# Deciding Ellipticity by Quantifier Elimination

Werner M. Seiler[1] and Andreas Weber[2]

[1] Interdisziplinäres Zentrum für Wissenschaftliches Rechnen, Universität Heidelberg, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany; E-mail: werner.seiler@iwr.uni-heidelberg.de
[2] Institut für Informatik II, Universität Bonn, Römerstr. 164, 53117 Bonn, Germany; E-mail: weber@cs.uni-bonn.de

**Abstract.** We show how ellipticity of partial differential systems in the sense of Douglis and Nirenberg can be decided algorithmically by quantifier elimination on real closed fields. A concrete implementation based on *MuPAD* and *Redlog* is presented.

## 1 Introduction

An important task in the theory of partial differential equations [15] is their classification into elliptic and hyperbolic systems. The distinction of these two classes[1] is fundamental not only for the theory but also for the numerical analysis, as it decides what kind of conditions (initial or boundary) we should impose. Furthermore, their solutions behave very differently.

For elliptic systems boundary value problems are usually well-posed and their solutions show typically a very high regularity. From an application point of view, they model stationary problems. In hyperbolic systems a distinguished direction ("time") exists and one considers initial value problems for them; thus they represent models for evolutionary problems. Even for regular data their solutions may exhibit shocks.

The classification is a local problem and thus for simplicity we consider here only linear systems. Nonlinear systems must be linearised along some solution and their behaviour may differ along different solutions. In particular, it is possible that a system is elliptic along one solution and hyperbolic along another one.

In order to resolve some issues arising in the reduction to a first order system of such classical elliptic equations like Laplace's equation, Douglis and Nirenberg [7] introduced a notion of ellipticity which we will call in this paper DN-ellipticity and which is based on finding an appropriate set of weights. In this article we will show how an algorithmic search for such weights and hence an effective decision procedure for DN-ellipticity may be performed by quantifier elimination.

In a recent article [17] it was shown that the concept of DN-ellipticity is not really necessary for the analysis of differential equations, as the main reason for the problem that motivated Douglis and Nirenberg is simply an insufficient treatment of overdetermined systems. If a system is DN-elliptic, then its involutive completion is elliptic in the ordinary sense. In fact, already Cosner [4] showed that any system that is DN-elliptic may be transformed into an equivalent first order system which is elliptic in the ordinary sense. Nevertheless, it is sometimes handy to use DN-ellipticity (e. g. as some regularity results are based on it) and for this reason we are treating here this more general notion of ellipticity.

## 2 Elliptic Systems

We consider linear differential operators of the form

$$L = \sum_{0 \leq |\mu| \leq q} A_\mu(x)\partial^\mu \,. \tag{1}$$

[1] Some readers may miss the class of parabolic systems. At the coarse level of our discussion here, parabolicity is a degenerate case of hyperbolicity and only appears when finer notions like strict hyperbolicity are introduced.

Here $\mu = [\mu_1, \dots, \mu_n] \in \mathbb{N}_0^n$ denotes a multi index with $n$ entries, i. e. we have $n$ independent variables $x_i$, and $|\mu| = \mu_1 + \cdots + \mu_n$ is the length of a multi index. The coefficients $A_\mu$ are matrix valued functions which for simplicity are assumed to be smooth and defined in some domain $\Omega \subseteq \mathbb{R}^n$, i. e. $A_\mu \in \left(C^\infty(\Omega)\right)^{k\times m}$. Thus, if $u$ is an $m$-dimensional vector of functions, the equation $Lu = 0$ represents a system of $k$ differential equations for $m$ unknown functions.

We assume that for some multi index $\mu$ with $|\mu| = q$ the matrix $A_\mu$ does not vanish so that $L$ is truly an operator of order $q$. We denote by

$$q_{\tau\alpha} = \max\bigl\{|\mu| \,:\, (A_\mu)_{\tau\alpha} \neq 0\bigr\} \tag{2}$$

the maximal order of a derivative of the $\alpha$th unknown function in the $\tau$th equation of the system $Lu = 0$. We furthermore introduce the notations $q_\tau = \max_\alpha\{q_{\tau\alpha}\}$ for the order of the $\tau$th equation and $\hat{q}_\alpha = \max_\tau\{q_{\tau\alpha}\}$ for the highest order of a derivative of the $\alpha$th unknown function in the whole linear system $Lu = 0$.

