to our choice of \hat{h} . But this implies that $\hat{h} \in \mathcal{I}_{[d]}$ contradicting our assumption. Hence $\widehat{\mathcal{I}} = \mathcal{I}_{[d]}$ and $\mathcal{I}_{[d]}$ is in quasi-stable position. \square

Remark 3.10. The same statement holds for the componentwise (strongly) stable case. The only difference is that now we must assume that the ideal \mathcal{I} is already in (strongly) stable position; the criterion itself does not change. As in this case the leading terms $\text{lt } \mathcal{H}$ form even the minimal basis of $\text{lt } \mathcal{I}$, we find that $\widehat{\mathcal{H}}_d = \mathcal{H}_d$ which simplifies the application of the criterion.

As a simple corollary, we find that componentwise quasi-stability is generic, too, as the intersection of finitely many Zariski open subsets is still Zariski open.

Corollary 3.11. *For verifying that a homogeneous ideal $\mathcal{I} \triangleleft \mathcal{P}$ in quasi-stable position is even in a componentwise quasi-stable position, it suffices to consider only finitely many ideals $\mathcal{I}_{\langle d \rangle}$. If the degree reverse lexicographic order is used, then we may restrict to $d \leq \text{reg } \mathcal{I}$.*

PROOF. It follows from the previous proposition that it suffices to restrict to $d \leq q$ where q is the maximal degree of a generator in the Pommaret basis of \mathcal{I} . If the degree reverse lexicographic order is used, then $q = \text{reg } \mathcal{I}$. \square

Example 3.12. The criterion of Proposition 3.9 is not necessary. Consider the ideal $\mathcal{I} = \langle x_1^5, x_1x_2^4, x_1^3x_2^3 \rangle \triangleleft \mathbb{k}[x_1, x_2]$. It is quasi-stable and its Pommaret basis is given by $\mathcal{H} = \{x_1^5, x_1x_2^4, x_1^3x_2^3, x_1^2x_2^4, x_1^4x_2^3\}$. The first two generators form the set \mathcal{H}_5 , adding the fourth one yields $\widehat{\mathcal{H}}_5$. Our criterion is *not* satisfied, as we find as involutive standard representation $x_1h_4 = x_2h_3$ and $h_3 \notin \widehat{\mathcal{H}}_5$. Nevertheless, one easily verifies that $\mathcal{I}_{\langle 5 \rangle} = \mathcal{I}_{[5]} = \langle x_1^5, x_1x_2^4 \rangle$ is quasi-stable.

3.4. Positive Characteristic

In principle, all above introduced notions of stability are independent of the characteristic of the base field. However, when we will discuss in Section 6 how to transform a given polynomial ideal into one of these positions, the characteristic will play a role. The simplest restriction will be that for finite base fields we will have to assume that the field is sufficiently large (the precise meaning of this will become apparent below). A more serious restriction will be that in positive characteristic, we can only guarantee that one can always reach the various variants of a quasi-stable position. For stability and strong stability only adapted “ p -versions” can be reached generally. The reason is simply that in positive characteristic many binomial coefficients vanish and hence many terms cannot be produced via linear transformations.

In order to define these “ p -versions”, we need the following notations – see e.g. (Eisenbud, 1995, §15.9.3). Let p be an arbitrary prime number. For two natural numbers k, ℓ , we say $k \prec_p \ell$, if $\binom{\ell}{k} \not\equiv 0 \pmod{p}$. Given a term \mathbf{x}^μ and natural numbers $i > j$ such that $\mu_i > 0$, we define for any natural number $s \leq \mu_i$ the s th elementary move as the term $e_{i,j}^{(s)}(\mathbf{x}^\mu) = x_j^s \mathbf{x}^\mu / x_i^s$ and this move is p -admissible, if and only if $s \prec_p \mu_i$.

The following definition of “ p -versions” covers only the classical stability notions. Of course, it is trivial to extend it to ℓ - and weak versions.

Definition 3.13. Assume that $\text{char } \mathbb{k} = p$ is positive. Then a monomial ideal $\mathcal{J} \trianglelefteq \mathcal{P} = \mathbb{k}[\mathbf{x}]$ is *p-stable*, if for every term $\mathbf{x}^\mu \in \mathcal{J}$ in it every p -admissible move $e_{i,j}^{(s)}(\mathbf{x}^\mu)$ with $j < i = m(\mathbf{x}^\mu)$ and $s \leq \mu_i$ yields again a term in \mathcal{J} . The ideal \mathcal{J} is *strongly p-stable*, if in the definition above every index i with $\mu_i > 0$ can be considered.

As above, it is sufficient to verify the conditions on some finite generating set of \mathcal{J} . It is easy to see that Definition 3.1 is equivalent to requiring that all elementary moves, i.e. without any condition on the exponent s , stay inside the ideal. Thus (strong) p -stability is a weaker notion than “ordinary” (strong) stability, as it simply ignores certain elementary moves.

4. Other Generic Positions

We now consider three classical generic positions and introduce a new fourth one. The material in the Subsections 4.2 and 4.3 is well-known and included only for the sake of completeness. Our main point in all cases is the relationship to the stability positions considered in the previous section.

4.1. Noether Position

Definition 4.1. The D -dimensional ideal $\mathcal{I} \triangleleft \mathcal{P}$ is in *Noether position*, if the variables x_1, \dots, x_D induce a Noether normalisation of \mathcal{I} .

Noether position is a classical concept in commutative algebra. The following well-known result provides a simple effective test via Gröbner bases.

Lemma 4.2 (e. g. (Bermejo and Gimenez, 2001, Lem. 4.1)). *Let $\mathcal{I} \triangleleft \mathcal{P}$ be a D -dimensional ideal. Then the following statements are equivalent:*

- (i) \mathcal{I} is in Noether position.
- (ii) There are integers s_i such that $x_i^{s_i} \in \text{lt } \mathcal{I}$ for all $1 \leq i \leq n - D$.
- (iii) $\dim(\mathcal{P}/\langle \mathcal{I}, x_{n-D+1}, \dots, x_n \rangle) = 0$.
- (iv) $\dim(\mathcal{P}/\text{lt } \langle \mathcal{I}, x_{n-D+1}, \dots, x_n \rangle) = 0$.

Remark 4.3. Bermejo and Gimenez (2006) proved that an ideal \mathcal{I} is quasi-stable, if and only if \mathcal{I} and all primary components of $\text{lt } \mathcal{I}$ are simultaneously in Noether position. In fact, it is easy to see that quasi-stability implies Noether position (Seiler, 2009b, Prop. 4.1), which immediately implies that the latter is a generic position, too.

Almost all algorithms proposed so far to get an ideal into Noether position are probabilistic – see e. g. (Greuel and Pfister, 2002, Algo. 3.4.5). An exception is the approach of Robertz (2009) using Janet bases. Furthermore, (Seiler, 2009b, Sect. 2) contains a method to obtain deterministically quasi-stable position and as mentioned above this entails Noether position. However, the result of Bermejo and Gimenez mentioned in Remark 4.3 shows that quasi-stability is stronger than Noether position. To the best of our knowledge, the following result represents the first combinatorial characterisation of Noether position. In particular, it implies that Noether position can also be achieved with the deterministic methods which will be presented in Section 6.

Theorem 4.4. *Let $\mathcal{I} \triangleleft \mathcal{P}$ be a D -dimensional ideal. It is in Noether position, if and only if it is in weakly D -quasi-stable position.*

PROOF. We first note the following simple consequence of the definition of weak D -quasi-stability for a monomial ideal \mathcal{J} with minimal basis B . If the term $\mathbf{x}^\mu \in \mathcal{J}$ lies in the ideal, then \mathcal{J} also contains any term of the form $x_1^{\mu_1+\nu_1} \cdots x_{n-\ell}^{\mu_{n-\ell}+\nu_{n-\ell}}$ with exponents ν_i that are multiples of $\deg B$ satisfying $\nu_1 + \cdots + \nu_{n-\ell} = k \deg B$ where $k = \#\{\mu_j \mid j > n - D \wedge \mu_j > 0\}$.

Assume now that $\mathcal{J} = \text{lt } \mathcal{I}$ is a weakly D -quasi-stable ideal. If there exists a term $x^\mu \in \mathcal{J} \cap \mathbb{k}[x_{n-D+1}, \dots, x_n]$, then we can immediately invoke the observation above to conclude that for each $1 \leq i \leq n - D$ a term $x_i^{s_i}$ is contained in \mathcal{J} , as $\mu_1 = \cdots = \mu_{n-D} = 0$. Thus \mathcal{I} is in Noether position by Lemma 4.2.

If the intersection $\mathcal{J} \cap \mathbb{k}[x_{n-D+1}, \dots, x_n]$ is empty, then the D -dimensional cone $1 \cdot \mathbb{k}[x_{n-D+1}, \dots, x_n]$ lies completely in the complement of \mathcal{J} . As for a D -dimensional ideal it is not possible that the complement contains a $(D + 1)$ -dimensional cone, the intersection $\mathcal{J} \cap \mathbb{k}[x_i, x_{n-D+1}, \dots, x_n]$ must be non-empty for any index $1 \leq i \leq n - D$. But if x^μ is a term in this intersection, then it follows again from the introductory remark that also a term $x_i^{s_i}$ lies in \mathcal{J} and thus that \mathcal{I} is in Noether position. \square

With the help of Theorem 4.4, we can now provide the postponed proof that Algorithm 1 for testing D -quasi-stability is indeed correct.

Proposition 4.5. *Algorithm 1 is correct.*

PROOF. We distinguish three cases:

1. \mathcal{J} is D -quasi-stable.
2. \mathcal{J} is not D -quasi-stable, but in Noether position.
3. \mathcal{J} is neither D -quasi-stable nor in Noether position.

In the first case, Theorem 4.4 entails that \mathcal{J} is in Noether position. Hence the number ℓ computed in Line 1 equals D by Lemma 4.2 and we will never reach Line 5 by the definition of D -quasi-stability. In the second case, we find again $\ell = D$ by the same argument. But as \mathcal{J} is not D -quasi-stable there must be an obstruction that leads us correctly to Line 5. In the last case, ℓ is greater than D (we know that $\ell \neq D$, since \mathcal{J} is not in Noether position; the assumption $\ell < D$ leads to a contradiction, since then $D \leq n - (n - \ell) = \ell < D$). As \mathcal{J} is not D -quasi-stable, there exists a term $\mathbf{x}^\mu \in G$ with $k = m(\mathbf{x}^\mu) \geq n - D > n - \ell$ such that $x_i^{\deg G} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \mathcal{J}$ for some $i < k$. Our algorithm will detect this obstruction and thus gives again the right answer. \square

4.2. Borel-Fixed Position

The next generic position which we consider is distinguished from all the other ones by the fact that it is the only one which depends on the characteristic of the underlying field \mathbb{k} . Recall that the subgroup $\mathfrak{B} \subseteq \mathrm{GL}(n, \mathbb{k})$ of all lower triangular invertible $n \times n$ matrices is called the *Borel group*. For any integer $0 \leq \ell < n$, we introduce the ℓ -Borel group as the subgroup $\mathfrak{B}_\ell \leq \mathfrak{B}$ consisting of all matrices $A \in \mathfrak{B}$ such that for $i < n - \ell$ we have $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$ (obviously, $\mathfrak{B}_{n-1} = \mathfrak{B}$).

Definition 4.6. The monomial ideal $\mathcal{J} \triangleleft \mathcal{P}$ is ℓ -Borel-fixed for an integer $0 \leq \ell < n$, if $A \cdot \mathcal{J} = \mathcal{J}$ for all $A \in \mathfrak{B}_\ell$. The polynomial ideal $\mathcal{I} \triangleleft \mathcal{P}$ is in ℓ -Borel-fixed position for a term order \prec , if $\mathrm{lt} \mathcal{I}$ is ℓ -Borel-fixed. If $\ell = n - 1$, then we drop the suffix ℓ and simply speak of a *Borel-fixed* ideal and position, respectively.

It is a classical result (e.g. Herzog and Hibi, 2011, Prop. 4.2.4) that any strongly stable ideal is Borel-fixed (which implies that we deal indeed with a generic position). In characteristic zero the converse is true, too. If the characteristic is a positive prime p , then $\langle x_1^p, x_2^p \rangle \triangleleft \mathbb{k}[x_1, x_2]$ is a simple example of a Borel-fixed ideal which is not strongly stable. However, it is easy to see that in any characteristic a Borel-fixed ideal is quasi-stable (Bayer and Stillman, 1987b, Cor. 2). (Hashemi et al., 2014, Prop. 9) generalised these assertions: in characteristic zero a monomial ideal \mathcal{J} is ℓ -Borel-fixed for some integer $0 \leq \ell < n$, if and only if \mathcal{J} is strongly ℓ -stable. In positive characteristic only one direction is true.

4.3. GIN Position

A classical result proven first by Galligo (1974) in characteristic zero and then later by Bayer and Stillman (1987b) in arbitrary characteristic asserts that almost all linear changes of coordinates applied to an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ lead to the same leading ideal which is then called the *generic initial ideal* $\text{gin } \mathcal{I}$ of \mathcal{I} . Again by Galligo (1979) in characteristic zero and by Bayer and Stillman (1987b) in arbitrary characteristic, it was shown that $\text{gin } \mathcal{I}$ is always Borel-fixed.

