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Overdetermined Elliptic Systems

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Abstract We consider linear overdetermined systems of partial differential equations. We show that the introduction of weights classically used for the definition of ellipticity is not necessary, as any system that is elliptic with respect to some weights becomes elliptic without weights during its completion to involution. Furthermore, it turns out that there are systems which are not elliptic for any choice of weights but whose involutive form is nevertheless elliptic. We also show that reducing the given system to lower order or to an equivalent one with only one unknown function preserves ellipticity.

Key words Overdetermined system, partial differential equation, symbol, elliptic system, completion, involution

1 Introduction

The definition of ellipticity for general overdetermined systems is quite rarely found in the literature, one accessible exception being the encyclopaedia article [15, Def. 2.1]. Without the general definition one may encounter conceptual problems already in very simple situations. For instance, consider the transformation of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ to the first order system (this is discussed in the recent textbook [42, Example 2.10]):

$$u_x = v\ ,\quad u_y = w\ ,\quad v_x + w_y = 0\ .$$

The transformed system is *not* elliptic, although it is obviously equivalent to Laplace's equation. The usual approach to resolve this issue [2,3,12] consists of introducing a weighted symbol where two sets of weights are attached to the equations and the dependent variables, respectively. It is straightforward to find weights such that the above first order system becomes elliptic (see Example 6.8 below).

However, a much simpler solution exists: if one adds the integrability condition $v_y = w_x$, one obtains an overdetermined system which is elliptic without weights.

Besides the already mentioned encyclopaedia article [15] and the research monograph [50], the question of defining ellipticity for overdetermined systems was taken up only by few authors [11, 22, 38]. Notable are here in particular the results of Cosner [11] who constructed for any system which is elliptic with weights an equivalent system which is also elliptic without weights. Within the theory of exterior differential systems, Bryant et al. [8, Chapt. V, §2] give a definition of an elliptic Pfaffian system; however, we are not aware of any extension of the approach via weighted symbols to exterior systems.

The purpose of this article is to show that the problems in defining ellipticity are solely related to the presence of hidden integrability conditions. For checking whether a formally integrable or passive system, i. e. a system explicitly containing all its integrability conditions, is elliptic, no weights are needed. It turns out that the main purpose of the weights is to simulate a partial completion: due to the addition of integrability conditions, terms which do not appear in the original symbol will show up in the symbol of the completed system. In some cases, weights can achieve the same effect. However, we will present explicit examples where it is not possible to find any weights such that the original system is elliptic with respect to them, although a completion shows that the system is in fact elliptic.

So the approach via weights has its limitations. On the other hand, the weights do contain some relevant information about the system, as they turn up in a rather natural way in the a priori estimates for systems which are elliptic with weights. Hence it may look like the weights are necessary. However, a completion does not really alter the solution space but only provides another (better) representation of it. Therefore we can readily obtain the same information from the a priori estimates of the completed system. But since these functional analytic considerations are not needed in the present article, we just refer to [4, 15, 50] for details.

The question of completion has attracted much interest since the middle of the 19th century and so many different approaches have been proposed that we can mention only some of the major directions. A more algebraic solution for linear systems[1] stems from Janet [25] and Riquier [43]. Within differential algebra (see [28] for a general introduction) Boulier et al. [6] presented an algorithmic solution for arbitrary ideals of differential polynomials; subsequent developments and improvements are contained in the survey by Hubert [24]. On the geometric side, Cartan [10] (and Kähler [27]) developed the notion of an involutive exterior differential system; some open points in the question of completion were settled by Kuranishi [30]. A modern presentation of this theory with many applications can be found in [8]. Later, ideas from Janet-Riquier and Cartan-Kähler theory, respectively, were merged into the formal theory of partial differential equations (see [14, 29, 37, 39, 45, 49] and references therein).

[1] The Janet-Riquier theory is often also applied to nonlinear systems. However, this requires some assumptions like that all equations (including the hidden integrability conditions appearing during the completion) can be solved for their leading derivatives.

It is perhaps worth while pointing out that the formal theory (or any other of these theories) is not only useful for studying analytic questions like defining ellipticity. As already demonstrated in a number of articles [20, 32, 34, 40, 44, 47, 51–53], completion is also important for a proper *numerical* treatment of overdetermined systems.

All these theories are quite involved with many technical subtleties. Fortunately, our results are independent of any concrete completion procedure, as they are based on analysing the syzygies of the (transposed) principal symbol and any completion procedure must treat, possibly in a rather hidden manner, all such syzygies. Thus in principle we could use any of the above mentioned approaches. Mainly for reasons of personal taste, we will use the language of the formal theory (emphasising its roots in Janet-Riquier theory). However, no deeper knowledge of it is required to understand our proofs; some familiarity with integrability conditions and the idea of completion is completely sufficient.

The article is organised as follows. In Section 2 we collect the necessary background material needed to formulate and prove our theorems; this includes some results from commutative algebra. Section 3 provides a brief introduction to a few basic ideas of the formal theory of differential equations. In Section 4 we make some general remarks about elliptic symbols and discuss their genericity. Section 5 introduces weighted symbols and their elementary properties. In Section 6 we prove our main result stating that given a system elliptic with respect to some weights its involutive form is elliptic without weights. In Section 7 we show that transforming a system to lower order or to an equivalent system with only one dependent unknown function preserves ellipticity. Finally, in Section 8 we conclude with some general remarks.

2 Basic definitions

2.1 Multi indices

Let $\mathbb{N}_0^n$ be the space of multi indices (or exponent vectors), i. e. the set of all ordered n-tuples $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathbb{N}_0$. The multi index where the jth component is one and all other ones vanish is denoted by $\mathbf{1}_j$. The *length* of a multi index is $|\mu| = \mu_1 + \cdots + \mu_n$. For a given μ and the variables $x^1, \dots, x^n$ we have the monomial $x^\mu = (x^1)^{\mu_1} \cdots (x^n)^{\mu_n}$ and the differential operator $\partial^\mu = \partial_{x^1}^{\mu_1} \cdots \partial_{x^n}^{\mu_n}$. The derivatives of a function y are denoted by $y_\mu = \partial^\mu y$. The number of distinct multi indices $\mu \in \mathbb{N}_0^n$ with length $|\mu| = q$ is

$$n_q = \binom{n+q-1}{q} \ .$$

In other words, n_q is the number of distinct derivatives of order q.

Assume that a total ordering $\prec$ on the set of multi indices satisfies the following conditions: for all ρ we have (1) $\mu \prec \mu + \rho$ and (2) $\mu \prec \nu$ implies $\mu + \rho \prec \nu + \rho$. Then $\prec$ is called a *ranking* (or *term order*) and can be used to order both monomials and derivatives. Finally, the integer $\operatorname{cls}\mu = \min\{i \mid \mu_i \neq 0\}$ is the *class* of the multi index μ (or the monomial x^μ or the derivative y_μ, respectively).

2.2 Maps and operators

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $E_0 = \Omega \times \mathbb{R}^m$ and $E_1 = \Omega \times \mathbb{R}^k$. Hence E_0 and E_1 are (trivial) vector bundles over Ω and we may identify the sections of E_0 (resp. E_1) with graphs of maps $\Omega \to \mathbb{R}^m$ (resp. $\Omega \to \mathbb{R}^k$). The coordinates in Ω are denoted by $x = (x^1, \dots, x^n)$ and in $\mathbb{R}^m$ by $y = (y^1, \dots, y^m)$. The tangent (resp. cotangent) bundle of Ω is denoted by $T\Omega$ (resp. $T^*\Omega$).

With these notations, the general qth order linear differential equation is

$$Ay = \sum_{|\mu|\le q} a_\mu(x)\partial^\mu y = f \tag{1}$$

where $x \in \Omega \subseteq \mathbb{R}^n$, $a_\mu(x) \in \mathbb{R}^{k\times m}$ and $\mu \in \mathbb{N}_0^n$. The corresponding differential operator is then a map $A : \mathcal{F}(E_0) \to \mathcal{F}(E_1)$ where $\mathcal{F}(E_i)$ are some convenient function spaces. For our purposes, it is not essential to define precisely the functional analytic setting, but we will make a few remarks about this question at appropriate places.

We will also need the special differential operator j^q which associates to a section of E_0 all of its derivatives up to order q. For example, if $m = 1$ and $n = q = 2$ we get

$$j^2 : y \longmapsto \big(y, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}\big) . \tag{2}$$

Elementary combinatorics shows that the number of components in $j^q y$ is md_q where

$$d_q = 1 + n_1 + \cdots + n_q = \binom{n+q}{q} .$$

2.3 Symbols

To each operator A we may associate two symbols: the geometric symbol and the principal symbol. As we will see, both contain essentially the same information but coded in different ways.

Definition 2.1 *The principal symbol of the operator A in* (1) *is*

$$\sigma A(x,\xi) = \sum_{|\mu|=q} a_\mu(x)\xi^\mu$$

where $\xi \in \mathbb{R}^n$ is a real vector.

The principal symbol is an intrinsic object which does not depend on the chosen coordinate system: we may regard ξ as a one-form, i. e. as a section of $T^*\Omega$, and in a fixed basis of $T^*\Omega$ the coefficients of this one-form define at each point $x \in \Omega$ a real vector $\xi \in \mathbb{R}^n$ as in the definition above. Then the principal symbol becomes a $k \times m$ matrix whose entries are homogeneous polynomials in ξ of degree q. Fixing some vector $\xi \in \mathbb{R}^n$ allows us to interpret σA also as a map $E_0 \to E_1$ or even as a map $\mathbb{R}^m \to \mathbb{R}^k$; this is the usual situation.

Definition 2.2 *The geometric symbol* $\mathcal{M}_q$ *of the system* (1) *is a family of vector spaces over* Ω *defined by the kernel of the matrix*

$$M_q = \left(a_{\mu^1}, \dots, a_{\mu^{n_q}}\right)$$

where $\mu^1, \dots, \mu^{n_q}$ *are the* n_q *distinct multi indices of length* q*, i. e.* $|\mu^i| = q$*.*

It is a customary abuse of language to call the matrix M_q geometric symbol, too, and we will do so in the sequel. From now on we suppose for the simplicity of notation that various properties of the symbols do not depend on the point $x \in \Omega$ and omit the reference to it. In particular, this implies that $\mathcal{M}_q$ is in fact a vector bundle.

In order to describe the connection between the two symbols, let us introduce the vector

$$\Xi^q = \left(\xi^{\mu^1}, \dots, \xi^{\mu^{n_q}}\right) .$$

Then we have the following formula which will be useful later on:

$$\sigma A = M_q(\Xi^q \otimes I_m) . \tag{3}$$

Here I_m is the unit matrix of size $m \times m$ and $\otimes$ is the tensor product.[2] For a coordinate free description of the connection between the two symbols see [49].