**Definition 1.** *Let $s_\tau$ with $1 \le \tau \le k$ and $t_\alpha$ with $1 \le \alpha \le m$ be some integer weights satisfying $s_\tau + t_\alpha \ge q_{\tau\alpha}$. The* weighted principal symbol *of the operator $L$ is the $k \times m$ matrix $\sigma_{(s,t)}L$ with entries*

$$\bigl(\sigma_{(s,t)}L(x,\xi)\bigr)_{\tau\alpha} = \sum_{|\mu|=s_\tau+t_\alpha} \bigl(A_\mu(x)\bigr)_{\tau\alpha}\xi^\mu \tag{3}$$

*where $\xi \in \mathbb{R}^m$ is a real vector.*

The weighted principal symbol may be considered as a matrix whose entries are (homogeneous) polynomials in $\xi$ with smooth functions of $x$ as coefficients. The maximal degree of a polynomial is the order $q$ of the differential operator $L$.

The weighted principal symbol does not change, if we subtract from all weights $s_\tau$ an integer $c$ and add at the same time $c$ to all weights $t_\alpha$. This implies that without loss of generality we may assume that

$$s_\tau \le 0\,, \qquad t_\alpha \ge 0\,. \tag{4}$$

The *classical principal symbol* $\sigma L$ is obtained for the weights

$$s_1 = \cdots = s_k = 0\,, \qquad t_1 = \cdots = t_m = q\,, \tag{5}$$

i. e. it considers only the highest order part of $L$. The *reduced principal symbol* $\sigma_r L$ corresponds to the choice

$$s_\tau = q_\tau - q\,, \qquad t_1 = \cdots = t_m = q\,, \tag{6}$$

i. e. the highest order part of each equation is considered separately. Finally, the *Petrovsky principal symbol* is characterised by the weights

$$s_1 = \cdots = s_k = 0\,, \qquad t_\alpha = \hat{q}_\alpha\,. \tag{7}$$

**Definition 2.** *The linear differential operator $L$ is* DN-elliptic *at the point $x \in \Omega$, if there exists a set of weights $s_\tau$, $t_\alpha$ such that the weighted principal symbol $\sigma_{(s,t)}L(x,\xi)$ has at $x$ full column rank for all non-vanishing vectors $\xi \in \mathbb{R}^n$. The operator $L$ is* elliptic, *if it is DN-elliptic for the weights (5). Finally, $L$ is* P-elliptic *or* elliptic in the sense of Petrovsky, *if it is DN-elliptic for the weights (7).*

Obviously, a necessary condition for DN-ellipticity is $k \ge m$, i. e. that we have more equations than unknown functions. In fact, an underdetermined system can never be DN-elliptic. This appears only natural, as in underdetermined systems one usually does not find the kind of regularity that one expects from elliptic systems. In the case that we are dealing with a square system, i. e. $k = m$, the condition of full column rank in Def. 2 is equivalent to the condition of a non-vanishing determinant of the corresponding principal symbol.

The most common notion of ellipticity is the one based on the standard weights (5). Ellipticity in the sense of Petrovsky is particularly popular in the Russian literature (see [1] and references therein). DN-ellipticity was introduced by Douglis and Nirenberg [7]. Classically, the definition is only for square systems; the extension to overdetermined system is discussed e. g. in the survey [8].

*Example 3.* The prototype of an elliptic equation is *Laplace's equation*: $Lu = \Delta u = u_{xx} + u_{yy} = 0$. Its classical principal symbol is given by[2] $\sigma L = \xi_x^2 + \xi_y^2$. Obviously, $\sigma L$ vanishes only for $\xi = 0$ so that $L$ is indeed an elliptic operator.

Rewriting Laplace's equation as a first order system, we obtain the $3 \times 3$ system $v = u_x$, $w = u_y$ and $v_x + w_y = 0$. Using the weights $s_1 = s_2 = -1$, $s_3 = 0$, $t_1 = 2$, $t_2 = t_3 = 1$, we find for the corresponding operator $L_1$

$$\sigma L_1 = \begin{pmatrix} \xi_x & 0 & 0 \\ \xi_y & 0 & 0 \\ 0 & \xi_x & \xi_y \end{pmatrix}, \qquad \sigma_{(s,t)} L_1 = \begin{pmatrix} \xi_x & -1 & 0 \\ \xi_y & 0 & -1 \\ 0 & \xi_x & \xi_y \end{pmatrix}. \tag{8}$$

As $\det \sigma L_1 \equiv 0$, we obtain the paradoxical result that seemingly the first order form of Laplace's equation is not elliptic, although the reduction to first order does not change the properties of the solution space. In contrast, we obtain for the used weights that $\det \sigma_{(s,t)} L_1 = \xi_x^2 + \xi_y^2 = \sigma L$. Hence the operator $L_1$ is DN-elliptic.