Theorem 4.7 (Galligo). *For any ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, there exists a nonempty Zariski open subset $\mathcal{U} \subseteq \text{GL}(n, \mathbb{k})$ such that $\text{lt}(A \cdot \mathcal{I}) = \text{lt}(B \cdot \mathcal{I})$ for all $A, B \in \mathcal{U}$.*

Definition 4.8. The ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ is in *GIN position* (for a term order \prec), if $\text{lt } \mathcal{I} = \text{gin } \mathcal{I}$.

GIN position is the strongest notion of genericity that we consider in this work. It implies all other positions with one exception: componentwise quasi-stability is an independent property (see Example 5.23 below). While the GIN position is very popular among theorists, as in it \mathcal{I} and $\text{lt } \mathcal{I}$ share many invariants, it should be noted that neither a simple effective criterion nor a simple deterministic algorithm is known for it. As far as we know, all computer algebra systems use a probabilistic approach to determine $\text{gin } \mathcal{I}$ by applying simply one or more random transformations. Such a computation may become quite expensive, as it inevitably leads to dense polynomials for which a Gröbner basis must be computed. Furthermore, it cannot be easily tested whether or not the result really is $\text{gin } \mathcal{I}$.

If one uses a parametric coordinate transformation instead of a random one, the computation becomes of course even more expensive, but the result is guaranteed to be the correct generic initial ideal. Let $A = (a_{ij})$ be an $n \times n$ parametric matrix and $\mathbb{k}(a_{ij})$ the field of fractions of $\mathbb{k}[a_{ij}]$. We consider the ideal $\hat{\mathcal{I}} = A \cdot \mathcal{I} \trianglelefteq \mathbb{k}(a_{ij})[x_1, \dots, x_n]$. It follows from Theorem 4.7 that $\text{lt } \hat{\mathcal{I}} = \text{gin } \mathcal{I}$. Hence a Gröbner basis of $\hat{\mathcal{I}}$ yields immediately $\text{gin } \mathcal{I}$.

Alternatively, we consider $\tilde{\mathcal{I}} = \hat{\mathcal{I}} \cap \tilde{\mathcal{P}}$ where $\tilde{\mathcal{P}} = \mathbb{k}[a_{ij}][x_1, \dots, x_n]$ and compute a Gröbner system for $\tilde{\mathcal{I}} \trianglelefteq \tilde{\mathcal{P}}$ imposing at the start the condition that $\det(A) \neq 0$. Again by Theorem 4.7, the generic branch (the only one for which the set N_i is empty) yields as leading ideal $\text{gin } \mathcal{I}$. Note that for finding the generic branch it is not necessary to determine the whole Gröbner system. It suffices to follow at each case distinction the “not equal zero” branch. This is equivalent to a fraction-free form of computing a Gröbner basis of $\hat{\mathcal{I}}$ and in practise probably more efficient.

Obviously, this approach requires to work with n^2 parameters. If one is interested in the generic initial ideal for the degree reverse lexicographic term order, then a slight optimisation is possible. In this case, it suffices to take for A a lower triangular matrix with all diagonal entries equal to 1 and thus one can reduce the number of parameters to $n(n - 1)/2$. Indeed, any regular matrix

A can be written as a product² $A = UDL$ where L is a lower triangular, U an upper triangular and D a diagonal matrix and where both L and U have only ones on the diagonal. While the transformation induced by D does trivially not change the leading term of any polynomial for arbitrary term orders, it follows from the definition of the degree reverse lexicographic order that here also the transformation induced by U does not affect any leading term. Hence we find that $\text{lt}(A \cdot \mathcal{I}) = \text{lt}(L \cdot \mathcal{I})$ and it suffices to work with the matrix L .

4.4. β -Maximal Position

Given a homogeneous ideal $\mathcal{I} \triangleleft \mathcal{P}$ and a degree q with $\mathcal{I}_q \neq 0$, we denote by $\mathcal{B}_q(\mathcal{I}) = (\text{lt } \mathcal{I})_q \cap \mathbb{T}$ the monomial \mathbb{k} -linear basis of $(\text{lt } \mathcal{I})_q$. We set $\beta_q^{(k)}(\mathcal{I}) = \#\{t \in \mathcal{B}_q(\mathcal{I}) \mid m(t) = k\}$. Then the β -vector of \mathcal{I} at degree q is defined as

$$\beta_q(\mathcal{I}) = (\beta_q^{(1)}(\mathcal{I}), \dots, \beta_q^{(n)}(\mathcal{I})) \in \mathbb{N}_0^n. \quad (11)$$

Remark 4.9. The β -vector provides a convenient way to compare the asymptotic behaviour of Hilbert polynomials. We call the set

$$\langle \mathcal{B}_q(\mathcal{I}) \rangle_P = \bigoplus_{t \in \mathcal{B}_q(\mathcal{I})} \mathbb{k}[\mathbf{x}_P(t)] \cdot t \subseteq \langle \mathcal{B}_q(\mathcal{I}) \rangle$$

the *Pommaret span* of $\mathcal{B}_q(\mathcal{I})$ and define $h_{\mathcal{I},q}^P(s) = \dim_{\mathbb{k}} (\langle \mathcal{B}_q(\mathcal{I}) \rangle_P)_s$. If $h_{\mathcal{I},q}$ denotes the Hilbert function of the monomial ideal $\langle \mathcal{B}_q(\mathcal{I}) \rangle$, then obviously $h_{\mathcal{I},q}^P(s) \leq h_{\mathcal{I},q}(s)$ for all degrees s and we have $h_{\mathcal{I},q}^P = h_{\mathcal{I},q}$, if and only if $\mathcal{B}_q(\mathcal{I})$ is the Pommaret basis of the ideal it generates. (Seiler, 2010, Prop. 8.2.6) showed that³

$$h_{\mathcal{I},q}^P(q+r) = \sum_{i=0}^{n-1} \left(\sum_{k=i}^{n-1} \frac{s_{k-i}^{(k)}(0)}{k!} \beta_q^{(n-k)}(\mathcal{I}) \right) r^i \quad (12)$$

where the modified Stirling numbers $s_i^{(j)}(\ell)$ are positive integers (see (Seiler, 2010, App. A.4) for more details). Thus $h_{\mathcal{I},q}^P$ is polynomial beyond degree q . If we write it as $\sum_i h_i r^i$, then its coefficient h_{n-i} is a linear combination of $\beta_q^{(1)}, \dots, \beta_q^{(i)}$ with positive coefficients. This simple observation entails that if \mathcal{I} and \mathcal{J} are two homogeneous ideals such that $\beta_q(\mathcal{I}) \prec_{\text{lex}} \beta_q(\mathcal{J})$, then $h_{\mathcal{J},q}^P(s) < h_{\mathcal{I},q}^P(s)$ for all sufficiently large degrees s , and motivates the following novel generic position.

Definition 4.10. The homogeneous ideal $\mathcal{I} \triangleleft \mathcal{P}$ is in β -maximal position (for a given term order \prec), if we have for all matrices $A \in \text{GL}(n, \mathbb{k})$ and all degrees $q \geq 0$ with $\mathcal{I}_q \neq 0$ the inequality

$$\beta_q(\mathcal{I}) \succeq_{\text{lex}} \beta_q(A \cdot \mathcal{I}). \quad (13)$$

²Classically, one uses decompositions $A = LDU$. But such a decomposition for the inverse A^{-1} yields immediately a decomposition of our form for A .

³Strictly speaking, (Seiler, 2010, Prop. 8.2.6) covered a slightly different situation than we consider here. In particular, it is there assumed that one deals with a Pommaret basis. However, the adaption to our case here is trivial.

We will now first show that β -maximality implies quasi-stability and then that the generic initial ideal has at all degrees the same β -vector as an ideal in β -maximal position (implying that β -maximality is a generic position). In both cases, the converse statement is not true. In particular, β -maximal position does not imply GIN position (see for instance Example 5.21 below).

Proposition 4.11. *The polynomial ideal $\mathcal{I} \triangleleft \mathcal{P}$ is in quasi-stable position, if and only if the following inequality holds for all matrices $A \in \mathrm{GL}(n, \mathbb{k})$ and all degrees $q \geq \mathrm{reg} \mathrm{lt} \mathcal{I}$:*

$$\beta_q(\mathcal{I}) \succeq_{\mathrm{lex}} \beta_q(A \cdot \mathcal{I}) \quad (14)$$

PROOF. Let us assume first that \mathcal{I} is in quasi-stable position. Then for any degree $q \geq \mathrm{reg} \mathcal{I} = \mathrm{reg} \mathrm{lt} \mathcal{I}$ the truncation $\mathcal{I}_{\geq q}$ is even in stable position (Seiler, 2009b, Prop. 9.6). Hence $\mathcal{B}_q(\mathcal{I})$ is a Pommaret basis of the ideal it generates and we find that $h_{\mathcal{I},q}^P = h_{\mathcal{I},q}$ which immediately implies (14) by Remark 4.9.

For the converse, note that (14) implies $h_{\mathcal{I},q}^P = h_{\mathcal{I},q}$, since there always exists a matrix A such that $A \cdot \mathcal{I}$ is in quasi-stable position. These Hilbert functions coincide beyond degree q , if and only if $\mathcal{B}_q(\mathcal{I})$ generates a stable ideal and thus if $\mathcal{I}_{\geq q}$ is in stable position (Seiler, 2009b, Prop. 9.6). But then the original ideal \mathcal{I} is in quasi-stable position (Seiler, 2009b, Lemma 2.2). \square

Corollary 4.12. *Any polynomial ideal $\mathcal{I} \triangleleft \mathcal{P}$ in β -maximal position is in quasi-stable position, too.*

Proposition 4.13. *If the polynomial ideal $\mathcal{I} \triangleleft \mathcal{P}$ is in GIN position, then \mathcal{I} is also in β -maximal position. In particular, if \mathcal{I} is in β -maximal position, then $\beta_q(\mathrm{gin} \mathcal{I}) = \beta_q(\mathcal{I})$ for all degrees q with $\mathcal{I}_q \neq 0$.*

PROOF. We exploit a result derived in the proof of Galligo's Theorem 4.7 presented by (Green, 1998, Thm. 1.27). For a given degree q , let the terms $\{t_1, \dots, t_{s_q}\}$ be a \mathbb{k} -basis of \mathcal{P}_q ordered according to the degree reverse lexicographic order: $t_1 \succ t_2 \succ \dots \succ t_{s_q}$. Then there exists a Zariski open subset $\mathcal{U} \subseteq \mathrm{GL}(n, \mathbb{k})$ such that for all matrices $A \in \mathcal{U}$, all degrees $q \geq 0$ and all indices $m \leq s_q$ the dimension of the \mathbb{k} -linear space

$$\mathcal{V}_{q,m}(A) = \langle \mathrm{lt} (A \cdot \mathcal{I})_q \rangle_{\mathbb{k}} \cap \langle t_1, \dots, t_m \rangle_{\mathbb{k}}$$

takes its maximal possible value (thus if $B \notin \mathcal{U}$, then for at least some values of q and m we have $\dim_{\mathbb{k}} \mathcal{V}_{q,m}(B) < \dim_{\mathbb{k}} \mathcal{V}_{q,m}(A)$ for any $A \in \mathcal{U}$). Now let a \mathbb{k} -basis of $\mathcal{B}_q(\mathrm{gin} \mathcal{I})$ be given by the terms $\{t_1, \dots, t_\ell\}$ and of $\mathcal{B}_q(\hat{A} \cdot \mathcal{I})$ for an arbitrarily chosen matrix $\hat{A} \in \mathrm{GL}(n, \mathbb{k})$ by $\{\hat{t}_1, \dots, \hat{t}_\ell\}$, respectively. In both cases, we assume again that the bases are ordered by the degree reverse lexicographic order. Then the above maximality condition implies that $\tilde{t}_i \succ \hat{t}_i$ for all $1 \leq i \leq \ell$. By definition of the degree reverse lexicographic order, we thus find that $m(\tilde{t}_i) \leq m(\hat{t}_i)$ for all indices i which is equivalent to $\beta_q(\mathrm{gin} \mathcal{I}) \succeq_{\mathrm{lex}} \beta_q(\hat{A} \cdot \mathcal{I})$. \square

Remark 4.14. In principle, these results provide us with a deterministic test for β -maximality. We first check whether or not we are in a quasi-stable position. If this is not the case, the position cannot be β -maximal by Corollary 4.12. Otherwise, we determine $\text{gin } \mathcal{I}$ deterministically (as discussed in Section 4.3) and then it suffices by Propositions 4.11 and 4.13 to compare the β -vectors $\beta_q(\mathcal{I})$ and $\beta_q(\text{gin } \mathcal{I})$ for the finitely many degrees $0 \leq q < \text{reg } \mathcal{I}$. Obviously, such a test is rather expensive. So far, no deterministic algorithm for finding a β -maximal position is known. One can only apply random transformations and then perform the above described check.