2.4 Rings and modules

In the analysis of the principal symbol it is convenient to introduce some basic notions of commutative algebra. All the relevant material can be found for example in [16,18]. Let $\mathbb{A} = \mathbb{K}[\xi] = \mathbb{K}[\xi_1, \dots, \xi_n]$ be a polynomial ring in n variables where $\mathbb{K}$ is some field of characteristic zero (in our applications $\mathbb{K}$ will always be $\mathbb{R}$ or $\mathbb{C}$). The Cartesian product $\mathbb{A}^k$ is then an $\mathbb{A}$-module of rank k. A module which is isomorphic to such a Cartesian product $\mathbb{A}^k$ is called *free*. A module M is *finitely generated*, if there is a finite number ν of elements $a_1, \dots, a_\nu \in M$ such that $M = \langle a_1, \dots, a_\nu \rangle$. Since $\mathbb{A}$ is a Noetherian ring by Hilbert's basis theorem, every submodule of $\mathbb{A}^k$ is finitely generated.

An $m \times k$ matrix B whose entries belong to the ring $\mathbb{A}$ defines a module homomorphism $B : \mathbb{A}^k \to \mathbb{A}^m$. We denote by $b^1, \dots, b^k \in \mathbb{A}^m$ the columns of B. If $M_0 = \mathrm{image}(B) = \langle b^1, \dots, b^k \rangle \subseteq \mathbb{A}^m$ is the submodule generated by the vectors b^i and $s \in \mathbb{A}^k$ is such that

$$Bs = s_1 b^1 + \dots + s_k b^k = 0 ,$$

then s is called a *syzygy* of M_0 (or B) and all such vectors s form the (first) *syzygy module* $M_1 \subseteq \mathbb{A}^k$ of M_0. Since $\mathbb{A}$ is Noetherian, there are generators $s^1, \dots, s^\ell \in \mathbb{A}^k$ such that $M_1 = \langle s^1, \dots, s^\ell \rangle$. We denote by S the matrix with columns $s^1, \dots, s^\ell$; it trivially satisfies $BS = 0$. One can compute generators of the syzygy module M_1 algorithmically using Gröbner bases, for example with the program SINGULAR [19].

[2] In the sequel we will use some elementary properties of the tensor or Kronecker product. The necessary material may be found in [23].

Remark 2.3 Let B be a $m \times k$ matrix with $k > m$. Then the module M_0 generated by the columns of B has a nonzero syzygy module, because it can easily be proved that in this case the system $Bs = 0$ has nonzero solutions. $\lhd$

The computation of the first syzygy module is the first step in the computation of a *free resolution* of the given module. Hilbert's syzygy theorem [16, p. 45] asserts that every finitely generated $\mathbb{A}$-module has a free resolution of length less than or equal to the number n of variables in the polynomial ring $\mathbb{A}$, i. e. for our module M_0 there exists an exact sequence of free $\mathbb{A}$-modules

$$0 \longrightarrow \mathbb{A}^{\ell_r} \xrightarrow{S_r} \mathbb{A}^{\ell_{r-1}} \longrightarrow \cdots \tag{4}$$

$$\cdots \longrightarrow \mathbb{A}^{\ell} \xrightarrow{S} \mathbb{A}^{k} \xrightarrow{B} \mathbb{A}^{m} \longrightarrow \mathbb{A}^{m}/M_0 \longrightarrow 0$$

with $r \leq n - 2$. Recall that exactness means that the image of one map in this sequence is equal to the kernel of the next map.

In general, the rank of a matrix B over some ring R is defined via determinantal ideals [7, Chapt. 4]. Let $I_j(B)$ denote the jth *Fitting ideal* of B generated by all $(j \times j)$-minors of B (it can be shown that the Fitting ideals depend only on the module $M_0 = \operatorname{im}(B)$). The *rank* of B in the sense of module theory, $\operatorname{rank}_R(B)$, is the largest nonnegative integer r such that $I_r(B) \neq \langle 0 \rangle$.[3] We put $I(B) = I_r(B)$.

The polynomial ring $\mathbb{A}$ is trivially an integral domain and thus possesses a field of fractions, the field $\mathbb{F} = \mathbb{K}(\xi_1, \ldots, \xi_n)$ of rational functions. Since $\mathbb{A} \subset \mathbb{F}$ and since it does not matter whether we compute minors over $\mathbb{A}$ or over $\mathbb{F}$, we find that $\operatorname{rank}_{\mathbb{A}}(B) = \operatorname{rank}_{\mathbb{F}}(B)$. But the latter rank is the classical rank of linear algebra and may be determined with Gaussian elimination.

Specialising each variable ξ_i to a field element $\bar{\xi}_i \in \mathbb{K}$ leads to a new matrix $B(\bar{\xi}) \in \mathbb{K}^{m \times k}$. Its rank (over the field $\mathbb{K}$) is denoted by $\operatorname{rank}\bigl(B(\bar{\xi})\bigr)$. Obviously,

$$\operatorname{rank}\bigl(B(\bar{\xi})\bigr) \leq \operatorname{rank}_{\mathbb{A}}(B)$$

and for generic vectors $\bar{\xi} \in \mathbb{K}^n$ equality holds. Thus the specialisation may affect the exactness of the sequence (4). From now on we will use the notation ξ for both the indeterminates of the polynomial ring $\mathbb{A}$ and vectors in $\mathbb{K}^n$. The intended meaning should be clear from the context.

Those vectors $\xi \in \mathbb{K}^n$ which lead to a smaller rank are called *characteristic* for the matrix B (they make denominators vanish which appear in the Gaussian elimination over $\mathbb{F}$). More formally, they are defined by the zeros of $I(B)$, i. e. they correspond to the points of the variety $V\bigl(I(B)\bigr)$. Recall that the radical $\operatorname{rad}(I)$ of an ideal $I \subseteq \mathbb{A}$ consists of all polynomials f such that $f^n \in I$ for some $n \in \mathbb{N}$ (thus trivially $I \subseteq \operatorname{rad}(I)$) and that $V(I) = V\bigl(\operatorname{rad}(I)\bigr)$. Furthermore, if I, J are two ideals with $I \subseteq J$, then the corresponding varieties satisfy $V(I) \supseteq V(J)$.

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[3] Some authors consider the annihilators of the Fitting ideals, but in our case this makes no difference, as the polynomial ring $\mathbb{A}$ does not contain zero divisors.

Lemma 2.4 *If the complex (4) is exact, then*

$$\operatorname{rad}\big(I(B)\big) \subseteq \operatorname{rad}\big(I(S)\big)\ . \tag{5}$$

For a proof we refer to [16, p. 504]. By the considerations above, it implies that any vector ξ that is characteristic for S is also characteristic for B, since

$$V\big(I(S)\big) = V\big(\operatorname{rad}\big(I(S)\big)\big) \subseteq V\big(\operatorname{rad}\big(I(B)\big)\big) = V\big(I(B)\big)\ . \tag{6}$$

Corollary 2.5 *Let the entries of B be homogeneous polynomials and*

$$\operatorname{rank}_{\mathbb{A}}(B) = \operatorname{rank}\big(B(\xi)\big) \qquad \forall \xi \in \mathbb{K}^n \setminus \{0\}\ . \tag{7}$$

Then we also have

$$\operatorname{rank}_{\mathbb{A}}(S) = \operatorname{rank}\big(S(\xi)\big) \qquad \forall \xi \in \mathbb{K}^n \setminus \{0\}\ . \tag{8}$$

Proof By definition, $\operatorname{rank}\big(B(\xi)\big) < \operatorname{rank}_{\mathbb{A}}(B)$ is equivalent to $\xi \in V\big(I(B)\big)$. Hence it follows from the hypothesis that $V\big(I(B)\big) = \{0\}$. But (6) implies that $V\big(I(S)\big) \subseteq \{0\}$ which yields (8). □

Lemma 2.6 *Under the assumptions of Corollary 2.5, the complex*

$$\mathbb{K}^\ell \xrightarrow{S(\xi)} \mathbb{K}^k \xrightarrow{B(\xi)} \mathbb{K}^m \tag{9}$$

is exact for all vectors $\xi \neq 0$.

Proof Since (4) is exact, $k = \operatorname{rank}(\mathbb{A}^k) = \operatorname{rank}_{\mathbb{A}}(B) + \operatorname{rank}_{\mathbb{A}}(S)$ [16, p. 500]. Using Corollary 2.5, we get

$$k = \operatorname{rank}\big(B(\xi)\big) + \operatorname{rank}\big(S(\xi)\big) = \dim\operatorname{im}\big(B(\xi)\big) + \dim\operatorname{im}\big(S(\xi)\big) \quad \forall \xi \neq 0\ . \tag{10}$$

Since $BS = 0$, we always have

$$\operatorname{im}\big(S(\xi)\big) \subseteq \ker\big(B(\xi)\big) \qquad \forall \xi \neq 0\ .$$

$B(\xi)$ also trivially satisfies $\dim\operatorname{im}\big(B(\xi)\big) = k - \dim\ker\big(B(\xi)\big)$ implying

$$\dim\ker\big(B(\xi)\big) = \dim\operatorname{im}\big(S(\xi)\big) \qquad \forall \xi \neq 0\ .$$

Together with the inclusion above, this observation entails

$$\operatorname{im}\big(S(\xi)\big) = \ker\big(B(\xi)\big) \qquad \forall \xi \neq 0$$

and hence the exactness of (9). □

If we apply the functor $\operatorname{Hom}_{\mathbb{K}}(\cdot, \mathbb{K})$ to an exact sequence of vector spaces, i. e. if we dualise the sequence, then by a standard result in homological algebra we obtain again an exact sequence [31] (note that generally this holds only for vector spaces and not even for free modules over a ring R, as $\operatorname{Hom}_R(\cdot, R)$ is only a left exact functor). At the level of matrices this yields the following corollary to the above lemma.

Corollary 2.7 *Under the assumptions of Corollary 2.5, the transposed complex*

$$\mathbb{K}^m \xrightarrow{B^T(\xi)} \mathbb{K}^k \xrightarrow{S^T(\xi)} \mathbb{K}^\ell$$

is exact for all $\xi \neq 0$, too.

3 Involutive Systems

3.1 Completion to Involution

Overdetermined systems usually still contain hidden integrability conditions; the process of their explicit construction is called *completion*. As already mentioned in the Introduction, many approaches to this problem exists; we will use the formal theory containing both geometric and algebraic elements. Since we study only linear systems, we emphasise the algebraic side and briefly describe the construction of involutive bases for linear differential systems [17]. More details and the precise connection of these bases to the formal theory can be found in [21]; for a general introduction to involutive bases see [9, 46].