An alternative (and from a theoretical point of view better) solution of this paradox consists of adding the hidden integrability condition $v_y = w_x$. This leads to an involutive $4 \times 3$ differential operator $L_1'$ with principal symbol

$$\sigma L_1' = \begin{pmatrix} \xi_x & 0 & 0 \\ \xi_y & 0 & 0 \\ 0 & \xi_x & \xi_y \\ 0 & \xi_y & -\xi_x \end{pmatrix}. \tag{9}$$

One easily checks that this matrix has full column rank for any non-vanishing vector $\xi \in \mathbb{R}^2$ and thus the completion $L_1'$ is elliptic.

*Remark 4.* A non-vanishing vector $\xi \in \mathbb{R}^n$ such that the principal symbol $\sigma L$ does *not* have full column rank is called a *characteristic* vector for the differential operator $L$. Thus an operator is elliptic, if and only if it does not possess any real characteristic vector.

## 3 Quantifier Elimination

The decidability of the first order theory of the ordered field $\mathbb{R}$ of the real numbers was first shown by Tarski [19]. Moreover, he showed that this theory admits *elimination of quantifiers*, i. e. that an equivalent quantifier free formula can be found for every (not necessarily closed) formula of this theory. Although Tarski's method is constructive, its complexity is prohibitive for any practical purpose, because it is not elementary recursive.

*Example 5.* A standard example demonstrating the use of quantifier elimination over the reals is the following formula, which states the existence of a solution of a quadratic equation:

$$\exists x\, (ax^2 + bx + c = 0)\,.$$

An equivalent quantifier free formula is the following one:

$$(b^2 - 4ac \geq 0) \wedge (b \neq 0 \vee a \neq 0 \vee c = 0)\,.$$

This quantifier free formula states the condition on the parameters, which have to be fulfilled so that the quadratic equation has a real solution.

The first algorithm having an elementary-recursive worst-case time bound was designed by Collins [3]. His algorithm—which was based on a new technique called *cylindrical algebraic decomposition* (CAD)—has a worst case running time which is doubly exponential in the number of

[2] Here the subscripts do not denote derivatives but label the components of the vector $\xi = (\xi_x, \xi_y)^t$ corresponding to the independent variables $x$, $y$.

variables. Many subsequent improvements of this method have culminated in its implementation in the *Qepcad* system [2, 13].

It can be shown [5] that the quantifier elimination problem has a lower bound that is doubly exponential in the number of quantifier alternations. This theoretical result still holds for many special cases such as the one of bound variables appearing only linearly [20]. However, for the latter case Weispfenning [14, 20] devised a new method that is much more efficient for most practically occurring cases. In particular, the complexity of the linear quantifier elimination is no longer dependant on the number of free variables. Similar methods have been developed for the quadratic and cubic case [21, 22]. Efficient implementations of all these algorithms exist in the system *Redlog* [6]. Quite recently, also an implementation of a partial cylindrical algebraic decomposition has been done in *Redlog* [16].

The method of *linear* quantifier elimination on real closed fields can be extended to theories interpreting a function *integer part* [23]. However, as shown in [23], this result does not apply for seemingly similar theories, such as the one interpreting an integer divisibility relation.

In the last decade, several applications of quantifier elimination methods have been found in the context of differential equations, see e. g. [12] for applications to the stability analysis of numerical methods or [9, 10] for reductions of bifurcation problems to quantifier elimination questions.