The ideal $\mathcal{I}_1 = \langle x_1^2, x_1x_2 + x_2^2, x_1x_3 \rangle \triangleleft \mathbb{k}[x_1, x_2, x_3]$ was already considered by (Green, 1998, Ex. 1.28) as an example where the leading ideal is strongly stable but nevertheless not the generic initial ideal. Indeed one finds $\text{lt } \mathcal{I}_1 = \langle x_1^2, x_1x_2, x_1x_3, x_2^3, x_2^2x_3 \rangle$, whereas $\text{gin } \mathcal{I}_1 = \langle x_1^2, x_1x_2, x_2^2, x_1x_3^2 \rangle$. It is easy to see that these two monomial ideals have different β -vectors and hence \mathcal{I}_1 is not in β -maximal position. On the other hand, $\mathcal{I}_2 = \text{lt } \mathcal{I}_1$ is strongly stable which implies $\text{gin } \mathcal{I}_2 = \mathcal{I}_2$ and thus \mathcal{I}_2 is in β -maximal position. This observation shows that two ideals \mathcal{I}_1 and \mathcal{I}_2 may have the same leading ideal and yet behave differently with respect to β -maximality. We conclude that there cannot exist a “simple” deterministic algorithm—meaning an algorithm solely based on the analysis of leading terms like the one developed in Section 6 for the various notions of stability—that produces a β -maximal position for arbitrary ideals.

In the context of a Pommaret basis of \mathcal{I} , one can roughly interpret β -maximality as a condition that generators with more multiplicative variables should have lower degrees (note, however, that componentwise quasi-stability admits the same rough interpretation and is nevertheless independent of β -maximality—see Examples 5.4 and 5.5 below). We will now show that this observation can be related to results by (Herzog and Hibi, 2011, Sect. 4.3) on the annihilator numbers of graded modules. In particular, we will prove that the genericity concept underlying their notion of generic annihilator numbers is exactly β -maximality.

Definition 4.15. A linear form $y \in \mathcal{P}_1$ is called *quasi-regular*⁴ for the graded \mathcal{P} -module \mathcal{M} , if the graded module $0 :_{\mathcal{M}} y = \{m \in \mathcal{M} \mid ym = 0\}$ is of finite length (i.e. if only finitely many graded components are non-vanishing). An ordered sequence $(y_1, \dots, y_k) \subset \mathcal{P}_1$ is *quasi-regular* for \mathcal{M} , if y_i is quasi-regular for $\mathcal{M}/\langle y_1, \dots, y_{i-1} \rangle \mathcal{M}$ for $1 \leq i \leq k$.

In the sequel, we will concentrate for notational simplicity on the case that $\mathcal{M} = \mathcal{P}/\mathcal{I}$ for a homogeneous ideal \mathcal{I} . However, all results can be straightforwardly extended to finitely presented modules $\mathcal{M} = \mathcal{P}^m/\mathcal{U}$ with a graded

⁴Following Aramova and Herzog (2000), Herzog and Hibi use the terminology *almost regular*. However, the same concept was introduced under the name quasi-regularity much earlier in a rather unknown letter of Serre appended to (Guillemin and Sternberg, 1964). Later, the same notion was reinvented by Schenzel et al. (1978) under the name *filter-regular*.

submodule \mathcal{U} . The following result by Seiler (2007) shows that quasi-regularity is actually just a different way to view quasi-stability and that quasi-regular sequences of lengths up to $n = \dim \mathcal{P}$ always exist.

Proposition 4.16 ((Seiler, 2007, Thm. 5.2)). *The sequence (x_n, \dots, x_2, x_1) is quasi-regular for $\mathcal{M} = \mathcal{P}/\mathcal{I}$, if and only if \mathcal{I} is in quasi-stable position.*

Given a quasi-regular sequence $\mathbf{y} = (y_1, \dots, y_n)$ of length n for the graded module $\mathcal{M} = \mathcal{P}/\mathcal{I}$, we introduce the graded modules

$$\begin{aligned} A_{i-1}(\mathbf{y}; \mathcal{M}) &= 0 :_{\mathcal{M}/\langle y_1, \dots, y_{i-1} \rangle \mathcal{M}} y_i \\ &\cong (\langle \mathcal{I}, y_1, \dots, y_{i-1} \rangle : y_i) / \langle \mathcal{I}, y_1, \dots, y_{i-1} \rangle \end{aligned}$$

and define the *annihilator numbers* of \mathcal{M} with respect to the sequence \mathbf{y} as $\alpha_{ij}(\mathbf{y}; \mathcal{M}) = \dim_{\mathbb{k}} A_{i-1}(\mathbf{y}; \mathcal{M})_j$ for all indices $0 \leq i < n$ and $j \geq 0$. The definition of quasi-regularity implies immediately that only finitely many of these numbers are non-zero. The following result shows that the annihilator numbers simply encode how the elements of the Pommaret basis of \mathcal{I} for the degree reverse lexicographic order distribute over the different degrees and the different numbers of multiplicative variables.

Theorem 4.17. *Let the finite set \mathcal{H} be the Pommaret basis of the homogeneous ideal $\mathcal{I} \triangleleft \mathcal{P}$ for the degree reverse lexicographic term order and $\mathcal{M} = \mathcal{P}/\mathcal{I}$. Then for all admissible indices i, j*

$$\alpha_{ij}(x_n, \dots, x_1; \mathcal{M}) = \#\{h \in \mathcal{H} \mid m(\text{lt } h) = n - i \wedge \deg(h) = j + 1\}. \quad (15)$$

PROOF. Consider the projection $\pi : \mathcal{P} = \mathbb{k}[x_1, \dots, x_n] \rightarrow \tilde{\mathcal{P}} = \mathbb{k}[x_1, \dots, x_{n-1}]$ defined by $\pi(f) = f|_{x_n=0}$. It is easy to see that if \mathcal{H} is the Pommaret basis of \mathcal{I} for the degree reverse lexicographic order, then $\pi(\mathcal{H}) \setminus \{0\}$ is the Pommaret basis of $\pi(\mathcal{I})$ for the same term order (we find $\pi(h) = 0$, if and only if $m(\text{lt } h) = n$). Because of the obvious isomorphism $\mathcal{P}/\langle \mathcal{I}, x_n \rangle \cong \tilde{\mathcal{P}}/\pi(\mathcal{I})$, it thus suffices to consider the case $i = 0$; the assertion for all other values of the index i follows by an easy induction.

The case $i = 0$ requires the analysis of the homogeneous polynomials $f \in (\mathcal{I} : x_n)_j \setminus \mathcal{I}_j$. For any such polynomial the product $x_n f \in \mathcal{I}_{j+1}$ possesses a unique involutive standard representation (Seiler, 2009a, Thm. 5.4): $x_n f = \sum_{h \in \mathcal{H}} P_h h$ with coefficients $P_h \in \mathbb{k}[\mathbf{x}_P(h)]$ satisfying $\text{lt}(P_h h) \preceq \text{lt}(x_n f)$. For any generator $h \in \mathcal{H}$ with $m(\text{lt } h) < n$, we must have $P_h \in \langle x_n \rangle$ whereas $m(\text{lt } h) = n$ entails $P_h \in \mathbb{k}[x_n]$. The assumption $f \notin \mathcal{I}_j$ implies that for at least one generator $h \in \mathcal{H}$ with $m(\text{lt } h) = n$ the coefficient P_h is a non-vanishing constant (which is only possible if $\deg h = j + 1$), as otherwise we could divide the above involutive standard representation by x_n and would obtain a standard representation of f . But this observation proves immediately our claim for $i = 0$. \square

Exploiting properties of Pommaret bases, we obtain the following two results of (Herzog and Hibi, 2011, Prop. 4.3.4, Thm. 4.3.6) as trivial corollaries.

Corollary 4.18. *Let $\mathcal{I} \triangleleft \mathcal{P}$ be a homogeneous ideal in quasi-stable position and set $\mathcal{M} = \mathcal{P}/\mathcal{I}$.*

- (i) $\sum_{j \geq 0} \alpha_{ij}(x_n, \dots, x_1; \mathcal{M}) = 0$, if and only if $i < \text{depth } \mathcal{I}$.
- (ii) *There exists a Zariski open subset $\mathcal{U} \subseteq \text{GL}(n, \mathbb{k})$ such that for all matrices $B \in \mathcal{U}$ the transformed ordered sequence $\mathbf{y} = B\mathbf{x}$ is again quasi-regular and for all admissible indices i, j we have the equality $\alpha_{ij}(y_n, \dots, y_1; \mathcal{M}) = \alpha_{ij}(x_n, \dots, x_1; \mathcal{P}/\text{gin } \mathcal{I})$.*

PROOF. The first assertion follows immediately from Theorem 4.17 and the fact that $\text{depth } \mathcal{I} = n - t$ with t the maximal value of $m(\text{lt } h)$ for a generator h in the Pommaret basis of \mathcal{I} for the degree reverse lexicographic order (Seiler, 2009b, Prop. 3.19). The second assertion follows from Proposition 4.13. \square

(Herzog and Hibi, 2011, Def. 4.3.9) call both a quasi-regular sequence \mathbf{y} as in Corollary 4.18(ii) and the corresponding annihilator numbers *generic*. According to Proposition 4.13, a generic quasi-regular sequence thus defines a β -maximal position and vice versa. (Herzog and Hibi, 2011, Sect. 4.3.2) conclude their discussion of the annihilator numbers by studying their relationship to the Betti numbers of \mathcal{M} . All these results follow again immediately from Theorem 4.17 and the resolution induced by a Pommaret basis (Seiler, 2009b, Thm. 6.1). In particular, the estimate given by (Herzog and Hibi, 2011, Prop. 4.3.12) is simply a bigraded version of the one contained in (Seiler, 2009b, Thm. 6.1).

5. Examples

The results in the previous sections entail certain relations between the above introduced generic positions. They are depicted in the diagram in Figure 1. In order to demonstrate that all positions are indeed different, we compile a series of examples separating them (for a field of characteristic zero). The numbers shown in the various fields of the diagram correspond to the numbering of the examples. The used abbreviations should be largely self-explanatory. “D” represents the dimension $D = \dim \mathcal{I}$, thus DS denotes D -stable ideals and WDS weakly D -stable ideals. Similarly, “C” stands for componentwise and “Q” for quasi.

Example 5.1. $\mathcal{I} = \langle x_1^2, x_2^2, x_1 x_4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4]$ is not quasi-stable, because $x_3^2 \frac{x_1 x_4}{x_4} = x_1 x_3^2 \notin \mathcal{I}$. $D = 2$ and \mathcal{I} is not weakly D -stable, as $x_2 \frac{x_1 x_4}{x_4} = x_1 x_2 \notin \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^2, x_1 x_2, x_2^2, x_1 x_3^2 \rangle$, we see that \mathcal{I} is not in β -maximal position as

$$\beta_2(\mathcal{I}) = (1, 1, 0, 1) \prec_{\text{lex}} (1, 2, 0, 0) = \beta_2(\text{gin } \mathcal{I}).$$

Example 5.2. $\mathcal{I} = \langle x_1 x_2, x_1^3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2]$ is quasi-stable, but not componentwise, as $\mathcal{I}_{(2)} = \langle x_1 x_2 \rangle$ is not quasi-stable. $D = 1$ and \mathcal{I} is not weakly D -stable, as $x_1 \frac{x_1 x_2}{x_2} = x_1^2 \notin \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^2, x_1 x_2^2 \rangle$, we see that \mathcal{I} is not in β -maximal position, as

$$\beta_2(\mathcal{I}) = (0, 1) \prec_{\text{lex}} (1, 0) = \beta_2(\text{gin } \mathcal{I}).$$

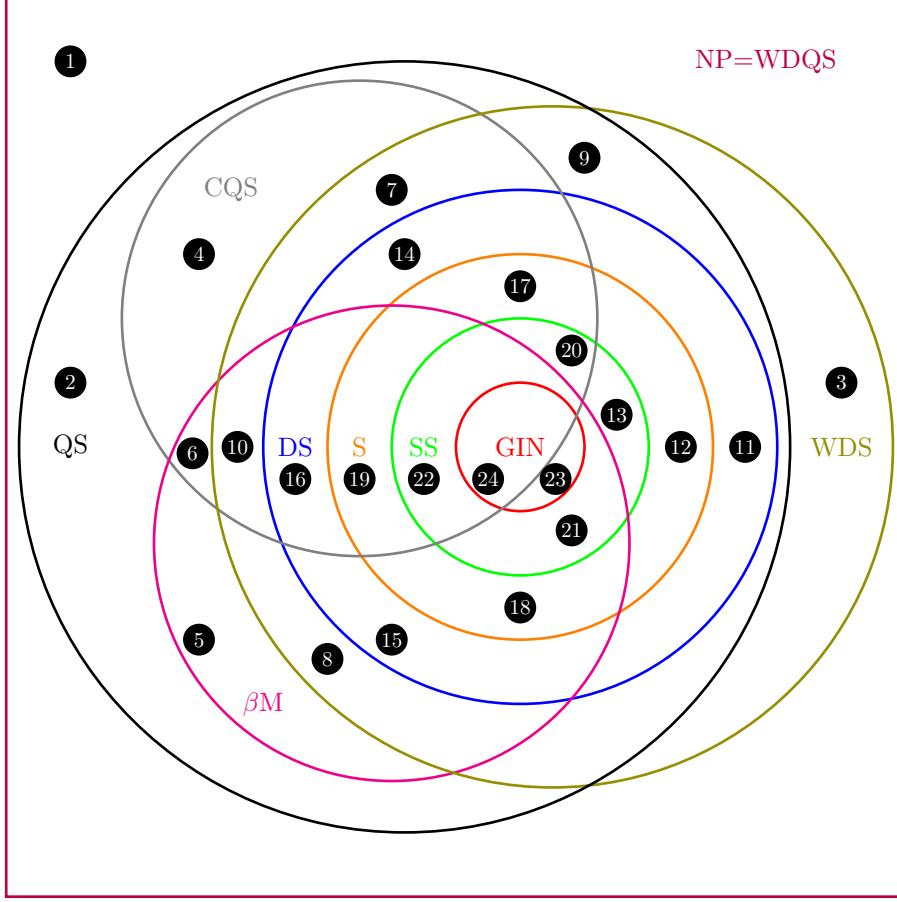


Figure 1: “Map of Positions”