Janet introduced the fundamental concept of *multiplicative variables*: we assign to each equation in the system a subset of the set of all independent variables as its multiplicative variables. Roughly speaking, a system is involutive, if it suffices to consider of each equation only the prolongations (i. e. differentiations) with respect to these variables. Another point of view is that this assignment of multiplicative variables permits us to generate in a systematic way all cross-derivatives which could lead to integrability conditions.[4]

A ranking $\prec$ distinguishes in each equation of the system a *leading derivative*, namely the one which is maximal with respect to $\prec$. By a Gaussian elimination, we may render any linear system triangular implying in particular that every equation has a different leading derivative. If an equation has the leading derivative y^j_μ with $\operatorname{cls}\mu = k$, then we assign it the multiplicative variables $x^1, \dots, x^k$. A ranking that is particularly useful in the context of the formal theory works as follows: $y^j_\mu \succ y^k_\nu$, if we have either that $|\mu| > |\nu|$ or that $|\mu| = |\nu|$ and the first non-vanishing entry of $\mu - \nu$ is positive or that $\mu = \nu$ and $j > k$.

We may now introduce the notions of (involutive) reduction and normal form, respectively. Assume that one of our equations contains a term y^j_μ and the leading derivative of another equation is y^j_ν with $\mu = \nu + \rho$. In principle, we could now reduce the first equation by subtracting ∂^ρ times the second one. However, we only allow this reduction, if the prolongation ∂^ρ requires only differentiations with respect to multiplicative variables of the second equation. Thus if $\operatorname{cls}\nu = k$ and $\rho_i > 0$ for some $i > k$, then the reduction is not permitted.

An equation is in *involutive normal form* with respect to a system, if it is not possible to involutively reduce any term in it. A system is *involutively autoreduced*, if any equation is in involutive normal form with respect to the remaining ones. The process of (involutive) autoreduction of a linear system may be thought of as a differential generalisation of Gaussian elimination.

Definition 3.1 *An involutively autoreduced system is* involutive*, if the involutive normal form of any differential consequence is zero. A differential consequence whose involutive normal form does not vanish is an* obstruction to involution*.*

[4] The word “multiplicative” might appear strange here, as we differentiate with respect to these variables. The reason is historical, as Janet formulated his theory in terms of monomials so that differentiation corresponds to a multiplication with these variables.

A more rigorous formulation of this definition is possible using some algebraic notions. Let $\mathbb{D} = \mathbb{F}[\partial_1, \dots, \partial_n]$ be the ring of linear differential operators with coefficients in some function field $\mathbb{F}$, say the rational functions $\mathbb{F} = \mathbb{K}(x^1, \dots, x^n)$. If there are m unknown functions, then our system defines a submodule S of the free $\mathbb{D}$-module $\mathbb{D}^m$ (in the case $m = 1$ this means of course that $S \subseteq \mathbb{D}$ is a differential ideal). An involutive system corresponds to a basis of S such that the involutive normal form of any element of S with respect to this basis vanishes.

Example 3.2 We illustrate these concepts with two simple systems of second order in two independent variables and one dependent variable. The first one is

$$y_{02} - y_{01} = 0\ , \quad y_{11} - cy_{10} = 0$$

where c is some real constant. As the first equation is of class 2 and the second one of class 1, we have one non-multiplicative variable, namely x^2 for the second equation. If we compute any differential consequence of the first equation, it is trivially involutively reducible, as all variables are multiplicative for the equation so that we may always reduce. The same holds, if we differentiate the second equation with respect to x^1.

Thus the only interesting differentiation is the x^2-derivative of the second equation. It yields $y_{12} - cy_{11} = 0$. We may now involutively reduce with the x^1-derivative of the first equation. For $c = 1$, the involutive normal form is 0 and thus our system is involutive. Otherwise, we have obtained an obstruction to involution ($y_{11} = 0$) and the system is not involutive. Obviously, this obstruction is a classical integrability condition obtainable also by simply taking the cross-derivative of the two equations in our system.

As second example we consider the seemingly similar system

$$y_{02} - y_{10} = 0\ , \quad y_{20} - y_{01} = 0\ .$$

We find the same classes as in the previous system, so that again only the x^2-derivative of the second equation is of interest. It yields $y_{21} - y_{02} = 0$. While we may involutively reduce the second term in it by simply adding the first equation of our system, it is not possible to simplify involutively the leading derivative y_{21}. Hence we have found an obstruction to involution and the system is not involutive. Note that in the classical sense this obstruction is *not* an integrability condition; it arises only because of our restriction to multiplicative differentiations. $\lhd$

If a system is not involutive, one may *complete* it to an involutive one by adding the arising obstructions to involution. One can show that this process terminates after a finite number of steps. Informally, we may describe the completion as follows. We always keep the system in an involutively autoreduced form. Each equation is differentiated with respect to its non-multiplicative variables and then the involutive normal form of the result is computed. If it does not vanish, it is added to the system as an integrability condition. The completion terminates as soon as no non-multiplicative differentiation yields a new equation.

Example 3.3 A completion may require surprisingly many steps, as demonstrated by the following classical second order system in one dependent variable y and three independent variables x^1, x^2, x^3 due to Janet:

$$y_{002} + x^2 y_{200} = 0 \;, \quad y_{020} = 0 \;.$$

We use a ranking such that in the first equation y_{002} is the leading derivative. So the first equation is of class 3 and the second one of class 2. Hence we must study only one non-multiplicative prolongation, namely the x^3-derivative of the second equation. It yields the new equation $y_{021} = 0$ which is already in involutive normal form with respect to our system.

This equation is again of class 2 and thus has x^3 as sole non-multiplicative variable. The equation $y_{022} = 0$ is not in involutive normal form, as it can be involutively reduced by the first equation. As one easily checks, its involutive normal form is $y_{210} = 0$. As this integrability condition is of class 1, we must check now two non-multiplicative prolongations. The one with respect to x^2 yields nothing new, as it is trivially reducible by the second equation. But the x^3-prolongation yields the new equation $y_{211} = 0$ which is in involutive normal form.

This integrability condition is of class 1, too, and therefore we must check two non-multiplicative prolongations. As before, the x^2-prolongation is trivially reducible but the x^3-prolongation yields after some computations the new equation $y_{400} = 0$. It leads to two further equations $y_{410} = 0$ and $y_{401} = 0$. The first one is involutively reducible with respect to the equation $y_{210} = 0$ and all non-multiplicative prolongations of the second one are involutively reducible, too, so that we are finally done.

Thus the involutive completion of our system has lead to the fifth order system:

$$\begin{gathered} y_{002} + x^2 y_{200} = 0 \;,\; y_{020} = 0 \;,\; y_{210} = 0 \;, \\ y_{400} = 0 \;,\; y_{021} = 0 \;,\; y_{211} = 0 \;,\; y_{401} = 0 \;. \end{gathered}$$

Only the first two obstructions to involution are integrability conditions in the classical sense; the remaining three are reducible although not involutively. ◁

Strictly speaking, we have described here the construction of a so-called Pommaret basis of the given system. Other kinds of involutive bases arise by using different rules for the assignment of multiplicative variables; for a detailed discussion of these notions we refer to [17,46]. Furthermore, we ignore here the problem of δ-regularity (which concerns the termination of the described completion algorithm in certain “bad” coordinate systems), as it is related to characteristics and thus of minor importance for elliptic systems. Details (and a constructive solution) are contained in [21].

Involutive systems possess many pleasant properties. For lack of space, we only mention one. In the analytic category, we have a general existence and uniqueness theorem for initial value problems, the *Cartan-Kähler theorem* generalising the well-known Cauchy-Kovalevskaya Theorem (for its proof all obstructions to involution and not only the classical integrability conditions are decisive). Not much is currently known about existence and regularity of solutions in larger function spaces. In the case of linear systems, it is not difficult to generalise the uniqueness theorem of Holmgren to arbitrary involutive systems. An existence and uniqueness theorem for smooth solutions of hyperbolic systems with elliptic constraints is contained in [47].

3.2 Completion and Equivalence

An important point in the completion to involution is to what extent we may say that the completed system is equivalent to the original one. Intuitively equivalence means that the solution space remains unchanged, but obviously this idea depends on what kind of solutions we are treating. The simplest class are *formal solutions*. Here it is clear that the completion does not change the solution space, as any formal solution trivially satisfies any integrability condition independent of its order. This extends trivially to *analytic solutions*, as these are nothing but converging formal solutions.

Furthermore, the same argument generalises to *smooth solutions*: because of their infinite differentiability, they automatically satisfy any integrability condition constructed during the completion. The same holds true for any *weak solution* that may be understood in a distributional sense, as distributions are again infinitely differentiable.

The situation is somewhat more complicated for solutions possessing only a finite differentiability. If we assume that the original system was of order q and that the completion lead to a system of order $q' > q$, then a strong solution of class $\mathcal{C}^q$ of the original system becomes a weak solution of the completed system.

Finally, we must discuss the effect of the completion on the data, i. e. the right hand side of a linear system and its coefficients. If we study an inhomogeneous system $Ay = f$, then the completion leads to a system $\tilde{A}y = \tilde{f}$ where the right hand side $\tilde{f}$ consists of linear combinations of components of f and their derivatives up to a finite order. Again this provides no real problems, if it is possible to interpret the derivatives in a distributional sense.

In contrast, the situation is much less clear, if the coefficients of the operator A are not sufficiently often differentiable. Here we cannot simply argue with distributional derivatives. Therefore we will assume in the sequel that the completion does not require more differentiations than the regularity of the coefficients permit.

More generally, we consider two systems of differential equations as equivalent, if a bijection between their solution spaces exists (requiring again a precise specification of the used function spaces). This notion of equivalence allows us to study more complex operations on differential equations like reduction to first order or to one dependent variable (see Section 7) where the number of independent and/or dependent variables changes. For a more formal definition of equivalence, see the discussion in [15].

3.3 Compatibility Conditions and the Fundamental Principle

Given an inhomogeneous overdetermined system $Ay = f$, it will generally not possess solutions for arbitrary right hand sides f. Solutions will exist only, if f

satisfies certain differential equations known as *compatibility conditions* (the differential analogue to syzygies). For an involutive system it is straightforward to determine a complete generating set of these conditions.