## 4 Deciding DN-Ellipticity

If we knew a priori the right choice of weights $\bar{s}_\tau$, $\bar{t}_\alpha$, then deciding DN-ellipticity would be very straightforward. Maximal column rank of the weighted principal symbol is equivalent to injectivity of the linear map defined by its matrix. This in turn is equivalent to the triviality of the kernel of the linear map. Thus the operator $L$ is DN-elliptic, if the formula

$$\exists \xi \in \mathbb{R}^n \, \exists w \in \mathbb{R}^m \, : \, \big(\xi \neq 0\big) \wedge \big(w \neq 0\big) \wedge \big(\sigma_{(\bar{s},\bar{t})} L(x,\xi) w = 0\big) \tag{10}$$

is *false*. Here $w \in \mathbb{R}^m$ is an auxiliary real vector on which the symbol matrix $\sigma_{(\bar{s},\bar{t})} L(x,\xi)$ acts. Another point of view is that it contains the coefficients of a linear combination of the columns of $\sigma_{(\bar{s},\bar{t})} L(x,\xi)$ and thus (10) may be interpreted as a check for their linear independence.

Note that in general, i. e. when we are dealing with an operator with variable coefficients, the truth of (10) will depend on $x$ and on the values of parameters, if $L$ contains some. If $L$ depends polynomially on $x$ (i. e. the operator $L$ is actually an element of a module over the Weyl algebra) and on the contained parameters, then the quantifier elimination will automatically determine those values of $x$ and the parameters for which $L$ is DN-elliptic. Otherwise, we can analyse the operator $L$ only at explicitly given points $x$ or for explicitly prescribed parameter values, respectively.

The situation is complicated by the fact that according to Def. 2 the operator $L$ is DN-elliptic, if *some* weights $s_\tau$, $t_\alpha$ exist such that (10) is false. The problem is that the matrix $\sigma_{(s,t)} L(x,\xi)$ may look very differently for different choices of the weights. Depending on whether $s_\tau + t_\alpha > q_{\tau\alpha}$ or $s_\tau + t_\alpha = q_{\tau\alpha}$, the entry in row $\tau$ and column $\alpha$ vanishes or not.

Each row of the weighted principal symbol yields one condition; but we have in general $2^m$ different possibilities for these conditions depending on whether or not the $\alpha$th column contributes a term. All these different cases must be contained as a conjunction of implications in our logical formula. As each case is characterised by $m$ conditions on the weights and the corresponding row equation, the simple equation $\sigma_{(s,t)} L(x,\xi) w = 0$ leads to a logical formula consisting of $k(m+1)2^m$ atomic formulae.

In concrete examples the situation is usually better. Typical differential equations are sparse so that some entries in the weighted principal symbol always vanish independent of the chosen weights. This considerably reduces the number of cases. Furthermore, one of the $2^m$ cases is always trivial and may thus be omitted, namely the case that the whole row vanishes.

*Example 6.* We continue with our discussion of Laplace's equation in Example 3. In its first order form, i. e. for the operator $L'$, we have $m = 3$ and one could expect that each row leads to six different cases. In fact, each row of $L'$ contains only two non-vanishing entries so that there are only four cases one of which is trivial. The first row yields thus the formula

$$\begin{aligned}\big((s_1+t_1=1)\wedge(s_1+t_2=0)\Longrightarrow(w_1\xi_1-w_2=0)\big)\wedge\\ \big((s_1+t_1=1)\wedge(s_1+t_2>0)\Longrightarrow(w_1\xi_1=0)\big)\wedge\\ \big((s_1+t_1>1)\wedge(s_1+t_2=0)\Longrightarrow(-w_2=0)\big)\,.\end{aligned}\tag{11}$$

The formulae for the different rows are logically combined with $\wedge$.

Reduction to first order always leads to very sparse systems, as the majority of the equations only introduces the new unknown functions. For Laplace's equation this is not so noticeable, as anyway only three unknown functions appear in $L_1$. So let us briefly consider the fourth order *biharmonic equation* $\Delta\Delta u = u_{xxxx}+2u_{xxyy}+u_{yyyy}=0$. If we rewrite it as a first order system and add all hidden integrability conditions, then we obtain a system of 16 equations for 10 unknown functions. In general, this would imply that each equation leads to 1024 different cases, each containing 11 atomic expressions. However, with one exception all equations consist only of two terms and thus lead to formulae with the same structure as (11). The exception is the equation obtained from the original equation containing three terms. This yields 52 different cases with a total of 142 atomic formulae.