Example 5.3. $\mathcal{I} = \langle x_1^2, x_1x_3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$ is not quasi-stable, as $x_2 \frac{x_1x_3}{x_3} = x_1x_2^2 \notin \mathcal{I}$. $D = 2$ and \mathcal{I} is not D -stable, as $x_2 \frac{x_1x_3}{x_3} = x_1x_2 \notin \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2 \rangle$, we see that \mathcal{I} is not in β -maximal position, as

$$\beta_2(\mathcal{I}) = (1, 0, 1) \prec_{\text{lex}} (1, 1, 0) = \beta_2(\text{gin } \mathcal{I}).$$

Example 5.4. For $\mathcal{I} = \langle x_1^2, x_2^2, x_3^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$ we have $D = 0$ and \mathcal{I} is not weakly D -stable, as $x_1 \frac{x_3^2}{x_3} = x_1x_3 \notin \mathcal{I}$. $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_2x_3^2, x_3^4 \rangle$ implies that \mathcal{I} is not in β -maximal position, as

$$\beta_2(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\text{gin } \mathcal{I})$$

Example 5.5. For $\mathcal{I} = \langle x_1^3, x_1x_2^2 + x_2^2x_3, x_2^4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$, we have $\text{lt } \mathcal{I} = \langle x_1^3, x_1x_2^2, x_2^4, x_2^2x_3^3 \rangle$ and $D = 1$. $\text{lt } \mathcal{I}$ is not weakly D -stable, as $x_1 \frac{x_1x_2^2}{x_2} = x_1^2x_2 \notin$

$\text{lt } \mathcal{I}$, and $\text{lt } \mathcal{I}_{\langle 3 \rangle} = \langle x_1^3, x_1x_2^2, x_2^2x_3^3 \rangle$ is not quasi-stable. \mathcal{I} is in β -maximal position, as $\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^3, x_2^4, x_1x_2^2x_3^2, x_1^2x_3^4 \rangle$ and

$$\begin{aligned}\beta_3(\mathcal{I}) &= (1, 1, 0) = \beta_3(\text{gin } \mathcal{I}), \\ \beta_4(\mathcal{I}) &= (1, 4, 2) = \beta_4(\text{gin } \mathcal{I}), \\ \beta_5(\mathcal{I}) &= (1, 5, 8) = \beta_5(\text{gin } \mathcal{I}).\end{aligned}$$

Example 5.6. For $\mathcal{I} = \langle x_1^2, x_2^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2]$ we have $D = 0$ and \mathcal{I} is not weakly D -stable, as $x_1 \frac{x_2^2}{x_2} = x_1x_2 \notin \mathcal{I}$. $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^3 \rangle$ implies that \mathcal{I} is in β -maximal position, as

$$\beta_2(\mathcal{I}) = (1, 1) = \beta_2(\text{gin } \mathcal{I}).$$

Example 5.7. For $\mathcal{I} = \langle x_1^2, x_1x_2, x_2^2 + x_3^2, x_1x_4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4]$ we have $\text{lt } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^2, x_1x_4, x_1x_3^2 \rangle$ and $D = 2$. $\text{lt } \mathcal{I}$ is not D -stable, as $x_3 \frac{x_1x_4}{x_4} = x_1x_3 \notin \mathcal{I}$. $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^2, x_1x_3, x_1x_4^2 \rangle$ entails that \mathcal{I} is not in β -maximal position, as

$$\beta_2(\mathcal{I}) = (1, 2, 0, 1) \prec_{\text{lex}} (1, 2, 1, 0) = \beta_2(\text{gin } \mathcal{I}).$$

Example 5.8. For the ideal $\mathcal{I} = \langle x_1^3 + x_1x_3^2, x_1^2x_2 + x_2x_4^2, x_1x_2^2, x_2^3, x_2^2x_3^2, x_2x_3^3, x_2x_4^2, x_1x_2x_3x_4^2, x_2x_3^2x_4^2, x_1x_2x_4^4, x_2x_3x_4^4, x_2x_4^6 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4]$ we have

$$\begin{aligned}\text{lt } \mathcal{I} &= \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_2x_3^2, x_2^2x_3^2, x_2x_3^3, x_2^2x_4^2, x_1x_2x_3x_4^2, \\ &\quad x_2x_3^2x_4^2, x_1x_2x_4^4, x_2x_3x_4^4, x_2x_4^6 \rangle\end{aligned}$$

and $D = 2$. $\text{lt } \mathcal{I}$ is not D -stable, as $x_3 \frac{x_2^2x_4^2}{x_4} = x_2^2x_3x_4 \notin \text{lt } \mathcal{I}$. Since $\text{lt } \mathcal{I}_{\langle 3 \rangle} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_2x_3^2, x_2^2x_4^2, x_2x_3^2x_4^2 \rangle$ is not quasi-stable, \mathcal{I} is not in componentwise quasi-stable position.

$$\begin{aligned}\text{gin } \mathcal{I} &= \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3^2, x_1x_2x_3^2, x_1x_3^3, x_1^2x_3x_4, x_1x_2x_3x_4^2, \\ &\quad x_1x_3^2x_4^2, x_1^2x_4^3, x_1x_2x_4^4, x_1x_3x_4^4, x_1x_4^6 \rangle\end{aligned}$$

entails that \mathcal{I} is in β -maximal position as

$$\begin{aligned}\beta_3(\mathcal{I}) &= (1, 3, 0, 0) = \beta_3(\text{gin } \mathcal{I}), \\ \beta_4(\mathcal{I}) &= (1, 4, 7, 5) = \beta_4(\text{gin } \mathcal{I}), \\ \beta_5(\mathcal{I}) &= (1, 5, 12, 20) = \beta_5(\text{gin } \mathcal{I}), \\ \beta_6(\mathcal{I}) &= (1, 6, 18, 40) = \beta_6(\text{gin } \mathcal{I}).\end{aligned}$$

Example 5.9. For $\mathcal{I} = \langle x_1^2, x_1x_2, x_2^2, x_1x_4, x_1x_3^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4]$ we have that $\mathcal{I}_{\langle 2 \rangle} = \langle x_1^2, x_1x_2, x_2^2, x_1x_4 \rangle$ is not quasi-stable and thus \mathcal{I} is not in componentwise quasi-stable position. $D = 2$ and $\text{lt } \mathcal{I}$ is not D -stable, as $x_3 \frac{x_1x_4}{x_4} = x_1x_3 \notin \mathcal{I}$. $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^2, x_1x_3, x_1x_4^2 \rangle$ entails that \mathcal{I} is not in β -maximal position as

$$\beta_2(\mathcal{I}) = (1, 2, 0, 1) \prec_{\text{lex}} (1, 2, 1, 0) = \beta_2(\text{gin } \mathcal{I}).$$

Example 5.10. Let $\mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^2x_3^2, x_2^2x_4^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4]$. Then $D = 2$ and \mathcal{I} is not D -stable since $x_3 \frac{x_2^2x_4^2}{x_4} = x_2^2x_3x_4 \notin \mathcal{I}$. As $\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_4^3 \rangle$, we see that \mathcal{I} is in β -maximal position since

$$\begin{aligned}\beta_3(\mathcal{I}) &= (1, 3, 0, 0) = \beta_3(\text{gin } \mathcal{I}), \\ \beta_4(\mathcal{I}) &= (1, 4, 5, 5) = \beta_4(\text{gin } \mathcal{I}).\end{aligned}$$

Example 5.11. Let $\mathcal{I} = \langle x_2^2, x_1x_3, x_2x_3, x_3^2, x_1^3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\mathcal{I}_{(2)} = \langle x_2^2, x_1x_3, x_2x_3, x_3^2 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. Furthermore, \mathcal{I} is not stable, as $x_1 \frac{x_2^2}{x_2} = x_1x_2 \notin \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2, x_3^4 \rangle$, we see that \mathcal{I} is not in β -maximal position, as $\beta_2(\mathcal{I}) = (0, 1, 3) \prec_{\text{lex}} (1, 2, 1) = \beta_2(\text{gin } \mathcal{I})$.

Example 5.12. Let $\mathcal{I} = \langle x_1^2, x_1x_2^2 + x_2x_3^2, x_2^5, x_2^4x_3, x_3^2x_3^2, x_2^2x_3^3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I}_{(3)} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_2x_3^2, x_2x_3^4 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. Furthermore,

$$\text{lt } \mathcal{I} = \langle x_1^2, x_1x_2^2, x_1x_2x_3^2, x_2^5, x_2^4x_3, x_3^2x_3^2, x_2^2x_3^3, x_2x_3^4 \rangle$$

is not strongly stable, as $x_1 \frac{x_1x_2x_3^2}{x_2} = x_1^2x_3^2 \notin \mathcal{I}$. Since

$$\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2^2, x_2^4, x_2^3x_3^2, x_1x_2x_3^3, x_2^2x_3^3, x_1x_3^4 \rangle,$$

\mathcal{I} is not in β -maximal position, as $\beta_4(\mathcal{I}) = (1, 3, 5) \prec_{\text{lex}} (1, 4, 4) = \beta_4(\text{gin } \mathcal{I})$

Example 5.13. Let $\mathcal{I} = \langle x_1^2, x_1x_2 + x_2x_3, x_1x_3, x_2^3, x_2^2x_3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I}_{(2)} = \langle x_1^2, x_1x_2, x_1x_3, x_2x_3^2 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. Furthermore, $\text{lt } \mathcal{I} = \langle x_1^2, x_1x_2, x_1x_3, x_2x_3^2, x_2^3, x_2^2x_3 \rangle \neq \langle x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_2x_3^2 \rangle = \text{gin } \mathcal{I}$ and so we see that \mathcal{I} is not in β -maximal position, as $\beta_2(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\text{gin } \mathcal{I})$.

Example 5.14. Let $\mathcal{I} = \langle x_1^3, x_2^3, x_1x_3^2, x_2x_3^2, x_3^3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then \mathcal{I} is not stable, as $x_1 \frac{x_2^3}{x_2} = x_1x_2^2 \notin \mathcal{I}$. Since

$$\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_1x_2x_3^2, x_2^2x_3^2, x_1x_3^4, x_2x_3^4, x_3^6 \rangle,$$

\mathcal{I} is not in β -maximal position, as $\beta_3(\mathcal{I}) = (1, 1, 3) \prec_{\text{lex}} (1, 3, 1) = \beta_3(\text{gin } \mathcal{I})$.

Example 5.15. Let $\mathcal{I} = \langle x_1^3, x_1x_2^2 + x_2^2x_3, x_2^4, x_1x_3^3, x_3^4, x_2x_3^3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I}_{(3)} = \langle x_1x_2^2, x_1^3, x_2^2x_3^3 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. $\text{lt } \mathcal{I} = \langle x_1^3, x_1x_2^2, x_2^4, x_2x_3^3, x_1x_3^3, x_3^4 \rangle$ is not stable, as $x_1 \frac{x_1x_2^2}{x_2} = x_1^2x_2 \notin \text{lt } \mathcal{I}$. With

$$\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^3, x_2^4, x_1x_2^2x_3, x_2^3x_3, x_1^2x_3^2, x_1x_2x_3^3, x_2^2x_3^3, x_1x_3^4, x_2x_3^5, x_3^6 \rangle,$$

we see that \mathcal{I} is in β -maximal position, since

$$\begin{aligned}\beta_3(\mathcal{I}) &= (1, 1, 0) = \beta_3(\text{gin } \mathcal{I}), \\ \beta_4(\mathcal{I}) &= (1, 4, 5) = \beta_4(\text{gin } \mathcal{I}), \\ \beta_5(\mathcal{I}) &= (1, 5, 13) = \beta_5(\text{gin } \mathcal{I}).\end{aligned}$$

Example 5.16. Let

$$\mathcal{I} = \langle x_1^3, x_1x_2^2, x_2^3, x_1^2x_2x_3, x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2, x_1x_3^3, x_2x_3^3, x_3^4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3].$$

Then \mathcal{I} is not stable, as $x_1 \frac{x_1x_2^2}{x_2} = x_1^2x_2 \notin \mathcal{I}$. Since

$$\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^4, x_2^3x_3, x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2, x_1x_3^3, x_2x_3^3, x_3^4 \rangle,$$

we see that \mathcal{I} is in β -maximal position, as $\beta_3(\mathcal{I}) = (1, 2, 0) = \beta_3(\text{gin } \mathcal{I})$.

Example 5.17. Let $\mathcal{I} = \langle x_1^3, x_1^2x_2 + x_2^3, x_1^2x_3, x_2^4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^3, x_2^4, x_2^3x_3 \rangle$ is not strongly stable, as $x_1 \frac{x_2^3x_3}{x_2} = x_1x_2^2x_3 \notin \text{lt } \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^4, x_1^2x_3^2 \rangle$, we see that \mathcal{I} is not in β -maximal position, as $\beta_3(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_3(\text{gin } \mathcal{I})$.