Recall from our discussion above that in an involutive system the involutive normal form of any equation obtained by a differentiation with respect to a non-multiplicative variable is zero. This implies that the equation can be written as a linear combination of multiplicative prolongations. Let us denote the class of the sth equation of the system by k_s. Then $Ay = 0$ is an involutive system, if and only if functions $B_{tl}^{sj}(x)$ and $C_t^{sj}(x)$ exist such that for all $j > k_s$

$$\partial_{x^j}(Ay)_s = \sum_t \left(\sum_{l \leq k_t} B_{tl}^{sj}(x)\partial_{x^l}(Ay)_t + C_t^{sj}(x)(Ay)_t \right) .$$

These relations trivially imply that a necessary condition for the existence of solutions of the inhomogeneous system $Ay = f$ is that the right hand side f satisfies the linear differential equations

$$\partial_{x^j} f_s = \sum_t \left(\sum_{l \leq k_t} B_{tl}^{sj}(x)\partial_{x^l} f_t + C_t^{sj}(x) f_t \right) . \tag{11}$$

Example 3.4 If we consider Maxwell's equations for the electric field E and the magnetic field B

$$E_t - \nabla \times B = J \ , \quad B_t + \nabla \times E = 0 \ , \quad \nabla \cdot E = \rho \ , \quad \nabla \cdot B = 0 \ , \tag{12}$$

then the compatibility condition is the well-known continuity equation

$$\rho_t - \nabla \cdot J = 0$$

describing the conservation of charge. $\lhd$

The *fundamental principle* states that the conditions (11) are not only necessary but also sufficient. Of course, the correctness of this statement depends again on the considered function spaces. Using the theory of involutive bases it is fairly straightforward to show that the principle is correct at the level of formal solutions. Ehrenpreis and Malgrange showed that the principle also holds for smooth and distributional solutions, if we restrict to linear equations with constant coefficients. These are, however, highly non-trivial results; see [35,36] for an extensive discussion of this and related issues.

We may express these considerations in a somewhat more abstract way using differential sequences. Let $\mathcal{F}_i(E_i)$ be some spaces of sections of bundles E_i. If A_1 represents the compatibility operator for a given linear differential operator A_0, then the sequence

$$\mathcal{F}_0(E_0) \xrightarrow{A_0} \mathcal{F}_1(E_1) \xrightarrow{A_1} \mathcal{F}_2(E_2)$$

defines by construction a complex, i. e. $\operatorname{im} A_0 \subseteq \ker A_1$. In other words, the differential equation $A_0 y = f$ may possess for a given right hand side $f \in \mathcal{F}_1(E_1)$

a solution $y \in \mathcal{F}_0(E_0)$ only, if $A_1 f = 0$. The fundamental principle concerns the question whether or not the sequence is exact, i. e. whether $\operatorname{im} A_0 = \ker A_1$. In this case every solution $f \in \mathcal{F}_1(E_1)$ of the equation $A_1 f = 0$ is of the form $f = A_0 y$ for some function $y \in \mathcal{F}_0(E_0)$.

4 Elliptic Symbols

4.1 Ellipticity and its Generalisations

Let us consider again the general linear qth order differential operator and its principal symbol:

$$Ay = \sum_{|\mu| \le q} a_\mu(x) \partial^\mu y \quad \text{and} \quad \sigma A = \sum_{|\mu| = q} a_\mu(x) \xi^\mu \,.$$

Definition 4.1 *The differential operator A or the principal symbol σA, resp., is called* elliptic*, if the map $\sigma A : E_0 \to E_1$ is injective for all $\xi \in \mathbb{R} \setminus \{0\}$.*

Note that, since ξ is intrinsically defined as a one-form, the property of being elliptic is independent of the choice of coordinates. Ellipticity is equivalent to the absence of characteristic vectors, so that we recover the familiar idea of an elliptic system as a system without real characteristics.

Note that Definition 4.1 excludes systems where $k < m$, i. e. systems with less equations than unknown functions; such a system is obviously underdetermined. While its symbol may still have full rank, it cannot have full *column* rank. In [15] such operators are called *operators with constant defect*. As a simple example of an underdetermined system with full rank, we take the system $\nabla \cdot y = 0$ defining divergence free vector fields in $\mathbb{R}^m$. For a given vector $\xi \in \mathbb{R}^m$ the principal symbol is simply the matrix ξ^t which has obviously full row rank for any $\xi \neq 0$.

Underdetermined systems with full rank appear mainly as subsystems of larger systems. In the Maxwell system (12) the first two equations (six scalar equations) form a (symmetric) hyperbolic system in Cauchy-Kovalevskaya form; the last two equations form an underdetermined system with full rank. This full rank condition is very important for the analysis of the whole system, see [47] or [45, Sect. 5.6] for a discussion of its role in proving an existence and uniqueness theorem for smooth solutions.

Independent of these considerations, we have the following interesting relation between a full rank symbol and involution (note that we do not require here full rank for all vectors ξ but only for one).

Proposition 4.2 *Let $k \le m$ and assume that there exists at least one vector $0 \neq \xi \in \mathbb{R}^n$ such that σA has full rank. Then A is involutive.*

Proof We perform a linear change of the independent variables $x \mapsto z$ subject to the sole condition that $z^n = \langle \xi, x \rangle$. Obviously this is always possible for a non-vanishing vector ξ. After such a change, we can transform the system $Ay = 0$

with the help of some linear operations and possibly a renumbering of the dependent variables y^α into a new system where the ith equation is $y^i_{0\dots 0d_i} = f^i$ and where the functions f^i do not depend on pure z^n-derivatives of the y^j of order greater than or equal to d_j for $1 \leq j \leq k$. We may consider this as an underdetermined Cauchy-Kovalevskaya form (for $k = m$ this *is* the classical Cauchy-Kovalevskaya form) and such a system is trivially involutive, as no equation has a non-multiplicative variable. □

In the sequel we will restrict to systems with $k \geq m$, as for applications this is the most interesting case. By the same reasoning as used in the above proof, one sees immediately that if such a system is elliptic, it must be either in Cauchy–Kovalevskaya form or overdetermined.[5]

4.2 On Genericity

From a certain degree of overdeterminacy on, linear systems are generically elliptic. The following result, although rather elementary, seems to be new. Let us consider the general qth order operator A as in (1).

Proposition 4.3 *The operator A is generically elliptic, if $n + m < k + 2$.*

Proof Recalling (3) linking the geometric and the principal symbol, we may state the condition of ellipticity as follows. The operator A is elliptic, if and only if the following algebraic system for $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ has only the trivial solutions $\xi = 0$, v arbitrary or $v = 0$, ξ arbitrary:

$$(\sigma A)v = M_q(\Xi^q \otimes I_m)v = M_q(\Xi^q \otimes v) = 0 \ . \tag{13}$$

It is convenient to write these equations in a different way. To this end let us introduce matrices $B_j \in \mathbb{R}^{n_q \times m}$ by writing the rows of M_q as matrices. More precisely, we set

$$(B_j)_i = (a_{\mu^i})_j$$

where $(B_j)_i$ denotes the ith row of B_j. With the help of the matrices B_j we can write the conditions in (13) as

$$\langle \Xi^q, B_j v \rangle = 0 \ , \quad 1 \leq j \leq k \ .$$

As these equations are homogeneous in ξ and linear in v, we may normalise $|\xi| = |v| = 1$. Together with the equations above this makes $k + 2$ equations. Since we have $n + m$ unknowns, the claim follows. □

[5] Opposed to common belief, a system with $k \geq m$ may very well be underdetermined. Examples are gauge theories in elementary particle physics; see e. g. [45, Sect. 3.3] for a rigorous discussion.

It is somewhat surprising that the result does not depend on the order of the system. Protter [38, p. 74] proved that a first order differential system is generically elliptic if $m(n+1)/2 \leq k$. Our result is sharper, except that for $m = 1$ we have the bound $k \geq n$ while Protter has $k \geq (n+1)/2$. However, Protter's statement is false in this case and our bound is in fact optimal.[6] This can be seen directly as follows. For $m = q = 1$ we have

$$\sigma A = M_1 \xi$$

where $M_1 \in \mathbb{R}^{k \times n}$. Ellipticity is now equivalent to the injectivity of M_1 which implies that $k \geq n$.

5 DN–Elliptic Systems

In order to generalise the notion of ellipticity (and to solve such problems like the reduction of the Laplace equation to first order mentioned in the introduction), Douglis and Nirenberg [12] introduced the concept of *weights* of a system, see also [4, §3.2b] for a discussion. The weights of a system are two sets of integers: we denote by s_i the weights for the equations, $1 \leq i \leq k$, and t_j the weights for the unknowns, $1 \leq j \leq m$. They must be chosen such that

$$s_i + t_j \geq q_{ij}$$

where q_{ij} is the maximal order of a derivative of the jth unknown function in the ith equation of the system.

Definition 5.1 *The* weighted (principal) symbol *of the differential operator A is*

$$\big(\sigma_w A\big)_{i,j} = \sum_{|\mu| = s_i + t_j} \big(a_\mu(x)\big)_{i,j} \xi^\mu \ .$$

Note that $\sigma_w A = \sigma A$, if we choose

$$s_1 = \cdots = s_k = 0 \quad \text{and} \quad t_1 = \cdots = t_m = q \ . \tag{14}$$

Obviously, the weighted symbol $\sigma_w A$ remains unchanged, if we replace all weights s_i by $s_i + c$ and all weights t_j by $t_j - c$ for some $c \in \mathbb{Z}$. Hence we may always suppose that $s_1 \leq s_2 \leq \cdots \leq s_k = 0$ and $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$. Furthermore, let us define indices i_l, I_l, j_l, J_l as follows:

$$\begin{aligned}
&s_1 = \cdots = s_{i_1} < s_{i_1+1} = \cdots = s_{i_1+i_2} < \cdots < s_{i_1+\cdots+i_{a-1}+1} = \cdots = s_k = 0 \ , \\
&t_1 = \cdots = t_{j_1} > t_{j_1+1} = \cdots = t_{j_1+j_2} > \cdots > t_{j_1+\cdots+j_{b-1}+1} = \cdots = t_m \ , \\
&\mathsf{I}_0 = 0 \ , \qquad \mathsf{I}_l = i_1 + \cdots + i_l \ , \\
&\mathsf{J}_0 = 0 \ , \qquad \mathsf{J}_l = j_1 + \cdots + j_l \ .
\end{aligned} \tag{15}$$

[6] For seeing why Protter's argument fails examine the matrix T in [38, p. 74].

Finally, we define i_a and j_b by $k = i_1 + \cdots + i_a$ and $m = j_1 + \cdots + j_b$. With these conventions, $\sigma_w A$ can be written as a block matrix:

$$\sigma_w A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1b} \\ A_{21} & A_{22} & \dots & A_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ A_{a1} & A_{a2} & \dots & A_{ab} \end{pmatrix} . \tag{16}$$

Here the block A_{lh} is an $i_l \times j_h$ matrix and its entries are homogeneous polynomials in ξ of degree $\nu_{lh} = s_{\mathsf{l}_l} + t_{\mathsf{J}_h}$. Now, conversely given some degrees ν_{lh}, can we solve for the corresponding weights?