The formula (10) handles only the situation where we know already the right weights which in general is of course not true (except if we want to decide classical or P-ellipticity which are defined with respect to a fixed set of weights). For DN-ellipticity we must decide whether integer vectors $s\in\mathbb{Z}^k$ and $t\in\mathbb{Z}^m$ exists such that all components of $s$ are non-positive and all components of $t$ are non-negative and the formula (10) is false. Example 8 below presents the resulting lengthy expression for a concrete instance. The additional restrictions $s_\tau+t_\alpha\geq q_{\tau\alpha}$ do not appear explicitly, as they are implicitly encoded in the case distinctions of the form (11).

If one is interested in obtaining concrete values for the weights, one may simply omit the existential quantifiers for the weights. This means that for the quantifier elimination the weights are considered as parameters and the resulting quantifier free formula characterises those weights for which the weighted principal symbol has full column rank.

*Remark 7.* Although we are only concerned with elliptic operators in this paper, it is worth while pointing out that this approach may also be applied for determining the characteristic directions of any differential operator. We must only omit the first quantifier in (10) in order to obtain an implicit description of all characteristic directions. As characteristics are always defined with respect to the classical weights (5), the obtained formula suffices.

Obviously, the formula (10) is linear in the components of the vector $w$ and for a differential operator $L$ of order $q$ its degree in the components of the vector $\xi$ is at most $q$. In particular, for a first order operator we have also linearity in the components of $\xi$. In view of the remarks in the previous section, an obvious question is whether for a higher order operator $L$ it is more efficient to rewrite it as an equivalent first order operator $L_1$. Although this transformation considerably increases the number $m$ of unknown functions (and thus the dimension of the vector $w$) and the number $k$ of rows, it also leads to a quantifier elimination problem with a much lower complexity.

### 4.1 Deciding the integer conditions

The weights $s_\tau$ and $t_\alpha$ are ranging over integer values only. In general, the quantifier elimination problem over the real field with the additional provision of some variables ranging over integer values only is undecidable.

However, in our problems the weights are only occurring in atomic subformulae of the form $s_\tau+t_\alpha\,\mathsf{relsymb}\,q_{\tau\alpha}$, where $q_{\tau\alpha}$ is an integer and relsymb is either an equality or a greater relation. Thus the variables ranging over the integers are separated from all other variables, because they do not occur jointly in the same atomic subformulae. The quantifier elimination methods developed in [20–22] show that in this case the equivalent quantifier free formulae have the property of having these variables in the same atomic subformulae only. Thus by mixed real-integer linear quantifier elimination these variables can be eliminated in a subsequent step.

In fact, the situation is even more favourable. Because of the particularly simple structure of the conditions on the integer weights, real quantifier elimination suffices: if real vectors $s \in \mathbb{R}^k$, $t \in \mathbb{R}^m$ exists for which the ellipticity conditions are satisfied, then among them are always some vectors containing only integer entries.

In order to see this one must distinguishes between equality and greater relations for the weights. Let us consider first the greater relations. For them it is obvious that the existence of real solutions implies the existence of integer solutions, as each of them defines a set isomorphic to a "halfspace". As we are always on the same side of the bordering hyperplane (we have only greater relations with positive right hand sides), the intersection of such sets is either empty or large enough to contain integer vectors.

All equality relations may be solved for the weights $s_\tau$: $s_\tau = q_{\tau\alpha} - t_\alpha$. Hence $s_\tau \in \mathbb{Z}$, if $t_\alpha \in \mathbb{Z}$. As all weights appear in these linear relations with the coefficient 1, one easily verifies that it is not possible to deduce relations of the form $p_1 t_\alpha = p_2$ with coefficients $p_1, p_2 \in \mathbb{Z}$ and $p_1 \neq 1$. One either obtains relations $t_\alpha = p_2 \in \mathbb{Z}$ or consistency conditions on the orders $q_{\tau\alpha}$ whose satisfaction is guaranteed by the quantifier elimination.

## 5 Examples

The computer algebra system *MuPAD*[3] contains an object-oriented environment for the symbolic treatment of differential equations [11]. Within its domain for linear differential equations we implemented a method for setting up the logical formula for DN-ellipticity as a string in the syntax of *Redlog*. Currently, the obtained formula is simply written into a text file and manually entered into *Redlog*.[4] We are working on a direct connection using *MuPAD*'s dynamic modules [18] or an integration in the more general Internet based component architecture that has already been used to access quantifier elimination packages [9].