Example 5.18. Let $\mathcal{I} = \langle x_1^2, x_1x_2 + x_2x_3, x_2^3, x_2^2x_3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I}_{(2)} = \langle x_1^2, x_1x_2, x_2x_3^2 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. Furthermore, $\text{lt } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_2x_3^2 \rangle$ is not strongly stable, as $x_1 \frac{x_2x_3^2}{x_2} = x_1x_3^2 \notin \text{lt } \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^2 \rangle$, we see that \mathcal{I} is in β -maximal position, as $\beta_2(\mathcal{I}) = (1, 3, 4) = \beta_2(\text{gin } \mathcal{I})$.

Example 5.19. Let $\mathcal{I} = \langle x_1^2, x_1x_2^2, x_2^3, x_2^2x_3^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then \mathcal{I} is not strongly stable, as $x_1 \frac{x_2^2x_3^2}{x_2} = x_1x_2x_3^2 \notin \mathcal{I}$. Since $\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2^2, x_2^3, x_1x_2x_3^2 \rangle$, we see that \mathcal{I} is in β -maximal position, as

$$\begin{aligned} \beta_2(\mathcal{I}) &= (1, 0, 0) = \beta_2(\text{gin } \mathcal{I}) \\ \beta_3(\mathcal{I}) &= (1, 3, 1) = \beta_3(\text{gin } \mathcal{I}). \end{aligned}$$

Example 5.20. Let $\mathcal{I} = \langle x_1^2, x_1x_2 + x_2x_3, x_2^3, x_2^2x_3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I} = \langle x_1^2, x_1x_2, x_1x_3, x_2^3, x_2^2x_3 \rangle \neq \langle x_1^2, x_1x_2, x_2^2, x_1x_3^2 \rangle = \text{gin } \mathcal{I}$ and so we see that \mathcal{I} is not in β -maximal position, as $\beta_2(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\text{gin } \mathcal{I})$.

Example 5.21. Consider $\mathcal{I} = \langle x_1^3, x_1^2x_2 + x_2x_3^2, x_1x_2^3, x_2^4, x_1x_2^2x_3, x_1^2x_3^2, x_1x_3^4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I}_{(3)} = \langle x_1^3, x_1^2x_2, x_1x_2x_3^2, x_2x_3^4 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. Furthermore,

$$\text{lt } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^3, x_2^4, x_1x_2^2x_3, x_1^2x_3^2, x_1x_2x_3^2, x_2^3x_3^2, x_2^2x_3^3, x_1x_3^4, x_2x_3^4 \rangle$$

does not equal

$$\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^3, x_2^4, x_1x_2^2x_3, x_2^3x_3, x_1^2x_3^2, x_1x_2x_3^3, x_2^2x_3^3, x_1x_3^4, x_2x_3^4 \rangle,$$

but \mathcal{I} is in β -maximal position, as

$$\begin{aligned} \beta_3(\mathcal{I}) &= (1, 1, 0) = \beta_3(\text{gin } \mathcal{I}), \\ \beta_4(\mathcal{I}) &= (1, 4, 5) = \beta_4(\text{gin } \mathcal{I}). \end{aligned}$$

Example 5.22. Let $\mathcal{K} = \langle x_2x_3 - x_1x_4, x_1^3 - x_2^2x_4, x_2^3 - x_1x_3^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4]$ and $\mathcal{I} = \Psi_2\Psi_1(\mathcal{K})$ with $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ and $\Psi_2 : (x_2 \mapsto x_2 + x_1)$. Then

$$\begin{aligned}\text{lt } \mathcal{I} &= \langle x_1^2, x_1x_2^2, x_2^3, x_1x_2x_3^2, x_1x_3^3, x_2^2x_3^3, x_2x_3^4 \rangle \neq \\ &\langle x_1^2, x_1x_2^2, x_2^3, x_1x_2x_3^2, x_2^2x_3^2, x_1x_3^4, x_2x_3^4 \rangle = \text{gin } \mathcal{I},\end{aligned}$$

but \mathcal{I} is in β -maximal position, as

$$\begin{aligned}\beta_2(\mathcal{I}) &= (1, 0, 0, 0) = \beta_2(\text{gin } \mathcal{I}), \\ \beta_3(\mathcal{I}) &= (1, 3, 1, 1) = \beta_3(\text{gin } \mathcal{I}), \\ \beta_4(\mathcal{I}) &= (1, 4, 7, 6) = \beta_4(\text{gin } \mathcal{I}).\end{aligned}$$

Example 5.23. Let $\mathcal{I} = \langle x_1^3, x_1^2x_2 + x_1x_2x_3, x_1x_2^3, x_1x_2^2x_3, x_1^2x_3^2 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Then $\text{lt } \mathcal{I}_{(3)} = \langle x_1^3, x_1^2x_2, x_1x_2x_3^2 \rangle$ is not quasi-stable, hence \mathcal{I} is not in componentwise quasi-stable position. Furthermore,

$$\text{lt } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^3, x_1x_2^2x_3, x_1^2x_3^2, x_1x_2x_3^2 \rangle = \text{gin } \mathcal{I}$$

and so \mathcal{I} is in β -maximal position.

Example 5.24. The final ideal that is in any position is simply $\langle x_1 \rangle \trianglelefteq \mathbb{k}[x_1]$.

6. A Deterministic Algorithm for Stable Positions

6.1. Description of the Algorithm

We discuss now the main computational result of this article: a deterministic algorithm that for a coefficient field of characteristic zero incrementally transforms into any of the generic positions related to stability⁵ and for a field of positive characteristic p into any of the corresponding p -variants. It performs at each step an *elementary move*, i.e. for a single pair (k, ℓ) of indices with $\ell < k$ we transform $x_k \mapsto x_k + x_\ell$ with all other variables unchanged, so that we obtain a fairly sparse transformation if not too many steps are necessary. Such a move transforms any term x^μ containing x_k into a linear combination of terms of which x^μ is the smallest with respect to the degree reverse lexicographic order (for this reason it is crucial that this order is used). While the algorithm itself is thus fairly simple, it turns out that quite some work is required to prove that it always terminates after a finite number of transformations.

The termination proof is based on the following simple observation. We proceed as in the above discussion of a deterministic way to compute $\text{gin } \mathcal{I}$: a linear coordinate transformation with undetermined coefficients is performed and then a Gröbner system is computed with the coefficients as parameters. By Remark 2.2, any ideal possesses only finitely many different leading ideals under arbitrary linear transformations. We define now an ordering on the set of these leading ideals and then show that our algorithm produces a strictly ascending sequence of leading ideals. Obviously, this implies termination.

⁵With the help of the criterion of Proposition 3.9, this also includes componentwise quasi-stability—see Remark 6.5 below for more details.

Definition 6.1. Let $F \subset \mathcal{P}$ be a finite set of polynomials with leading terms $\text{lt } F = \{t_1, \dots, t_\ell\}$ such that $t_1 \succ_{\text{revlex}} \dots \succ_{\text{revlex}} t_\ell$ where now \prec_{revlex} denotes the pure reverse lexicographic order.⁶ Then we denote the ordered tuple of these leading terms by $\mathcal{L}(F) = (t_1, \dots, t_\ell)$. If $F, \tilde{F} \subset \mathcal{P}$ are two finite sets of polynomials with $\mathcal{L}(F) = (t_1, \dots, t_\ell)$ and $\mathcal{L}(\tilde{F}) = (\tilde{t}_1, \dots, \tilde{t}_{\tilde{\ell}})$, then we define an ordering by setting

$$\mathcal{L}(F) \prec_{\mathcal{L}} \mathcal{L}(\tilde{F}) \iff \begin{cases} \exists j \leq \min(\ell, \tilde{\ell}) \forall i < j : t_i = \tilde{t}_i \wedge t_j \prec_{\text{revlex}} \tilde{t}_j \quad \text{or} \\ \forall j \leq \min(\ell, \tilde{\ell}) : t_j = \tilde{t}_j \wedge \ell < \tilde{\ell}. \end{cases}$$

For notational simplicity, we present our Algorithm 2 for the special case of strongly stable position. If the algorithm terminates, then its correctness is obvious, as the condition in Line 2 just encodes the definition of a strongly stable ideal. The only not so obvious part of the algorithm is the **while** loop in Line 5. It will become later evident why we need it. In fact, it only works, if $\text{char } \mathbb{k} = 0$. We will discuss later the modifications required for positive characteristic.

Algorithm 2 SS-TRAFO: Transformation to strongly stable position

Input: reduced Gröbner basis G of homogeneous ideal $\mathcal{I} \trianglelefteq \mathcal{P}$
Output: a linear change of coordinates Ψ such that $\text{lt } \Psi(\mathcal{I})$ is strongly stable

```

1:  $\Psi := \text{id};$ 
2: while  $\exists g \in G, 1 \leq j \leq n, 1 \leq i < j : x_j \mid \text{lt } g \wedge x_i^{\frac{\text{lt } g}{x_j}} \notin \langle \text{lt } G \rangle$  do
3:    $\psi := (x_j \mapsto x_j + x_i); \Psi = \psi \circ \Psi$ 
4:    $\tilde{G} := \text{REDUCEDGRÖBNERBASIS}(\psi(G))$ 
5:   while  $\mathcal{L}(G) \succeq_{\mathcal{L}} \mathcal{L}(\tilde{G})$  do
6:      $\psi := (x_j \mapsto x_j + x_i); \Psi = \psi \circ \Psi$ 
7:      $\tilde{G} := \text{REDUCEDGRÖBNERBASIS}(\psi(\tilde{G}))$ 
8:   end while
9:    $G := \tilde{G}$ 
10: end while
11: return  $\Psi$ 
```

To apply the algorithm for a different notion of stability, one only has to modify the condition in Line 2 so that it encodes the corresponding stability criterion. Then again the correctness is obvious and the precise nature of the stability criterion will play no role in the termination proof below.

Example 6.2. Let $\mathcal{I} = \langle x_1^3, x_2^3, x_2^2 x_3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. \mathcal{I} is not strongly stable, as $x_1 \frac{x_2^2 x_3}{x_1^3} = x_1 x_2^2 \notin \mathcal{I}$. We perform the coordinate transformation $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ and obtain

$$\text{lt } \Psi_1(\mathcal{I}) = \langle x_1^3, x_1 x_2^2, x_2^3, x_2^2 x_3^3 \rangle.$$

⁶Note that opposed to the *degree* reverse lexicographic order, \prec_{revlex} is not a term order. Since we are, however, exclusively considering homogeneous polynomials, we may always pretend that the leading term has been selected via \prec_{revlex} .

Since $(x_1^3, x_2^3, x_2^2 x_3) \prec_{\mathcal{L}} (x_1^3, x_1 x_2^2, x_2^3, x_2^2 x_3^2)$, we do not enter the `while` loop in Line 5. But $\text{lt } \Psi_1(\mathcal{I})$ is still not strongly stable, as $x_1 \frac{x_1 x_2^2}{x_2} = x_1^2 x_2 \notin \text{lt } (\Psi_1(\mathcal{I}))$. Thus we perform as second coordinate transformation $\Psi_2 : (x_2 \mapsto x_2 + x_1)$ leading to

$$\text{lt } \Psi_2(\Psi_1(\mathcal{I})) = \langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^4, x_1^2 x_3^3 \rangle.$$

Again we do not enter the inner `while` loop, as this time $(x_1^3, x_1 x_2^2, x_2^3, x_2^2 x_3^2) \prec_{\mathcal{L}} (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^4, x_1^2 x_3^3)$. Now there are no obstructions left, i.e. $\text{lt } \Psi_2(\Psi_1(\mathcal{I}))$ is strongly stable (in this case we even have $\text{lt } \Psi_2(\Psi_1(\mathcal{I})) = \text{gin } \mathcal{I}$).

The next example shows explicitly that the result of Algorithm 2 is not unique. More precisely, in the outer `while` loop one finds generally more than one obstruction (i, j) and each choice will lead to a different transformations.

Example 6.3. Let $\mathcal{I} = \langle x_1^2, x_1 x_2, x_2 x_3, x_2^3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$. Since both $x_1 \frac{x_2 x_3}{x_2} = x_1 x_3$ and $x_2 \frac{x_2 x_3}{x_3} = x_2^2$ are not contained in \mathcal{I} , we have the choice to perform either $\Psi_1 : (x_2 \mapsto x_2 + x_1)$ or $\Psi_2 : (x_3 \mapsto x_3 + x_1)$. Since

$$\begin{aligned} \text{lt } \Psi_1(\mathcal{I}) &= \langle x_1^2, x_1 x_2, x_1 x_3, x_2^3, x_2^2 x_3 \rangle \\ \text{lt } \Psi_2(\mathcal{I}) &= \langle x_1^2, x_1 x_2, x_2^3, x_2 x_3 \rangle, \end{aligned}$$

we see that applying Ψ_1 directly leads to a strongly stable ideal whereas $\text{lt } \Psi_2(\mathcal{I})$ is still not strongly stable ($x_1 \frac{x_2 x_3}{x_2} = x_1 x_3^2$ is not contained). But

$$\text{lt } \Psi_1(\Psi_2(\mathcal{I})) = \langle x_1^2, x_1 x_2, x_2^2, x_1 x_3^2 \rangle$$

is strongly stable and not equal to $\text{lt } \Psi_1(\mathcal{I})$.