Lemma 5.2 *If we fix $s_k = 0$, choose arbitrary values for ν_{1h} and ν_{l1}, and set $\nu_{lh} = \nu_{l1} + \nu_{1h} - \nu_{11}$, then there exist unique weights s_i and t_j corresponding to this choice.*

Proof By definition, $s_l + t_h = \nu_{lh} = \nu_{l1} + \nu_{1h} - \nu_{11}$. Fixing $s_k = 0$ leaves us with $k + m - 1$ unknowns (weights). We obtain now the solution simply as follows: first $s_k + t_j = t_j = \nu_{k1} + \nu_{1j} - \nu_{11}$; then $s_i = \nu_{i1} + \nu_{1j} - \nu_{11} - t_j = \nu_{i1} - \nu_{k1}$. □

For a fixed vector $\xi \in \mathbb{R}^n$, the weighted symbol may also be interpreted as a map $\sigma_w A : E_0 \to E_1$. This leads to the following generalised notion of ellipticity.

Definition 5.3 *The differential operator A is* DN–elliptic*, if we can find weights s_i and t_j such that its weighted symbol $\sigma_w A$ is injective for all $\xi \in \mathbb{R} \setminus \{0\}$.*

Note that an operator is DN–elliptic, if *some* choice of relevant weights exists and in general there are many different possible choices. In particular, the property of being DN–elliptic is *not* independent of the choice of coordinates. Also it may not be easy to effectively find suitable weights/coordinates. Quantifier elimination allows an algorithmic solution of the problem of weight determination [48].

In particular, a system is elliptic in the usual sense, if it is DN–elliptic with respect to the weights (14). Two other special cases are worth mentioning. Let us denote by q_i the order of the ith equation and by $\tilde{q}_j$ the maximal order of the variable y^j in the whole system. Hence, by our conventions, $q = \max q_i = \max \tilde{q}_j$.

Definition 5.4 *A* reduced (principal) symbol *of the operator A, denoted by $\sigma_r A$, is a weighted symbol with all weights t_j equal. A* Petrovskij (principal) symbol *of the operator A, denoted by $\sigma_p A$, is a weighted symbol with all weights s_i equal. If $\sigma_p A$ is injective, the operator A is said to be* P–elliptic *(elliptic in the sense of Petrovskij [4]).*

Of course, in the reduced case the most natural choice of weights is

$$s_i = q_i - q \quad \text{and} \quad t_1 = \cdots = t_m = q \tag{17}$$

and in the Petrovskij case

$$s_i = 0 \quad \text{and} \quad t_j = \tilde{q}_j \ , \tag{18}$$

respectively. If we speak in the sequel of *the* reduced or *the* Petrovskij symbol, we always mean the weighted symbol with respect to this particular choice. Referring to the block matrix (16), we see that in the "reduced case" we have $b = 1$ while in the "Petrovskij case" we have $a = 1$.

Remark 5.5 Let $\sigma_r A$ be the reduced symbol of the operator A and let $s \in \mathbb{A}^k$ be a syzygy of the transposed matrix $(\sigma_r A)^t$. We associate with s a differential operator $\hat{s}$ by substituting $\partial^{\mathbf{1}_j}$ for ξ_j. Then the expression $\hat{s}^t Ay$ is a linear combination of differential consequences of the original system $Ay = 0$ and, because of the fact that s is a syzygy, the highest order terms cancels. Thus such linear combinations may be considered as generalised cross-derivatives and the result is possibly an integrability condition (depending on whether or not it reduces to zero modulo $Ay = 0$). In particular, adding the equation $\hat{s}^t Ay = 0$ to the original system may increase the column rank of the reduced symbol, as we have already seen in the introductory example of the first order form of the Laplace equation.

We will later formulate the proof of our main theorem solely on the basis of such syzygy considerations. As the purpose of any completion method is the detection of all hidden integrability conditions, it must check for all syzygies of $(\sigma_r A)^t$ whether they lead to an integrability condition. Gröbner-like approaches are explicitly formulated this way (recall that S-polynomial is an abbreviation for syzygy polynomial); in other approaches like exterior systems theory this fact is rather obscured. Nevertheless, this technique of proof ensures that our results remain true for any completion theory. $\lhd$

It is easily seen that differentiating (some of) the equations of a system preserves DN–ellipticity.

Lemma 5.6 *Suppose that the operator A is DN–elliptic. Let the weight of the ith equation be s_i. Let A' be the operator obtained from A by adding all equations obtained by differentiating the ith equation c times with respect to each variables. Then A' is DN–elliptic for the following weights: s_i is set to zero, the weights for the new equations are $s_i + c$, and all other weights are as for A.*

Proof Let $v = \big(\xi_1^c, \dots, \xi_n^c\big)$. Let us denote by $\big(\sigma_w A\big)_i$ the ith row in $\sigma_w A$. Now apply the derivative ∂_j^c to the ith equation and set the weight of this new equation to $s_i + c$. Doing this for each j and adding all these equations to the original system we get the new operator A'. But clearly in terms of the symbols, this corresponds to adding the rows $v \otimes \big(\sigma_w A\big)_i$ to the original weighted symbol. Hence, choosing weights for A' as described in the statement of the Lemma, we see that if $\sigma_w A$ has full rank, then $\sigma_w A'$ has full rank, too. $\square$

Informally, we may describe the content of the Lemma as follows. From the point of view of analysing the rank properties of the symbol we may replace $\big(\sigma_w A\big)_i$ by $v \otimes \big(\sigma_w A\big)_i$. As a further consequence, one may without loss of generality suppose that all equations in a DN-elliptic system are of order q whenever it is convenient. In particular, the above Lemma gives the following simple result.

Corollary 5.7 *Let a reduced symbol $\sigma_r A$ be elliptic and let $s = \max_i s_i$. Then the operator A' obtained by differentiating the ith equation $s - s_i$ times with respect to all variables (including all mixed derivatives) has an elliptic symbol $\sigma A'$.*

Lemma 5.8 *Assume that all rows in A are of order q and that the weights are ordered as in* (15). *If $\sigma_w A$ is DN–elliptic, then*

(i) $s_1 + t_1 = q$ and $A_{l1} = 0$ for $1 < l \leq a$;
(ii) A_{11} is an elliptic symbol; i. e. $\operatorname{rank}\big(A_{11}\big) = j_1$ and in particular $i_1 \geq j_1$;
(iii) $t_m \geq 0$ and without loss of generality we may suppose that $t_m \geq 1$.

Proof If $s_1 + t_1 < q$, then the first i_1 equations could not be of order q. On the other hand, if $s_1 + t_1 > q$, then the first block column would be zero, so the system could not be DN–elliptic. Thus $s_1 + t_1 = q$ and for all $i > i_1$ we have $s_i + t_1 > q$ implying $A_{l1} = 0$ for $1 < l \leq a$. If $\operatorname{rank}\big(A_{11}\big) < j_1$, then $(\sigma_w A)v = 0$ for some nonzero vector v of the form

$$v = (v_1, \ldots, v_{j_1}, 0, \ldots, 0)\ .$$

As this is impossible for an DN–elliptic symbol, we must have $i_1 \geq j_1$. Finally, for $t_m < 0$ the last block column would be zero, and the system could not be DN–elliptic.

Suppose that $t_m = 0$ and let us call the variables $y^{1+\mathsf{J}_{b-1}}, \ldots, y^m$ algebraic variables because no derivatives of these variables appear in the system. Moreover, the first I_{a-1} equations do not depend on these variables. Hence the first I_{a-1} equations form a DN–elliptic system with variables $y^1, \ldots, y^{\mathsf{J}_{b-1}}$. Because A_{ab} is of full column rank, the algebraic variables can be solved in terms of other variables. In case $b = 1$, A_{a1} is of full column rank and we can again solve the algebraic variables in terms of other variables, and hence obtain a system without algebraic variables. □

So, whenever it is convenient, one may suppose that

$$\sigma_w A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1b} \\ 0 & A_{22} & \ldots & A_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & A_{a2} & \ldots & A_{ab} \end{pmatrix} \quad \text{and} \quad \sigma A = \begin{pmatrix} A_{11} & 0 & \ldots & 0 \\ A'_{21} & A'_{22} & \ldots & A'_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ A'_{a1} & A'_{a2} & \ldots & A'_{ab} \end{pmatrix}\ .$$

6 Ellipticity and Completion

Our goal in this section is to show that if weights exists such that the linear differential operator A is DN–elliptic, then the completion of A leads to an equivalent operator that is elliptic without weights. Thus we may dispense with the introduction of weights, if we always complete to involution before the classification. In addition, we will show with some concrete examples that the approach via weights is not sufficient, as it sometimes fails to properly recognise elliptic systems.

6.1 Preliminary results

Let us consider the general linear system $Ay = f$ defined in (1), and from now on we will suppose that $k \geq m$.

Proposition 6.1 *If during the completion to involution a reduced symbol becomes elliptic at some stage, then it will remain elliptic until the end of the completion.*

Proof The completion to involution is based on the addition of the arising obstructions to involution. At the level of reduced symbols this leads to the addition of further rows. If a reduced symbol has already full column rank, then such further rows cannot change the rank and the completion does not affect its ellipticity. Note that involutive head autoreductions and similar algebraic computations performed during the completion do not matter here, as they correspond at the level of reduced symbols to elementary row operations. □

Remark 6.2 By Corollary 5.7, it is trivial to go from an operator with an elliptic reduced symbol to an equivalent elliptic operator: we must only add derivatives of the lower order equations. Hence for all practical purposes it suffices to show that a reduced symbol becomes elliptic at some stage of the completion process. ◁

Example 6.3 Consider the system

$$A: \quad \begin{cases} y^1_{20} - y^2_{02} = 0 \;, \\ y^1 + y^2 = 0 \;, \end{cases} \qquad \text{and} \quad \sigma_r A = \begin{pmatrix} \xi_1^2 & -\xi_2^2 \\ 1 & 1 \end{pmatrix} \;.$$

Obviously the reduced symbol $\sigma_r A$ is elliptic. Differentiating the last equation twice with respect to both variables we obtain the elliptic system:

$$A^{(1)}: \quad \begin{cases} y^1_{20} - y^2_{02} = 0 \;, \\ y^1_{20} + y^2_{20} = 0 \;, \\ y^1_{02} + y^2_{02} = 0 \;, \end{cases} \qquad \text{and} \quad \sigma A^{(1)} = \begin{pmatrix} \xi_1^2 & -\xi_2^2 \\ \xi_1^2 & \xi_1^2 \\ \xi_2^2 & \xi_2^2 \end{pmatrix} \;.$$