*Example 8.* As a trivial test case where the full formula can be displayed, we consider a variation of Laplace's equation: the *Tricomi equation* $Lu = u_{yy} - y u_{xx} = 0$. Obviously, it is only elliptic for $y < 0$. The logical formula for its DN-ellipticity is obtained in *MuPAD* as follows:

MuPAD

```
>> LDF := Dom::LinearDifferentialFunction(Vars=[[x,y],[u]],
>>                                        Rest=[Types="Indep"]):
>> tricomi := LDF(u([y,y])-y*u([x,x])):
>> LDF::ellCond(tricomi)
```

Output

```
ex(s1,
ex(t1,
  (s1 <= 0) and
  (0 <= t1) and
  all(xi1,
  all(xi2,
  all(w1,
    not(
      ((xi1 <> 0) or (xi2 <> 0)) and
      ((w1 <> 0)) and
      ((((s1 + t1 = 2)) impl (-w1*(xi1**2*y - xi2**2) = 0)))
    )
  )))
))
```

[3] See `http://www.mupad.de` for more information.

[4] Under Linux this can be automatised using the *MuPAD* `system` command.

The first input line creates a domain for linear differential functions in the two independent variables $x$, $y$ and the unknown function $u$. The coefficients may be arbitrary expressions in $x$ and $y$. The second line defines the Tricomi equation using an abbreviated syntax for the derivatives. Finally, the method `ellCond` is called which generates the formula using default names for the weights and vectors (with additional optional arguments the names for these variables may be prescribed and a file for the output specified). The shown output is not the one obtained in a screen session where a long string with control sequences for the line breaks (which are anyway only introduced for better human readability) is generated but the form in which it appears in an output text file. In fact, we generate not only the logical formula itself but a whole small *Redlog* program for performing quantifier elimination on it.

The generated formula has one free parameter (the variable $y$, as we are dealing with a variable coefficient equation), an inner block of three universally quantified variables and an outer block of two existentially quantified variables and consists of 7 atomic subformulae. The corresponding quantifier free formula was found by *Redlog* in less than 1 sec of computation time and is exactly the one we expect, namely $y < 0$.

The first order form of Tricomi's equation is defined by the operator

$$L_1 = \begin{pmatrix} \partial_x & -1 & 0 \\ \partial_y & 0 & -1 \\ 0 & -y\partial_x & \partial_y \end{pmatrix} . \tag{12}$$

The generated formula has one free parameter, an inner block of 5 universally quantified variables, an outer block of 6 existentially quantified variables and consists of 38 atomic subformulae. The corresponding quantifier free formula was again found by *Redlog* in less than 1 sec of computation time and was of course identically with the one obtained above.

*Example 9.* The next system depends on two real parameters $a$, $d$ and is defined by the operator

$$L = \begin{pmatrix} \partial_x^3 & a & 0 \\ \partial_y^3 & 0 & d \\ \partial_x^2\partial_y & \partial_x^2 & \partial_x\partial_y \\ \partial_x\partial_y^2 & \partial_x\partial_y & \partial_y^2 \end{pmatrix} . \tag{13}$$

The generated formula has two free parameters (the parameters $a, d$ of the operator $L$), an inner block of 5 universally quantified variables and an outer block of 7 existentially quantified variables and consists of 86 atomic subformulae.

In the inner block of quantifiers 3 of 5 universal quantifiers can be eliminated by the special methods for linear and quadratic quantifier elimination. The entire problem cannot be handled by these special algorithms alone, as the obtained equivalent formula at this point of the computation does no longer fulfill the restrictions on the degree. However, one can continue using the general partial cylindrical algebraic decomposition method, and this fallback is automatically done by *Redlog*. Using this hybrid technique an equivalent quantifier free formula is found after a total computation time of 130 sec (on a 1 GHz Pentium III PC running under Linux):

*Redlog*
```
a < 0 and d < 0 or a > 0 and d > 0
```

Thus the differential operator $L$ is elliptic, if and only if $ad > 0$.

The use of the hybrid technique is essential for the success of the computation: when trying to perform quantifier elimination on the input formula by partial cylindrical algebraic decomposition alone neither its implementation in *Qepcad* nor its implementation in *Redlog* have been successful within 24 h of computation time.