Remark 6.4. Although in this article we are only concerned with the principal question of deterministically obtaining generic positions, we want to comment briefly on some efficiency issues. In a concrete computer realisation of Algorithm 2, any optimisation will aim at reducing either the number of checks for obstructions or the total number of transformations. One can think of quite a number of natural strategies to achieve these goals. However, for each of them one can provide counter examples (Schweinfurter, 2016, Sect. 2.3), so that none of them is always successful. The relative merits of these strategies can thus be assessed only in extensive benchmarks.

We consider here only one particularly natural strategy, namely to attack always the obstructions of lowest degree. The logic behind this strategy is the expectation that no transformation introduces obstructions in lower degrees and thus that each degree must be considered only once. However, this expectation is wrong, as the following example demonstrates. Consider the ideal $\mathcal{I} = \langle x_1^3, x_1^2 x_2 + x_2^3, x_2^2 x_3 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3]$ with leading ideal

$$\text{lt } \mathcal{I} = \langle x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2^3, x_2^3 x_3, x_2^5 \rangle.$$

There are no obstructions in degree 3, which is the lowest degree of a generator. But since $x_2 \frac{x_2^3 x_3}{x_3} = x_2^4 \notin \text{lt } \mathcal{I}$, there exists one in degree 4. We can remove it by applying the transformation $\Psi : (x_3 \mapsto x_3 + x_2)$. The new leading ideal

$$\text{lt } \Psi(\mathcal{I}) = \langle x_1^3, x_1^2 x_2, x_2^3, x_1^2 x_3^2 \rangle$$

has no obstructions in degree 4 or 5, which is the highest degree of a minimal generator. But $\text{lt } \Psi(\mathcal{I})$ is not strongly stable, since now an obstruction appears in degree 3: $x_1 \frac{x_2^3}{x_2} = x_1 x_2^2 \notin \text{lt } \Psi(\mathcal{I})$.

Remark 6.5. The definition of a componentwise quasi-stable position is quite different from the one of a quasi-stable position, as it uses the component ideals $\mathcal{I}_{\langle d \rangle}$ (which are truly polynomial) instead of the monomial ideal $\text{lt } \mathcal{I}$. Thus a straightforward algorithm for obtaining a componentwise quasi-stable position would analyse all these ideals simultaneously which is very expensive. Our results in Section 3.3 allow us to modify Algorithm 2 in such a way that it can be directly used for this task.

First of all, we use the obvious variant of Algorithm 2 to put \mathcal{I} into a quasi-stable position. Then we start Algorithm 2 again with the condition in Line 2 replaced by the sufficient criterion derived in Proposition 3.9. The implementation of this criterion requires two further modifications: Instead of reduced Gröbner bases we compute Pommaret bases in the Lines 4 and 7 (their finiteness is ensured, as we are in a quasi-stable position) and this computation must be performed in such a way that we also obtain all the syzygies corresponding to the involutive standard representations (10).

As already mentioned in Remark 3.10, we can similarly transform into a componentwise (strongly) stable position. We only have to put \mathcal{I} in the first step into a (strongly) stable position. Then we can use the same modified algorithm as for a componentwise quasi-stable position.

6.2. The Termination Proof

Let $F = \{f_1, \dots, f_\ell\} \subset \mathcal{P}$ be a finite set of polynomials. We call F *completely autoreduced*, if no term contained in the support of a polynomial f_i is divisible by a leading term $\text{lt } f_j$ with $j \neq i$. F is *head autoreduced*, if no leading term $\text{lt } f_i$ is divisible by another leading term $\text{lt } f_j$. By an obvious algorithm, any set F can be rendered either completely or head autoreduced. We denote the results by F^\blacktriangleleft and by F^\triangleleft , respectively. Furthermore, if $0 \neq f \in \mathcal{P}$ is an arbitrary non-vanishing polynomial and $t \in \text{supp}(f)$ a term appearing in it, then we denote the coefficient of t in f by $C_f(t)$.

Lemma 6.6. *Let $F \subseteq \mathcal{P}$ be a completely autoreduced set of polynomials. Let $\Psi : (x_j \mapsto x_j + ax_i)$ be a linear coordinate transformation with $i < j$ and a parameter $a \in \mathbb{k}^\times$. If the field \mathbb{k} possesses more than $2 \deg F$ elements, then there exists a value a such that*

$$\mathcal{L}(F) \preceq_{\mathcal{L}} \mathcal{L}(\Psi(F)^\triangleleft).$$

If \mathbb{k} is an infinite field, then this inequality will hold for any (Zariski) generic choice of the parameter a .

PROOF. We order $F = \{f_1, \dots, f_\ell\}$ such that $\text{lt } f_k \prec_{\text{revlex}} \text{lt } f_l$ whenever $k > l$. Furthermore we set $t_k = \text{lt } f_k$ and $s_k = \text{lt } \Psi(f_k)$ for each k . Without loss of generality, we assume that $\text{lc } f_k = 1$ for each k . It is easy to see that $t_k \preceq_{\text{revlex}} s_k$ for all k , as $i < j$. If $t_k = s_k$ for all k , then there is nothing to prove, since then $\text{lt } F = \text{lt } \Psi(F) = \text{lt } \Psi(F)^\Delta$. Otherwise let α be the smallest index such that $t_\alpha \neq s_\alpha$. In other words: $t_k = s_k$ for all $k < \alpha$, $t_\alpha \prec_{\text{revlex}} s_\alpha$ and $t_k \preceq_{\text{revlex}} s_k$ for all $k > \alpha$. Let h_α be the remainder of $\Psi(f_\alpha)$ after reducing it by the set $\{\Psi(f_1), \dots, \Psi(f_{\alpha-1})\}$ — note that this set is head but in general not completely auto-reduced. We want to show that $t_\alpha \in \text{supp}(h_\alpha)$, as then obviously $\text{lt } h_\alpha \succeq_{\text{revlex}} t_\alpha$.

If $h_\alpha = \Psi(f_\alpha)$, we are done, since then $t_\alpha \in \text{supp}(\Psi(f_\alpha))$. Otherwise there exists an index $\beta < \alpha$ such that $s_\beta = t_\beta$ divides s_α . So the question arises whether or not t_α remains in the support of

$$h_\beta = \Psi(f_\alpha) - \frac{\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)s_\alpha}{\mathbf{C}_{\Psi(f_\beta)}(t_\beta)t_\beta} \Psi(f_\beta).$$

Let us assume that this was not the case. Hence in $\Psi(f_\beta)$ a monomial $m_\beta = \mathbf{C}_{\Psi(f_\beta)}(t_{m_\beta})t_{m_\beta}$ exists which causes the cancellation of t_α . Clearing denominators, we arrive thus at the equality

$$\mathbf{C}_{\Psi(f_\alpha)}(t_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_\beta)t_\alpha t_\beta = \mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{m_\beta})s_\alpha t_{m_\beta}. \quad (16)$$

We analyse now the appearing coefficients as elements of $\mathbb{k}[a]$, i. e. as polynomials in the parameter a . Because of the form of the transformation Ψ , the term 1 is contained in both $\text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(t_\alpha))$ and $\text{supp}(\mathbf{C}_{\Psi(f_\beta)}(t_\beta))$ and hence also in $\text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(t_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_\beta))$. But our assumption $s_\alpha \succ_{\text{revlex}} t_\alpha$ implies that $1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha))$ and thus $1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{m_\beta}))$. This argument shows that as polynomials in a the two decisive coefficients $\mathbf{C}_{\Psi(f_\alpha)}(t_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_\beta)$ and $\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{m_\beta})$ cannot be equal. For any value a outside the set

$$\mathcal{V}(\mathbf{C}_{\Psi(f_\alpha)}(t_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_\beta) - \mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{m_\beta})) \subseteq \mathbb{k}$$

therefore the equality (16) cannot hold which contradicts our assumption that $t_\alpha \notin \text{supp}(h_\beta)$. Each coefficient in $\Psi(f_\alpha)$ is a polynomial in a with its degree bounded by $\deg f_\alpha$ and analogously for $\Psi(f_\beta)$. Thus there are at most $2\deg F$ “bad” values a and for a sufficiently large field \mathbb{k} we can always find a “good” one.

Clearing denominators in the equation for the coefficient of t_α in h_β , we obtain the equality

$$\mathbf{C}_{\Psi(f_\beta)}(t_\beta)\mathbf{C}_{h_\beta}(t_\alpha) = \mathbf{C}_{\Psi(f_\beta)}(t_\beta)\mathbf{C}_{\Psi(f_\alpha)}(t_\alpha) - \mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{m_\beta}).$$

With the arguments from above, we find $1 \in \text{supp}(\mathbf{C}_{\Psi(f_\beta)}(t_\beta)\mathbf{C}_{h_\beta}(t_\alpha))$ and thus

$$1 \in \text{supp}(\mathbf{C}_{h_\beta}(t_\alpha)). \quad (17)$$

If already $h_\beta = h_\alpha$, we are done. Otherwise there exists an index $\gamma < \alpha$ such that $s_\gamma = t_\gamma$ divides $\text{lt } h_\beta = t_{h_\beta}$. The existence of such a divisor shows that t_{h_β} cannot be equal to t_α since F is a completely autoreduced set—note that we could not argue like this if F was only head autoreduced—and therefore

$$t_{h_\beta} \succ_{\text{revlex}} t_\alpha. \quad (18)$$

As above we must show that t_α remains in the support of

$$h_\gamma = h_\beta - \frac{\mathbf{C}_{h_\beta}(t_{h_\beta})t_{h_\beta}}{\mathbf{C}_{\Psi(f_\gamma)}(t_\gamma)t_\gamma} \Psi(f_\gamma).$$

Let us assume that this was not the case. Hence in $\Psi(f_\gamma)$ a monomial $m_\gamma = \mathbf{C}_{\Psi(f_\gamma)}(t_{m_\gamma})t_{m_\gamma}$ exists such that—after clearing denominators—

$$\mathbf{C}_{h_\beta}(t_\alpha)\mathbf{C}_{\Psi(f_\gamma)}(t_\gamma)t_\alpha t_\gamma = \mathbf{C}_{h_\beta}(t_{h_\beta})\mathbf{C}_{\Psi(f_\gamma)}(t_{m_\gamma})t_{h_\beta}t_{m_\gamma}. \quad (19)$$

Let us again analyse the coefficients. As above, we immediately find that $1 \in \text{supp}(\mathbf{C}_{\Psi(f_\gamma)}(t_\gamma))$ because of the form of the transformation Ψ . In (17) we already saw that $1 \in \text{supp}(\mathbf{C}_{h_\beta}(t_\alpha))$, hence $1 \in \text{supp}(\mathbf{C}_{h_\beta}(t_\alpha)\mathbf{C}_{\Psi(f_\gamma)}(t_\gamma))$. We are done, if we are able to show that

$$1 \notin \text{supp}(\mathbf{C}_{h_\beta}(t_{h_\beta})), \quad (20)$$

as then $1 \notin \text{supp}(\mathbf{C}_{h_\beta}(t_{h_\beta})\mathbf{C}_{\Psi(f_\gamma)}(t_{m_\gamma}))$ and so again the equality (19) cannot hold for all values a in a sufficiently large field \mathbb{k} .

To show (20), we recall the construction of h_β ,

$$h_\beta = \Psi(f_\alpha) - \frac{\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)s_\alpha}{\mathbf{C}_{\Psi(f_\beta)}(t_\beta)t_\beta} \Psi(f_\beta),$$

which implies the equality

$$\mathbf{C}_{h_\beta}(t_{h_\beta})\mathbf{C}_{\Psi(f_\beta)}(t_\beta) = \mathbf{C}_{\Psi(f_\alpha)}(t_{h_\beta})\mathbf{C}_{\Psi(f_\beta)}(t_\beta) - \mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{h_\beta}). \quad (21)$$

On one hand we note that $1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(t))$ for all terms $t \in \text{supp}(\Psi(f_\alpha))$ with $t \succ_{\text{revlex}} t_\alpha$. Thus, since $t_{h_\beta} \succ_{\text{revlex}} t_\alpha$ by (18), it follows that if $t_{h_\beta} \in \text{supp}(\Psi(f_\alpha))$, then $1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(t_{h_\beta}))$ and therefore

$$1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(t_{h_\beta})\mathbf{C}_{\Psi(f_\beta)}(t_\beta)).$$

On the other hand, $1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha))$ as we have seen above and so

$$1 \notin \text{supp}(\mathbf{C}_{\Psi(f_\alpha)}(s_\alpha)\mathbf{C}_{\Psi(f_\beta)}(t_{h_\beta})).$$

Since at least one of the coefficients $\mathbf{C}_{\Psi(f_\alpha)}(t_{h_\beta})$ and $\mathbf{C}_{\Psi(f_\beta)}(t_{h_\beta})$ must be nonzero, we conclude from (21) that $1 \notin \text{supp}(\mathbf{C}_{h_\beta}(t_{h_\beta})\mathbf{C}_{\Psi(f_\beta)}(t_\beta))$. Now (20) follows from the fact that $1 \in \text{supp}(\mathbf{C}_{\Psi(f_\beta)}(t_\beta))$.