Remark 6.4 In Section 3.1 we gave an algebraic introduction to the notion of involution. There also exists a geometric approach based on jet bundles. Within this approach, the completion consists of two basic operations: prolongation and projection. A projection corresponds to the addition of integrability conditions; hence it preserves ellipticity by the same argument as in the proof of Proposition 6.1. In a prolongation, all equations in the system are differentiated with respect to all independent variables. It also preserves ellipticity, as the following simple argument shows. Let A be a linear differential operator and A' the operator obtained by adding to A all the differentiated equations. Then clearly

$$\sigma A' = \begin{pmatrix} \xi_1 \sigma A \\ \vdots \\ \xi_n \sigma A \end{pmatrix} = \xi \otimes \sigma A \;.$$

Thus the prolonged symbol $\sigma A'$ has full column rank for all $\xi \neq 0$, if and only if the original σA has full column rank. This implies that Proposition 6.1 holds for the geometric approach, too. ◁

6.2 The Petrovskij Case

Before treating the general case, let us make a few remarks about the Petrovskij case. Using the indices J_l defined in (15), we introduce vectors $y^{(l)}$ by

$$y^{(l)} = \left(y^{1+\mathsf{J}_{l-1}}, \ldots, y^{\mathsf{J}_l}\right) , \quad l = 1, \ldots, b \; . \tag{19}$$

In this way, the general operator A in (1) may be written as

$$Ay = \sum_{l=1}^{b} A_l y^{(l)} \; . \tag{20}$$

Using this decomposition, the Petrovskij symbol may be written as

$$\sigma_p A = \begin{pmatrix} \sigma A_1 & \sigma A_2 & \cdots & \sigma A_b \end{pmatrix} \; .$$

Lemma 6.5 *If A is P–elliptic, then each operator A_l in* (20) *is elliptic.*

Proof Suppose that some A_l is not elliptic. Then there is some nonzero $v^{(l)} \in \mathbb{R}^{j_l}$ such that $\sigma A_l v^{(l)} = 0$. Let

$$v = \left(0, \ldots, 0, v^{(l)}, 0, \ldots, 0\right) \in \mathbb{R}^{\mathsf{J}_b}$$

Then v is nonzero and $(\sigma_p A)v = 0$. □

The converse to this result is obviously false. Anyway, since P–elliptic systems are constructed as sums of systems which are elliptic in the ordinary sense, it seems natural that the completed system should also be elliptic.

Example 6.6 Consider the system

$$A : \quad \begin{cases} y^1_{20} - y^2 = 0 \; , \\ y^1_{02} + y^2 = 0 \; , \end{cases} \qquad \text{and} \quad \sigma_p A = \begin{pmatrix} \xi_1^2 & -1 \\ \xi_2^2 & 1 \end{pmatrix} \; .$$

Obviously A is P–elliptic. Taking cross derivatives we obtain an elliptic system:

$$A^{(1)} : \quad \begin{cases} y^1_{20} - y^2 = 0 \; , \\ y^1_{02} + y^2 = 0 \; , \\ y^2_{20} + y^2_{02} = 0 \; , \end{cases} \qquad \text{and} \quad \sigma A^{(1)} = \begin{pmatrix} \xi_1^2 & 0 \\ \xi_2^2 & 0 \\ 0 & \xi_1^2 + \xi_2^2 \end{pmatrix} \; .$$

6.3 General case

Let us first consider some examples.

Example 6.7 Consider the system

$$A: \quad \begin{cases} y^1_{30} + y^1 + ay^2 + by^3 = 0 \ , \\ y^1_{03} + cy^2 + dy^3 = 0 \ , \\ y^1_{11} + y^2_{10} + y^3_{01} = 0 \ , \end{cases}$$

depending on four real parameters a, b, c and d. With the following choice of weights $s_1 = s_2 = -1$, $s_3 = 0$, $t_1 = 4$, $t_2 = t_3 = 1$ the symbols are

$$\sigma_w A = \begin{pmatrix} \xi_1^3 & a & b \\ \xi_2^3 & c & d \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \quad \text{and} \quad \sigma A = \begin{pmatrix} \xi_1^3 & 0 & 0 \\ \xi_2^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

It is easy to see that A is DN–elliptic, if and only if the following polynomial has no real zeros:

$$p(z) = az^4 - bz^3 - cz + d \ . \tag{21}$$

Clearly this is possible for suitable values of the parameters, e. g. $b = c = 0$ and $ad > 0$.

Then by differentiating the system with convenient operators which can easily be found by inspection and then eliminating the highest order equations we obtain

$$A^{(1)}: \quad \begin{cases} y^1_{30} + y^1 + ay^2 + by^3 = 0 \ , \\ y^1_{03} + cy^2 + dy^3 = 0 \ , \\ y^1_{11} + y^2_{10} + y^3_{01} = 0 \ , \\ -y^1_{01} + y^2_{30} - ay^2_{01} + y^3_{21} - by^3_{01} = 0 \ , \\ y^2_{12} - cy^2_{10} + y^3_{03} - dy^3_{10} = 0 \ . \end{cases}$$

So the symbols are now

$$\sigma_w A^{(1)} = \begin{pmatrix} \xi_1^3 & a & b \\ \xi_2^3 & c & d \\ 0 & \xi_1 & \xi_2 \\ 0 & \xi_1^3 & \xi_1^2\xi_2 \\ 0 & \xi_1\xi_2^2 & \xi_2^3 \end{pmatrix} \quad \text{and} \quad \sigma A^{(1)} = \begin{pmatrix} \xi_1^3 & 0 & 0 \\ \xi_2^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \xi_1^3 & \xi_1^2\xi_2 \\ 0 & \xi_1\xi_2^2 & \xi_2^3 \end{pmatrix} \ .$$

with weights $s_1 = s_2 = -3$, $s_3 = -2$, $s_4 = s_5 = 0$, $t_1 = 6$, $t_2 = t_3 = 3$. Note that $\sigma A^{(1)}$ is still not elliptic, as the second and the third columns are linearly dependent. But differentiating the last two equations of the system $A^{(1)}y = 0$ and subtracting one from another yields

$$A^{(2)}: \quad \begin{cases} y^1_{30} + y^1 + ay^2 + by^3 = 0 \ , \\ y^1_{03} + cy^2 + dy^3 = 0 \ , \\ y^1_{11} + y^2_{10} + y^3_{01} = 0 \ , \\ -y^1_{01} + y^2_{30} - ay^2_{01} + y^3_{21} - by^3_{01} = 0 \ , \\ y^2_{12} - cy^2_{10} + y^3_{03} - dy^3_{10} = 0 \ , \\ y^1_{03} + ay^2_{03} - cy^2_{30} + by^3_{03} - dy^3_{30} = 0 \ . \end{cases}$$

The principal symbol is now

$$\sigma A^{(2)} = \begin{pmatrix} \xi_1^3 & 0 & 0 \\ \xi_2^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \xi_1^3 & \xi_1^2\xi_2 \\ 0 & \xi_1\xi_2^2 & \xi_2^3 \\ \xi_2^3 & a\xi_2^3 - c\xi_1^3 & b\xi_2^3 - d\xi_1^3 \end{pmatrix} ,$$

which is elliptic, if and only if the polynomial (21) has no real zeros. Thus, we have transformed the operator A to an equivalent operator $A^{(2)}$ which is elliptic, if and only if the operator A is DN-elliptic. $\lhd$

Example 6.8 Let us rewrite the Helmholtz operator $Hy = \Delta y + cy = 0$ for $n = 2$ as a first order operator:

$$\hat{H} : \quad \begin{cases} y^1_{10} - y^2 = 0 \ , \\ y^1_{01} - y^3 = 0 \ , \\ cy^1 + y^2_{10} + y^3_{01} = 0 \ . \end{cases} \tag{22}$$

Then choosing $s_1 = s_2 = -1$, $s_3 = 0$, $t_1 = 2$, $t_2 = t_3 = 1$ gives

$$\sigma_w \hat{H} = \begin{pmatrix} \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \quad \text{and} \quad \sigma \hat{H} = \begin{pmatrix} \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} .$$

So the operator $\hat{H}$ is DN–elliptic. Adding the hidden integrability condition gives an elliptic system:

$$\hat{H}^{(1)} : \quad \begin{cases} y^1_{10} - y^2 = 0 \ , \\ y^1_{01} - y^3 = 0 \ , \\ cy^1 + y^2_{10} + y^3_{01} = 0 \ , \\ y^2_{01} - y^3_{10} = 0 \ , \end{cases} \qquad \text{and} \quad \sigma \hat{H}^{(1)} = \begin{pmatrix} \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ 0 & \xi_1 & \xi_2 \\ 0 & \xi_2 & -\xi_1 \end{pmatrix} .$$

As the following two examples demonstrate, it is not only that the completion to involution avoids the search for appropriate weights. In some cases the original system is not DN–elliptic, although the system becomes elliptic after the completion to involution. Thus we may conclude that the weights are neither necessary nor sufficient for deciding ellipticity of a differential operator.

Example 6.9 Consider the system $Ay = \nabla \times y + y = 0$. While A is not DN–elliptic, adding the integrability condition $\nabla \cdot y = 0$ gives the symbol

$$\sigma A^{(1)} = \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}$$

which is obviously elliptic. $\lhd$

Example 6.10 Let us consider the following system

$$A: \quad \begin{cases} y^1_{20} + y^2_{20} + ay^2_{01} + y^3_{11} = 0 \ , \\ y^1_{11} + y^2_{11} + cy^2_{10} + y^3_{02} = 0 \ , \\ y^1_{01} + y^2_{10} - y^3_{10} = 0 \ . \end{cases}$$

Some relevant information is contained in the first order terms in the first two equations. As second order derivatives of y^2 are present in these equations, it is not possible to choose weights such that these terms enter the symbol, and therefore the system cannot be DN–elliptic. Adding the integrability condition gives

$$A^{(1)}: \quad \begin{cases} y^1_{20} + y^2_{20} + ay^2_{01} + y^3_{11} = 0 \ , \\ y^1_{11} + y^2_{11} + cy^2_{10} + y^3_{02} = 0 \ , \\ y^1_{01} + y^2_{10} - y^3_{10} = 0 \ , \\ cy^2_{20} - ay^2_{02} = 0 \ . \end{cases}$$

Then the reduced principal symbol is:

$$\sigma_r A^{(1)} = \begin{pmatrix} \xi_1^2 & \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_1\xi_2 & \xi_2^2 \\ \xi_2 & \xi_1 & -\xi_1 \\ 0 & c\xi_1^2 - a\xi_2^2 & 0 \end{pmatrix} \ .$$

This is evidently DN–elliptic whenever $ac < 0$. Hence simply differentiating once the third equation produces an elliptic system. $\triangleleft$

In all the examples we have considered we found suitable operators by inspection and applying these operators to the original system we obtained the integrability conditions. As already indicated in Remark 5.5, the general procedure can be conveniently described with syzygies. In the proof of the main result we will need the following technical lemma. Recall that $\mathbb{A} = \mathbb{K}[\xi]$.