If we remove the outer block of existential quantifiers, the result of *Redlog* is a lengthy formula consisting of 88 atomic subformulae. This formula can be simplified by the *Qepcad* based simplification program of C. Brown[5] [2] to the following result:

[5] See `http://www.cs.usna.edu/~qepcad/SLFQ/Home.html`.

Qepcad

```
An equivalent formula is:
[ t1 - 5 >= 0 /\ t2 - t1 + 3 = 0 /\
  t3 - t1 + 3 = 0 /\ s1 + t1 - 3 = 0 /\
  s2 + t1 - 3 = 0 /\ s3 + t2 - 2 = 0 /\
  s4 + t2 - 2 = 0 /\
  [ [ a > 0 /\ d > 0 ] \/ [ a < 0 /\ d < 0 ] ] ]
There were 175 qepcadd runs!
```

The obtained relations for the weights are readily solved. Although we have in principle a one parameter family of solutions ($t_1$ must only satisfy an inequality), all its members are equivalent to the one solution

$$s_1 = s_2 = -2\,,\ s_3 = s_4 = 0\,,\ t_1 = 5\,,\ t_2 = t_3 = 2\,, \tag{14}$$

as all other ones simply correspond to the addition of a positive constant $c$ to all $t_\alpha$ and the subtraction of the same constant from all $s_\tau$. These weights yields the weighted principal symbol

$$\sigma_{(s,t)}L(x,\xi) = \begin{pmatrix} \xi_x^3 & a & 0 \\ \xi_y^3 & 0 & d \\ 0 & \xi_x^2 & \xi_x\xi_y \\ 0 & \xi_x\xi_y & \xi_y^2 \end{pmatrix}\,. \tag{15}$$

One readily checks that for $ad > 0$ this matrix has indeed full column rank for all non-vanishing vectors $\xi \in \mathbb{R}^2$.

The corresponding first order form gives a formula having two free parameters, an inner block of 14 universally quantified variables and an outer block of 29 existentially quantified variables and consisting of 244 atomic subformulae. With 32 MB of allocated memory *Redlog* did not succeed to find a equivalent quantifier free formula. Attempts to make runs with more allocated memory failed due to errors in the memory management of the underlying Reduce system.

Note that from the point of view of the quantifier elimination there is no difference between the analysis of a linear operator with variable coefficients like in Example 8 or of an operator depending on parameters as in the last example. The coefficient $y$ in the Tricomi equation appears like a parameter. In both cases it is crucial that only a polynomial dependency of the parameters is permitted.

In these examples the reduction to first order operators $L_1$ did not give quantifier elimination problems that could be solved more easily than the problems corresponding to the original higher order operator $L$. As this reduction gives a “polynomial growth” of the number of formulae only, this result is not to be expected for all cases, cf. the remarks on the complexity of linear quantifier elimination and cylindrical algebraic decomposition in Section 3. So the fine tuning of algorithms and the use of hybrid methods in *Redlog* seems to be important for the successful elimination of quantifiers in the above examples.

*Example 10.* Finally, we consider the system $Lu = \nabla \times u + u = 0$ where $u$ represents a three-dimensional real vector. The closed formula corresponding to $L$ has 6 existentially and 6 universally quantified variables and consists of 39 atomic subformulae. It is reduced by *Redlog* to the equivalent `false` in less than 1 sec. Thus no weights exist such that the corresponding principal symbol has full column rank and the system is not DN-elliptic.

However, if we add the hidden integrability condition $\nabla \cdot u = 0$, a different picture emerges. The closed formula corresponding to $L'$ has 7 existentially and 6 universally quantified variables and consists of 68 atomic subformulae. It is reduced by *Redlog* to the equivalent `true` in less than 2 sec. Thus weights exist such that $\sigma_w L'$ possesses full column rank for any non-vanishing vector $\xi$ and $L'$ is a DN-elliptic operator. In fact, one readily checks that one solution for the weights consists of the classical weights (5) so that $L'$ is even elliptic.

This example demonstrates that for deciding ellipticity of overdetermined systems it is generally crucial to consider only complete systems, as otherwise one might obtain wrong results. In fact, as [17] shows then one does not even need the more general concept of DN-ellipticity and thus weights. However, DN-ellipticity can often be earlier detected than ordinary ellipticity and for this reason it is still useful to be able to treat this case.

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