We can repeat this procedure for each reduction step until we end up at the final result h_α and the arguments imply then that $t_\alpha \in \text{supp}(h_\alpha)$. Hence either $t_\alpha \prec_{\text{revlex}} t_{h_\alpha}$ or $t_\alpha = t_{h_\alpha}$. Let us first assume that $t_\alpha \prec_{\text{revlex}} t_{h_\alpha}$. It is not clear that the set $\{\Psi(f_1), \dots, \Psi(f_{\alpha-1}), h_\alpha\}$ is head autoreduced, as it could happen that there is an index $\delta < \alpha$ such that $t_{h_\alpha} \mid s_\delta = t_\delta$. Since $t_{h_\alpha} \neq t_\delta$ by the construction of h_α , we know that $t_{h_\alpha} \succ_{\text{revlex}} t_\delta$. In this case we check whether or not the set $\{\Psi(f_1), \dots, \Psi(f_{\delta-1}), h_\alpha\}$ is head autoreduced. If it is not, then there is an index $\epsilon < \delta$ such that $t_{h_\alpha} \mid s_\epsilon = t_\epsilon$ and we check again whether or not the set $\{\Psi(f_1), \dots, \Psi(f_{\epsilon-1}), h_\alpha\}$ is head autoreduced. We continue like this until we reach an index $\zeta < \epsilon$ such that the set $\{\Psi(f_1), \dots, \Psi(f_{\zeta-1}), h_\alpha\}$ is head autoreduced. It is still not clear whether this set is a subset of $\Psi(F)^\Delta$, but we can see that $\text{lt } f_\zeta \prec_{\text{revlex}} \text{lt } h_\alpha$ and thus

$$\mathcal{L}(f_1, \dots, f_\zeta) \prec_{\mathcal{L}} \mathcal{L}(\Psi(f_1), \dots, \Psi(f_{\zeta-1}), h_\alpha).$$

If $\Psi(F)^\Delta = \{\hat{f}_1, \dots, \hat{f}_{\hat{m}}\}$, then of course

$$\mathcal{L}(\Psi(f_1), \dots, \Psi(f_{\zeta-1}), h_\alpha) \preceq_{\mathcal{L}} \mathcal{L}(\hat{f}_1, \dots, \hat{f}_\zeta)$$

and this inequality suffices to prove our claim $\mathcal{L}(F) \prec_{\mathcal{L}} \mathcal{L}(\Psi(F)^\Delta)$.

There remains the case $t_\alpha = t_{h_\alpha}$. Now we have to look for the smallest index $\alpha' > \alpha$ such that $t_{\alpha'} \neq s_{\alpha'}$. Then we reduce $\Psi(f_{\alpha'})$ by the set

$$\{\Psi(f_1), \dots, \Psi(f_{\alpha-1}), h_\alpha, \Psi(f_{\alpha+1}), \dots, \Psi(f_{\alpha'-1})\} \quad (22)$$

to the polynomial $h_{\alpha'}$ in the same way as above — note that (22) is head autoreduced since the leading terms did not change in comparison to the completely autoreduced set F . It is clear that if we go on like this, then we will either end up at $\Psi(F)^\Delta$ with $\text{lt } \hat{f}_k = \text{lt } f_k$ for all k which would mean that $\mathcal{L}(F) = \mathcal{L}(\Psi(F)^\Delta)$ or we find a generator h_ω with $t_\omega \prec_{\text{revlex}} t_{h_\omega}$ which finishes our proof. \square

Lemma 6.7. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be an ideal and G its reduced Gröbner basis. Let $\Psi : (x_j \mapsto x_j + ax_i)$ be a linear coordinate transformation with $i < j$ and a parameter $a \in \mathbb{k}^\times$. Furthermore, let \tilde{G} be the reduced Gröbner basis of the transformed ideal $\Psi(\mathcal{I})$. Then*

$$\mathcal{L}(\Psi(G)^\Delta) \preceq_{\mathcal{L}} \mathcal{L}(\tilde{G}).$$

PROOF. Suppose that $\mathcal{L}(\Psi(G)^\Delta) = (t_1, \dots, t_\ell)$ and $\mathcal{L}(\tilde{G}) = (\tilde{t}_1, \dots, \tilde{t}_{\tilde{\ell}})$. By definition of a Gröbner basis, there exists for any leading term $t_k \in \text{lt}(\Psi(G)^\Delta) \subseteq \text{lt}(\Psi(G)^\Delta) = \langle \text{lt } \tilde{G} \rangle$ a generator $\tilde{g}_k \in \tilde{G}$ such that $\text{lt } \tilde{g}_k$ divides t_k and therefore⁷ $\text{lt } \tilde{g}_k \succeq_{\text{revlex}} t_k$. Now we compare the two lists beginning with the first entry.

Let $\text{lt } \tilde{g}_1 = \tilde{t}_\alpha$. If $\alpha > 1$, we are done since then $\tilde{t}_1 \succ_{\text{revlex}} \tilde{t}_\alpha = \text{lt } \tilde{g}_1 \succeq_{\text{revlex}} t_1$. So we assume $\text{lt } \tilde{g}_1 = \tilde{t}_1$. We are again done, if $\tilde{t}_1 \succ_{\text{revlex}} t_1$. Thus we further

⁷As \prec_{revlex} is not a term order, it shows a quite different behaviour compared to the partial order defined by divisibility: $s \mid t$ trivially implies $s \succeq_{\text{revlex}} t$.

assume that $t_1 = \tilde{t}_1$ and go on with the next entry. We note that $\tilde{g}_1 \neq \tilde{g}_2$, since otherwise $t_1 = \tilde{t}_1 = \text{lt } \tilde{g}_1 = \text{lt } \tilde{g}_2$ divides t_2 which contradicts $\Psi(G)^\Delta$ being head autoreduced. Now we have to check which position $\text{lt } \tilde{g}_2 = \tilde{t}_\beta$ has in the list $\mathcal{L}(\tilde{G})$. Since \tilde{G} is reduced $\text{lt } \tilde{g}_1 \neq \text{lt } \tilde{g}_2$ and therefore $\beta > 1$. If $\beta > 2$, we again have the situation $\tilde{t}_2 \succ_{\text{revlex}} \tilde{t}_\beta = \text{lt } (\tilde{g}_2) \succeq_{\text{revlex}} t_2$ and are done. Otherwise $\beta = 2$ and so either $\tilde{t}_2 \succ_{\text{revlex}} t_2$ or $\tilde{t}_2 = t_2$. In the first case, our assertion follows and in the second one we go on with the next entry. Thus sooner or later we either find an index ω with $\tilde{t}_\omega \succ_{\text{revlex}} t_\omega$ which shows that $\mathcal{L}(\Psi(G)^\Delta) \prec_{\mathcal{L}} \mathcal{L}(\tilde{G})$ or

$$\tilde{t}_k = t_k \text{ for all } k \leq \min(\tilde{\ell}, \ell) \quad (23)$$

Assuming that (23) holds, we note that since \tilde{G} is a Gröbner basis of $\langle \Psi(G)^\Delta \rangle$ and both $\Psi(G)^\Delta$ and \tilde{G} are reduced sets we must have $\ell \leq \tilde{\ell}$. Hence it follows that $\mathcal{L}(\Psi(G)^\Delta) = \mathcal{L}(\tilde{G})$ if $\ell = \tilde{\ell}$ and $\mathcal{L}(\Psi(G)^\Delta) \prec_{\mathcal{L}} \mathcal{L}(\tilde{G})$ if $\ell < \tilde{\ell}$. \square

The next, rather elementary lemma studies the effect of our basic coordinate transformations on a polynomial. It encapsulates the dependence of our approach on the characteristic of the base field \mathbb{k} and shows why for a positive characteristic in general only the p -version of our stability notions are reachable: some terms simply cannot be generated by linear coordinate transformations.

Lemma 6.8. *Let $f \in \mathcal{P} \setminus \mathbb{k}$ be a non-constant polynomial and $\Psi : (x_j \mapsto x_j + ax_i)$ a linear coordinate transformation with $i < j$ and a parameter $a \in \mathbb{k}^\times$. Furthermore, let $\mathbf{x}^\mu \in \text{supp}(f)$ be a term in the support of f with $\mu_j > 0$. If $\text{char } \mathbb{k} = 0$, then, for a generic choice of a , all terms of the form $x_i^{\mu_j-s} \mathbf{x}^\mu / x_j^{\mu_j-s}$ with $1 \leq s \leq \mu_j$ appear in the support of $\Psi(f)$. If $\text{char } \mathbb{k} = p > 0$ and \mathbb{k} has more than $\deg f$ elements, then for each term of this form with $s \prec_p \mu_j$ at least one value of a exists such that the term appears in $\text{supp}(\Psi(f))$.*

PROOF. An arbitrary term $\mathbf{x}^\nu \in \text{supp } f$ is transformed into the polynomial

$$\Psi(\mathbf{x}^\nu) = \sum_{s=0}^{\nu_j} \binom{\nu_j}{s} a^{\nu_j-s} x_i^{\nu_j-s} \frac{\mathbf{x}^\nu}{x_j^{\nu_j-s}}. \quad (24)$$

Thus all terms in the transformed polynomial $\Psi(f)$ have as coefficients polynomials in $\mathbb{k}[a]$ of degree at most $\deg f$. Now we analyse the coefficients of the terms $x_i^{\mu_j-s} \mathbf{x}^\mu / x_j^{\mu_j-s}$. Each of these terms appears in $\Psi(x^\mu)$ with coefficient a^{μ_j-s} . If such a term also appears in $\Psi(\mathbf{x}^\nu)$ with $\nu \neq \mu$, then the exponent vector ν must satisfy $\nu_k = \mu_k$ for all $k \neq i, j$ and $\nu_i + \nu_j = \mu_i + \mu_j$. This implies that the coefficient a^{ν_j-s} is different from a^{μ_j-s} . Hence none of the terms we consider has a zero polynomial as coefficient. It is now straightforward to verify our assertion. \square

Proposition 6.9. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be an ideal and G its reduced Gröbner basis. Assume that for a generator $g \in G$ with $\text{lt } g = \mathbf{x}^\mu$ there exist indices i, j with $i < j$*

and $\mu_j > 0$ and an exponent $1 \leq s \leq \mu_j$ (satisfying additionally $s \prec_p \mu_j$ if $\text{char } \mathbb{k} = p > 0$) such that

$$x_i^{\mu_j-s} \frac{\text{lt } g}{x_j^{\mu_j-s}} \notin \text{lt } \mathcal{I}. \quad (25)$$

If $\text{char } \mathbb{k} > 0$, assume in addition that \mathbb{k} contains more than $\deg g$ elements. Finally, let $\Psi : (x_j \mapsto x_j + ax_i)$ be a linear coordinate transformation with a parameter $a \in \mathbb{k}^\times$ and \tilde{G} the reduced Gröbner basis of the transformed ideal $\Psi(\mathcal{I})$. Then there exists at least one value $a \in \mathbb{k}^\times$ such that

$$\mathcal{L}(G) \prec_{\mathcal{L}} \mathcal{L}(\tilde{G}).$$

In the case of an infinite coefficient field \mathbb{k} , this estimate holds for a (Zariski) generic choice of a .

PROOF. Lemmata 6.6 and 6.7, respectively, assert that

$$\mathcal{L}(G) \preceq_{\mathcal{L}} \mathcal{L}(\Psi(G)^\Delta) \preceq_{\mathcal{L}} \mathcal{L}(\tilde{G}).$$

To prove our assertion, we show that (25) implies that $\mathcal{L}(G) \neq \mathcal{L}(\Psi(G)^\Delta)$ for a suitable choice of the parameter a . Let us assume that this was not the case. Further let $G = \{g_1, \dots, g_\ell\}$ and $\Psi(G)^\Delta = \{\hat{g}_1, \dots, \hat{g}_\ell\}$. Without loss of generality, suppose that $\text{lt } g_k \prec_{\text{revlex}} \text{lt } g_l$ and $\text{lt } \hat{g}_k \prec_{\text{revlex}} \text{lt } \hat{g}_l$ if $k > l$. Our assumption implies that $\text{lt } g_k = \text{lt } \hat{g}_k$ for all k . Suppose that $g = g_r$ with $\text{lt } g_r = \mathbf{x}^\mu$ and denote $t = x_i^{\mu_j-s} \text{lt } g_r / x_j^{\mu_j-s}$. For $s = \mu_j$ the term t was equal to $\text{lt } g_r \in \text{lt } \mathcal{I}$ contradicting (25). Thus we may assume $s < \mu_j$ and then for the reverse lexicographic order $\text{lt } g_r \prec_{\text{revlex}} t$.

Lemma 6.8 asserts that for a suitable choice of a every term of the form $x_i^{\mu_j-\hat{s}} \text{lt } g_r / x_j^{\mu_j-\hat{s}}$ with $0 \leq \hat{s} \leq \mu_j$ lies in the support of $\Psi(g_r)$, in particular $t \in \text{supp}(\Psi(g_r))$. Since $\text{lt } g_r = \text{lt } \hat{g}_r$, any term in $\Psi(g_r)$ that is greater than $\text{lt } g_r$ must be reduced. Since t is one of these terms, there must be an element in $\{\text{lt } g_1, \dots, \text{lt } g_\ell\}$ that divides t . But this means that $t \in \langle \text{lt } g_1, \dots, \text{lt } g_\ell \rangle = \text{lt } \mathcal{I}$ which is a contradiction to (25). \square

Remark 6.10. Proposition 6.9 encapsulates the central part of our termination proof. As mentioned above, it is formulated for the case of strongly stable position. Indeed, (25) simply represents an obstruction to strong stability of the leading ideal $\text{lt } \mathcal{I}$ (for $\text{char } \mathbb{k} = p > 0$ to strong p -stability). With suitable adaptations, one easily obtains analogous propositions for any of the stable positions introduced in Section 3.