Lemma 6.11 *Suppose that* $B = \bigl(B_1, C_1\bigr)$, $B_1 \in \mathbb{A}^{k\times m_1}$, $C_1 \in \mathbb{A}^{k\times m_2}$, $m = m_1 + m_2$, *and let*

$$B' = \begin{pmatrix} B_1 & 0 \\ B_2 & S^T C_1 \end{pmatrix}$$

where S is the syzygy matrix of B_1^T and B_2 an arbitrary matrix of appropriate size. Then $\ker\bigl(B(\xi)\bigr) = \{0\}$ *for all* $\xi \neq 0$ *implies* $\ker\bigl(B'(\xi)\bigr) = \{0\}$ *for all* $\xi \neq 0$.

Proof Note that, by Remark 2.3, $S \neq 0$. Suppose now that there is a vector $\hat{\xi} \neq 0$ such that $\ker\bigl(B'(\hat{\xi})\bigr) \neq \{0\}$. Then there exists a $v = \bigl(\tilde{v}, \hat{v}\bigr) \neq 0$ with $B'(\hat{\xi})v = 0$ implying that $B_1(\hat{\xi})\tilde{v} = 0$. Since $\ker\bigl(B(\hat{\xi})\bigr) = \{0\}$, we have $\ker\bigl(B_1(\hat{\xi})\bigr) = \{0\}$ and then it follows that $\tilde{v} = 0$. Thus we get $S^T(\hat{\xi})C_1(\hat{\xi})\hat{v} = 0$ and $C_1(\hat{\xi})\hat{v} \in \ker\bigl(S^T(\hat{\xi})\bigr)$. Since $\ker\bigl(B_1(\xi)\bigr) = \{0\}$ for all $\xi \neq 0$, we may apply Proposition 2.7 which implies that $\ker\bigl(S^T(\hat{\xi})\bigr) = \operatorname{im}\bigl(B_1(\hat{\xi})\bigr)$. So there is some $\hat{u}$ such that $B_1(\hat{\xi})\hat{u} + C_1(\hat{\xi})\hat{v} = 0$. Putting $u = \bigl(\hat{u}, \hat{v}\bigr) \neq 0$ implies that $B(\hat{\xi})u = 0$. But this contradicts our assumption that $\ker\bigl(B(\xi)\bigr) = \{0\}$ for all $\xi \neq 0$. $\square$

Finally, we are in the position to prove the main result of this article.

Theorem 6.12 *If an operator A is DN–elliptic, then its completion to involution will lead to an equivalent elliptic operator.*

Proof Consider a DN–elliptic operator A. Then using the decomposition (20) with variables $y^{(\ell)}$ partitioned as in (19), the weighted principal symbol of the operator A may be decomposed into reduced symbols

$$\sigma_w A = (\sigma_r A_1, \ldots, \sigma_r A_b)$$

with weights $t_{\mathsf{J}_1} > t_{\mathsf{J}_2} > \cdots > t_{\mathsf{J}_b}$ and some s_i, $i = 1, \ldots, k$.

Let S be the syzygy matrix of the matrix $(\sigma_r A_1)^T$. By Remark 2.3, $S \neq 0$. Let l be the number of the columns of the matrix S. The columns of S are denote by v^r, $r = 1, \ldots, l$. Since the entries of $(\sigma_r A_1)^T$ are homogeneous polynomials, for each r there is some m_r such that the degree of v_i^r is $m_r - s_i$ or v_i^r is zero.

Substituting $\partial^{\mathbf{1}_j}$ for the variable ξ_j in the matrix S, we construct the differential operator $\hat{S}$. Let us now consider the operator $A^{(1)} = (A, \hat{S}^T A)$. If we choose $t_j^{(1)} = t_j + 1$ for $j > \mathsf{J}_1$, $s_{k+r}^{(1)} = m_r - 1$, $r = 1, \ldots, l$, and all other weights as in $\sigma_w A$, then its weighted principal symbol is of the form

$$\sigma_w A^{(1)} = \begin{pmatrix} \sigma_r A_1 & 0 \\ B & S^T\bigl(\sigma_r A_2, \ldots, \sigma_r A_b\bigr) \end{pmatrix}$$

with some matrix B of appropriate size. This choice of weights is consistent with the definition of weights since the orders of derivatives of variables $y^{(1)}$ in the rth equation of the system $\hat{S}^T A y = 0$ is smaller than or equal to $t_{\mathsf{J}_1} + m_r - 1$.

Since the symbol $\sigma_w A$ is DN–elliptic, using Lemma 6.11 in the case $\mathbb{K} = \mathbb{R}$ we get that the symbol $\sigma_w A^{(1)}$ is also DN–elliptic. So we can apply the same arguments to the operator $A^{(1)}$ and so on until we obtain an operator $A^{(\nu)}$ such that $t_{\mathsf{J}_1} = t_{\mathsf{J}_2}^{(\nu)}$.[7] Thus in a finite number of steps we have reduced a DN–elliptic operator with b block columns to an equivalent operator with $b-1$ block columns.

Continuing in this fashion, we get after a finite number of steps an operator which is equivalent to the original operator and which has an elliptic reduced symbol. But, by Remark 6.2, this suffices to prove our claim. □

Remark 6.13 Of course, in the proof of Theorem 6.12 for a DN–elliptic operator A an equivalent elliptic operator $\tilde{A}$ was constructed in a very different manner than the completion to involution outlined in Section 3.1. However, every equation appearing in the final operator $\tilde{A}$ is a differential consequence of the original system A. Thus by Definition 3.1 of an involutive system, the involutive normal form of every equation in $\tilde{A}$ with respect to the involutive completion A' of A vanishes. At the level of the principal symbols this implies that any row in $\sigma\tilde{A}$ equals a linear combination of rows in $\sigma A'$ with coefficients that are polynomials in ξ. Thus if $\sigma\tilde{A}$ has full column rank, then $\sigma A'$ must possess full column rank, too.

[7] In our general case $\nu = t_{\mathsf{J}_1} - t_{\mathsf{J}_2}$. But sometimes it is possible to set $s_{k+r}^{(i)} = m_r - c$ with some $c > 1$ for all $r = 1, \ldots, l$. In this case we have $t_{\mathsf{J}_1} = t_{\mathsf{J}_2}^{(\nu)}$ for some $\nu < t_{\mathsf{J}_1} - t_{\mathsf{J}_2}$.

Moreover, it is not really necessary for our purposes that the involutive normal form of each equation in $\tilde{A}$ vanishes. We only need that each row in the principal symbol $\sigma\tilde{A}$ is expressible as a linear combination of the rows in $\sigma A'$. This property holds not only for involutive systems as defined in Section 3.1 but also for systems obtained via other approaches to completion. This includes in particular passive systems in Janet-Riquier theory [25,43], Mansfield's differential Gröbner bases [33], Reid's reduced involutive form [41], or the geometric Cartan-Kuranishi completion [45]. $\lhd$

Example 6.14 Consider the system

$$A: \quad \begin{cases} y^1_{02} + y^2_{10} - y^3 = 0 \,, \\ y^1_{10} + y^2_{20} + y^2_{02} + y^3 = 0 \,, \\ y^1_{40} + y^2_{12} + y^3_{02} + y^4_{01} = 0 \,, \\ y^2_{33} + y^4_{40} + y^4_{04} = 0 \end{cases}$$

and its weighted principal symbol

$$\sigma_w A = \begin{pmatrix} \xi_2^2 & 0 & -1 & 0 \\ 0 & \xi_1^2 + \xi_2^2 & 1 & 0 \\ \xi_1^4 & 0 & \xi_2^2 & 0 \\ 0 & \xi_1^3 \xi_3^3 & 0 & \xi_1^4 + \xi_2^4 \end{pmatrix}$$

with weights $t_1 = t_2 = 4$, $t_3 = t_4 = 2$, $s_1 = s_2 = -2$, $s_3 = 0$, $s_4 = 2$. This system is DN–elliptic since $det(\sigma_w A) = (\xi_1^2 + \xi_2^2)(\xi_1^4 + \xi_2^4)^2$. We write the system and its symbol as $Ay = A_1 y^{(1)} + A_2 y^{(2)}$ and $\sigma_w A = \big(\sigma_r A_1, \sigma_r A_2\big)$.

Computing with SINGULAR [19] the syzygy matrix of $(\sigma_r A_1)^T$, we get

$$S = \begin{pmatrix} \xi_1^4 & 0 \\ 0 & \xi_1^3 \xi_2^3 \\ -\xi_2^2 & 0 \\ 0 & -\xi_1^2 - \xi_2^2 \end{pmatrix} \,.$$

Thus in the notation of Theorem 6.12 we have $m_1 = 2$ and $m_2 = 4$. Computing further with SINGULAR, we find that

$$\operatorname{rad}\big(I(S)\big) = \operatorname{rad}\Big(I\big((\sigma_r A_1)^T\big)\Big) = \langle \xi_1, \xi_2 \rangle \,.$$

So in this example we have in fact equality and not just inclusion as in (5).

Using the differential operator $\hat{S}$ corresponding to S, we obtain

$$\hat{S}^T A: \quad \begin{cases} y^2_{50} - y^2_{14} - y^3_{40} - y^3_{04} - y^4_{03} = 0 \,, \\ y^1_{43} + y^3_{33} - y^4_{60} - y^4_{42} - y^4_{24} - y^4_{06} = 0 \,. \end{cases}$$

The weighted principal symbol of the operator $A^{(1)} = \big(A, \hat{S}^T A\big)$ is

$$\sigma_w A^{(1)} = \begin{pmatrix} \xi_2^2 & 0 & 0 & 0 \\ 0 & \xi_1^2 + \xi_2^2 & 0 & 0 \\ \xi_1^4 & 0 & 0 & 0 \\ 0 & \xi_1^3 \xi_3^3 & 0 & 0 \\ 0 & \xi_1^5 - \xi_1 \xi_2^4 & -\xi_1^4 - \xi_2^4 & 0 \\ \xi_1^4 \xi_2^3 & 0 & \xi_1^3 \xi_2^3 & -(\xi_1^2 + \xi_2^2)(\xi_1^4 + \xi_2^4) \end{pmatrix}$$

with weights $t_3^{(1)} = t_4^{(1)} = 3$, $s_5^{(1)} = m_1 - 1 = 1$, $s_6^{(1)} = m_2 - 1 = 3$ and all other weights as in $\sigma_w A$.