Theorem 6.11. *If $\text{char } \mathbb{k} = 0$, then Algorithm 2 terminates after finitely many steps and returns a coordinate transformation Ψ such that $\Psi(\mathcal{I})$ is in strongly stable position.*

PROOF. Let \mathcal{I} be the given ideal and G its reduced Gröbner basis. According to Remark 2.2, \mathcal{I} has only finitely many different leading ideals under linear

coordinate transformations. We denote the minimal bases of these leading ideals by B_1, \dots, B_ℓ and assume without loss of generality that

$$\mathcal{L}(B_1) \prec_{\mathcal{L}} \cdots \prec_{\mathcal{L}} \mathcal{L}(B_\ell).$$

In particular, there must be an index $1 \leq \alpha \leq \ell$ such that $\text{lt } \mathcal{I} = \langle B_\alpha \rangle$ and thus $\mathcal{L}(G) = \mathcal{L}(B_\alpha)$.

If $\text{lt } \mathcal{I}$ is not strongly stable, there exists a generator $g \in G$ and integers $i, j \in \{1, \dots, n\}$ with $i < j$ such that x_j divides $\text{lt } g = \mathbf{x}^\mu$ and $x_i \text{lt}(g)/x_j \notin \text{lt } \mathcal{I}$. Consider the transformation $\Psi_1 : (x_j \mapsto x_j + x_i)$ and let \tilde{G}_1 be the reduced Gröbner basis of the transformed ideal $\Psi_1(\mathcal{I})$. There is an index $1 \leq \beta \leq \ell$ such that $\text{lt } \Psi_1(\mathcal{I}) = \langle B_\beta \rangle$ and thus $\mathcal{L}(\tilde{G}_1) = \mathcal{L}(B_\beta)$. If $a = 1$ is a generic value in Proposition 6.9, then $\alpha < \beta$. Otherwise, we enter the `while` loop in line 5 and perform the transformation Ψ_1 a second time. The two transformations together are equivalent to the single transformation $(x_j \mapsto x_j + 2x_i)$. Thus the effect of the inner `while` loop is that we try for the parameter a consecutively the values $1, 2, 3, \dots$. We know from Proposition 6.9 that there are only a finite number of “bad” values of a and thus after finitely many iterations we will reach a “good” one. Hence there is an integer r such that the reduced Gröbner basis \tilde{G}_r of $\Psi_1^r(\mathcal{I})$ satisfies $\mathcal{L}(\tilde{G}_r) = \mathcal{L}(B_\gamma)$ with $\alpha < \gamma \leq \ell$.

Since there are only finitely many different leading ideals possible, it is obvious that also the outer `while` loop is iterated only a finite number of times. However, the termination of this loop is equivalent to the fact that the final transformed ideal is in a strongly stable position. \square

Example 6.12. In the situation of the proof of Theorem 6.11 one could be tempted to think that if B_δ is the minimal basis of a strongly stable leading ideal, then all bases B_ϵ with $\epsilon > \delta$ also generate strongly stable ideals. This is, however, not true. Consider the ideal

$$\mathcal{I} = \langle x_1^3, x_1^2 x_2 + x_1 x_2^2 + x_1 x_3^2, x_1^2 x_3, x_1^2 x_4 \rangle \trianglelefteq \mathbb{k}[x_1, x_2, x_3, x_4].$$

Its leading ideal

$$\text{lt } \mathcal{I} = \langle x_1^3, x_1^2 x_2, x_1^2 x_3, x_1^2 x_4, x_1 x_2^3, x_1 x_2^2 x_3, x_1 x_2^2 x_4 \rangle$$

is strongly stable. After the transformation $\Psi : (x_3 \mapsto x_3 + x_2)$, we find

$$\text{lt } \Psi(\mathcal{I}) = \langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_1^2 x_4, x_1^2 x_3^2 \rangle$$

which is no longer strongly stable, as $x_3(x_1^2 x_4)/x_4 = x_1^2 x_3 \notin \text{lt } \Psi(\mathcal{I})$. However, $\text{lt } \mathcal{I} \prec_{\mathcal{L}} \text{lt } \Psi(\mathcal{I})$.

6.3. An Algorithm for Positive Characteristic

The adaption of Algorithm 2 to a field \mathbb{k} of positive characteristic p faces two problems. Firstly, the strategy for the choice of the parameter a realised by the inner `while` loop is no longer valid, as it obviously fails as soon as the

loop is iterated the p th time. If \mathbb{k} is an infinite field, then one uses simply an enumeration of a countable subset of \mathbb{k} , i.e. a procedure that returns for each natural number $\ell \in \mathbb{N}$ a different element $a_\ell \in \mathbb{k}$. In the case of a finite field, one uses an enumeration of the whole field. Then the transformation Ψ_1 in the proof of Theorem 6.11 is defined as $(x_j \mapsto x_j + a_1 x_i)$ and in the loop we do not apply the same transformation again and again, but instead of the transformation Ψ_1^ℓ we use $(x_j \mapsto x_j + a_\ell x_i)$ in the ℓ th iteration.

Secondly, in positive characteristic all our auxiliary statements require that the base field is sufficiently large (this also affects the modified strategy for the inner `while` loop where one needs for each iteration a new field element). In each of the statements, it was straightforward to specify precisely what the minimal required size is and this number could be easily read off from the input data. In the context of Algorithm 2, one can still easily state a bound: the maximal degree of a generator in one of the minimal bases B_i . However, since we do not compute the Gröbner system, we do not know this number. On the other hand, the bounds in the various lemmata and propositions are worst case estimates and will in practice almost never be realised. Hence in an implementation one simply checks in the inner `while` loop whether one still has new field elements to try. If this is not the case, one must perform a field extension.

These two modifications lead to Algorithm 3 for transforming an ideal over a base field \mathbb{k} of characteristic p into strongly p -stable position (i.e. into p -Borel-fixed position). It uses an enumeration procedure `enum` for generating new field elements as discussed above. The proof of its correctness and termination for a sufficiently large field is now completely analogous to the one of Theorem 6.11 and therefore omitted. Again it is straightforward to adapt the algorithm to other notions of p -stability.

Remark 6.13. We mentioned without justification in Section 3 that one does not need a p -version of quasi-stability. The reason is simply that even in positive characteristic one can always reach a quasi-stable position. Indeed, if one considers the behaviour of a single term under the simple transformations we use, i.e. (24), then the idea underlying our algorithms is to replace the old term \mathbf{x}^ν by one of the new terms appearing in $\Psi(\mathbf{x}^\nu)$. For obtaining a (strongly) stable position, the relevant term is generally one “in the middle” of $\Psi(\mathbf{x}^\nu)$ and thus is multiplied by a binomial coefficient which may be zero in positive characteristic (the “ p -versions” are defined in exactly such a way that these terms never become relevant). For obtaining a quasi-stable position, we always need the last term whose binomial coefficient is one. Thus even in positive characteristic we never encounter a problem, provided the field \mathbb{k} is sufficiently large.

6.4. Implementations and Experiments

An efficient implementation of the algorithm described in this work is highly non-trivial, as many aspects have to be considered. A first point concerns the strategy by which the next transformation is chosen, as often several obstructions exist simultaneously and each may propose a different elementary move.

Algorithm 3 BF-TRAFO: Transformation to p -Borel-fixed position with $\text{char } \mathbb{k} = p > 0$

Input: Reduced Gröbner basis G of ideal $\mathcal{I} \trianglelefteq \mathcal{P}$

Output: a linear change of coordinates Ψ such that $\text{lt } \Psi(\mathcal{I})$ is p -Borel-fixed

```

1:  $\Psi := \text{id}$ 
2: while  $\exists g \in G$ ,  $1 \leq j \leq n$ ,  $1 \leq i < j$ ,  $1 \leq s \leq \mu_j$  :
    $x_j \mid \text{lt } g = \mathbf{x}^\mu \wedge \binom{\mu_j}{u} \not\equiv 0 \pmod{p} \wedge x_i \frac{\text{lt } g}{x_j} \notin \langle \text{lt } G \rangle$  do
3:    $k := 1$ ;  $\psi := (x_j \mapsto x_j + \text{enum}(k)x_i)$ 
4:    $\tilde{G} := \text{REDUCEDGRÖBNERBASIS}(\psi(G))$ ;
5:   while  $\mathcal{L}(G) \succeq_{\mathcal{L}} \mathcal{L}(\tilde{G})$  do
6:      $k := k + 1$ 
7:     if  $k > |\mathbb{k}|$  then
8:       error: field too small
9:     else
10:       $\psi := (x_j \mapsto x_j + \text{enum}(k)x_i)$ 
11:       $\tilde{G} := \text{REDUCEDGRÖBNERBASIS}(\psi(G))$ ;
12:    end if
13:   end while
14:    $\Psi := \psi \circ \Psi$ ;  $G := \tilde{G}$ 
15: end while
16: return  $\Psi$ 

```

Then one must decide whether one performs in each iteration only one elementary move or whether one combines several moves into a larger transformation. Obviously, the first approach gives a better chance to preserve sparsity while the second approach might reduce the number of Gröbner bases computations. These two points will require extensive experiments. We have mentioned already above that to many natural strategies one can construct counter examples where it fares badly. Hence only by experiments one can study the average behaviour for classical examples typical for applications.

Finally, one must discuss how these repeated Gröbner bases computations can be done most efficiently. One should note that one always considers the same ideal, however, in different coordinates. Thus the question arises how a Gröbner (or involutive) basis of an ideal in one coordinate system can be efficiently transformed into one for the same ideal expressed in another coordinate system. In particular from an involutive basis, many invariants of the ideal like its Hilbert function can be easily read off and, in principle, one even knows a basis of the first syzygy module (Seiler, 2009b). Thus ideas like a Hilbert-driven Buchberger algorithm (Traverso, 1996) or exploiting syzygies for the detection of reductions to zero (Möller et al., 1992) (see more generally (Eder and Faugère, 2017) for a recent survey on signature based algorithms) can significantly increase the efficiency. Binaei et al. (2016) report on some preliminary results in particular concerning the first point.

As the design of a new specialised algorithm for computing Gröbner or involutive bases is outside of the scope of this work, we only briefly describe the results of four small test computations performed with a prototype implementation⁸ of our algorithm in MAPLE. In this simple implementation at each iteration the first found elementary move is taken (with the leading terms sorted according to our term order). Instead of the strategy described here, a random integer value between -2 and 2 is chosen for the parameter a (in our experience this suffices for small examples as considered here).

The following examples are taking from standard test suites for Gröbner bases computations. They can e.g. be found at <http://invo.jinr.ru/ginr/>. To demonstrate the flexibility of the algorithm, we go in each example for a different generic position.

Example 6.14. The Butcher ideal is generated by seven polynomials in eight variables with degrees up to 4 and of dimension 3. Our implementation finds that the single elementary move $x_8 \mapsto x_8 - x_4$ transform it into Noether position. By comparison, MAGMA's command `NoetherNormalisation` delivers the much denser linear change of coordinates $x_6 \mapsto x_6 - 2x_1 - x_2 - x_3, x_7 \mapsto x_7 + 3x_2 + x_3 + x_5, x_8 \mapsto x_8 - 3x_1 + 4x_2 - 2x_4 + 2x_5 + x_6 + x_7$ using the probabilistic method of (Greuel and Pfister, 2002).

Example 6.15. The Vermeer ideal is generated by four polynomials in six variables with degrees up to 5 and of dimension 3. Our implementation finds a single elementary move $x_6 \mapsto x_6 + x_3$ to transform it into quasi-stable position where one could immediately read off many of its invariants from a Pommaret basis.

Example 6.16. The Noon ideal is generated by four polynomials in five variables of degree 3 and of dimension 1. For putting it into stable position, our implementation produces the following sequence of seven elementary moves: $x_4 \mapsto x_4 + x_1, x_4 \mapsto x_4 + 2x_3, x_3 \mapsto x_3 + x_1, x_3 \mapsto x_3 + 2x_2, x_4 \mapsto x_4 + 2x_3, x_5 \mapsto x_5 - 2x_1, x_5 \mapsto x_5 - x_4$. In total this corresponds to a linear change with the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 1 & 0 \\ -1 & 4 & 4 & -1 & 1 \end{pmatrix}.$$

Thus here we obtain an almost dense lower triangular matrix which more or less represents the worst case for our algorithm. At least the coefficients are very small. By contrast, a call of CoCoA's command `gin` yields usually a linear transformation which consists of a dense lower triangular matrix where each non-zero entry is an integer with five to six digits. This is a typical behaviour for probabilistic approaches.

⁸The code and the used examples are available at <http://amirhashemi.iut.ac.ir/software.html> (we therefore refrain from giving explicitly the generators). To be consistent with the assumptions of this article, we homogenised all examples.

Example 6.17. The Weispfenning94 ideal is generated by three polynomials in four variables with degrees up to 5 and of dimension 2. For putting it into strongly stable position, our implementation produces the following sequence of four elementary moves: $x_2 \mapsto x_2 - x_1$, $x_4 \mapsto x_4 - 2x_3$, $x_3 \mapsto x_3 + x_1$, $x_4 \mapsto x_4 + 2x_3$. In total this corresponds to a linear change with the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Thus this time we end up with a fairly sparse transformation. A probabilistic computation indicates that it actually even yields $\text{gin } \mathcal{I}$.

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