Since t_1 and $t_3^{(1)}$ are not equal, we now compute the syzygy matrix S_1 of $(\sigma_r A_1^{(1)})^T$. This yields

$$S_1 = \begin{pmatrix} 0 & 0 & \xi_1^4 & 0 \\ 0 & \xi_1^3 - \xi_1\xi_2^2 & 0 & \xi_1\xi_2^5 \\ \xi_2^3 & 0 & -\xi_2^2 & 0 \\ 0 & 0 & 0 & -\xi_1^2 - \xi_2^2 \\ 0 & -1 & 0 & \xi_2^3 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

Thus we deduce that $m_1^{(1)} = 3$, $m_2^{(1)} = 1$, $m_3^{(1)} = 2$ and $m_4^{(1)} = 4$. In this case we also find the same Fitting ideals as before:

$$\operatorname{rad}\bigl(I(S_1)\bigr) = \operatorname{rad}\Bigl(I\bigl((\sigma_r A_1^{(1)})^T\bigr)\Bigr) = \langle \xi_1, \xi_2 \rangle .$$

Operating now with $\hat{S}_1^T$, we get

$$\hat{S}_1^T A^{(1)} : \quad \begin{cases} y_{15}^2 - y_{33}^3 - y_{05}^3 + y_{60}^4 + y_{42}^4 + y_{24}^4 + y_{06}^4 + y_{04}^4 = 0 , \\ y_{40}^1 - y_{22}^1 + y_{40}^3 + y_{04}^3 + y_{30}^3 - y_{12}^3 + y_{03}^4 = 0 , \\ y_{50}^2 - y_{14}^2 - y_{40}^3 - y_{04}^3 - y_{03}^4 = 0 , \\ y_{25}^1 - y_{43}^3 - y_{07}^3 + y_{15}^3 - y_{60}^4 - y_{42}^4 - y_{24}^4 - 2y_{06}^4 = 0 . \end{cases}$$

This gives for the operator $A^{(2)} = \bigl(A^{(1)}, \hat{S}_1^T A^{(1)}\bigr)$ the weighted principal symbol

$$\sigma_w A^{(2)} = \sigma_r A^{(2)} = \begin{pmatrix} \xi_2^2 & 0 & 0 & 0 \\ 0 & \xi_1^2 + \xi_2^2 & 0 & 0 \\ \xi_1^4 & 0 & 0 & 0 \\ 0 & \xi_1^3\xi_3^3 & 0 & 0 \\ 0 & \xi_1^5 - \xi_1\xi_2^4 & 0 & 0 \\ \xi_1^4\xi_2^3 & 0 & 0 & 0 \\ 0 & \xi_1\xi_2^5 & -\xi_1^3\xi_2^3 & (\xi_1^2 + \xi_2^2)(\xi_1^4 + \xi_2^4) \\ \xi_1^4 - \xi_1^2\xi_2^2 & 0 & \xi_1^4 + \xi_2^4 & 0 \\ 0 & \xi_1^5 - \xi_1\xi_2^4 & 0 & 0 \\ \xi_1^2\xi_2^5 & 0 & -\xi_1^4\xi_2^3 - \xi_2^7 & 0 \end{pmatrix}$$

with weights $t_3^{(2)} = t_4^{(2)} = 4$, $s_7^{(2)} = m_1^{(1)} - 1 = 2$, $s_8^{(2)} = m_2^{(1)} - 1 = 0$, $s_9^{(2)} = m_3^{(1)} - 1 = 1$, $s_{10}^{(2)} = m_4^{(1)} - 1 = 3$ and all other weights as in $\sigma_w A^{(1)}$.

So the reduced symbol of the operator $A^{(2)}$ is elliptic, i. e. we have transformed the DN-elliptic operator A into an equivalent operator $A^{(2)}$ with an elliptic reduced symbol. By Lemma 6.2, the involutive form of the operator $A^{(2)}$ is elliptic. $\triangleleft$

Example 6.15 Consider the system

$$A : \quad \begin{cases} y_{30}^1 + y_{20}^1 - y_{01}^2 + y^3 = 0 , \\ y_{03}^1 + y_{11}^1 + y_{10}^2 = 0 , \\ y_{12}^1 + y^3 = 0 , \end{cases} \qquad \text{and} \qquad \sigma_p A = \begin{pmatrix} \xi_1^3 & -\xi_2 & 1 \\ \xi_2^3 & \xi_1 & 0 \\ \xi_1\xi_2^2 & 0 & 1 \end{pmatrix} .$$

In this case we have $Ay = A_1y^1 + A_2y^2 + A_3y^3$. This is clearly a P–elliptic system with weights $t_1 = 3$, $t_2 = 1$ and $t_3 = 0$. Computing with SINGULAR the syzygy matrix of $(\sigma A_1)^T$, we get

$$S = \begin{pmatrix} 0 & -\xi_2^2 \\ \xi_1 & 0 \\ -\xi_2 & \xi_1^2 \end{pmatrix}$$

and thus $m_1 = 1$ and $m_2 = 2$. So we have the system

$$\hat{S}^T A: \quad \begin{cases} y^1_{21} + y^2_{20} - y^3_{01} = 0 \,, \\ -y^1_{22} + y^2_{03} + y^3_{20} - y^3_{02} = 0 \,. \end{cases}$$

We set $A^{(1)} = (A, \hat{S}^T A)$ as usual and get for the weighted principal symbol

$$\sigma_w A^{(1)} = \begin{pmatrix} \xi_1^3 & 0 & 0 \\ \xi_2^3 & 0 & 0 \\ \xi_1\xi_2^2 & 0 & 0 \\ \xi_1^2\xi_2 & \xi_1^2 & -\xi_2 \\ -\xi_1^2\xi_2^2 & \xi_2^3 & \xi_1^2 - \xi_2^2 \end{pmatrix}$$

with weights $t_2^{(1)} = 2$, $t_3^{(1)} = 1$, $s_4 = 0$, $s_5 = 1$, and all other weights as in $\sigma_p A$. It is evident that the operator $A^{(1)}$ is DN-elliptic. Note that its weighted principal symbol has still the form $\sigma_w A^{(1)} = \big(\sigma_r A_1^{(1)}, \sigma_r A_2^{(1)}, \sigma_r A_3^{(1)}\big)$ but now we have $t_1 - t_2^{(1)} < t_1 - t_2$. Now we get for the syzygy matrix of $\sigma_r A_1^{(1)}$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & -\xi_2 \\ 0 & \xi_1 & 0 & 0 \\ 0 & -\xi_2 & \xi_1 & 0 \\ \xi_2 & 0 & -\xi_2 & \xi_1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

This yields $m_r^{(1)} = 1$ for all $r = 1, \ldots, 4$. Now operating with $\hat{S}_1^T$ gives

$$\hat{S}_1^T A^{(1)}: \quad \begin{cases} y^2_{21} + y^2_{03} + y^3_{20} - 2y^3_{02} = 0 \,, \\ y^1_{21} + y^2_{20} - y^3_{01} = 0 \,, \\ -y^2_{21} + y^3_{02} + y^3_{10} = 0 \,, \\ -y^1_{21} + y^2_{30} + y^2_{02} - y^3_{11} - y^3_{01} = 0 \,. \end{cases}$$

We set $A^{(2)} = (A^{(1)}, \hat{S}_1^T A^{(1)})$. The second equation in the system $\hat{S}_1^T A^{(1)} y = 0$ is equal to the first equation in the system $\hat{S}^T A y = 0$. So we can drop one of these equations and the weighted principal symbol of the operator $A^{(2)}$ is

$$\sigma_w A^{(2)} = \begin{pmatrix} \xi_1^3 & 0 & 0 \\ \xi_2^3 & 0 & 0 \\ \xi_1\xi_2^2 & 0 & 0 \\ \xi_1^2\xi_2 & 0 & 0 \\ -\xi_1^2\xi_2^2 & 0 & 0 \\ 0 & \xi_1^2\xi_2 + \xi_2^3 & \xi_1^2 - 2\xi_2^2 \\ 0 & -\xi_1^2\xi_2 & \xi_2^2 \\ -\xi_1^2\xi_2 & \xi_1^3 & -\xi_1\xi_2 \end{pmatrix}$$

with weights $t_1 = t_2^{(2)} = 3$, $t_3^{(2)} = 2$, $s_i = 0$, $i = 6, \ldots 8$, and all other weights as in $\sigma_w A^{(1)}$. It is evident that the operator $A^{(2)}$ is DN-elliptic. Hence, we transformed the P–elliptic system with three blocks into the DN-elliptic system $A^{(2)}y = A_1^{(2)} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} + A_2^{(2)} y^3$ with two blocks with the help of syzygy matrices.

Computing the syzygy matrix of $\sigma A_1^{(2)}$, we find

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & -\xi_2 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 & 0 & 0 \\ 0 & -\xi_2 & \xi_1 & 0 & 0 & 0 \\ \xi_2 & 0 & -\xi_2 & \xi_1 & \xi_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & \xi_1 \\ 0 & 0 & 0 & 0 & 0 & \xi_1^2 \\ 0 & 0 & 0 & 0 & \xi_1 & \xi_2^2 \\ 0 & 0 & 0 & 0 & \xi_2 & -\xi_1\xi_2 \end{pmatrix} .$$

So $m_r^{(2)} = 1$, $r = 1, \ldots, 5$ and $m_6^{(2)} = 2$. Then we get the system

$$\hat{S}_2^T A^{(2)} : \quad \begin{cases} y^2_{21} + y^2_{03} + y^3_{20} - 2y^3_{02} = 0 \,, \\ y^1_{21} + y^2_{20} - y^3_{01} = 0 \,, \\ -y^2_{21} + y^3_{02} + y^3_{10} = 0 \,, \\ -y^1_{21} + y^2_{30} + y^2_{02} - y^3_{11} - y^3_{01} = 0 \,, \\ y^2_{21} + y^2_{03} + y^3_{20} - 2y^3_{02} = 0 \,, \\ y^3_{40} - y^3_{22} + y^3_{04} + y^3_{30} + y^3_{12} = 0 \,. \end{cases}$$

Consider the operator $A^{(3)} = (A^{(2)}, \hat{S}_2^T A^{(2)})$. Again in the system $A^{(3)}y = 0$ some equations appear twice. Removing the duplicates, we obtain

$$A^{(3)} : \quad \begin{cases} y^1_{30} + y^1_{20} - y^2_{01} + y^3 = 0 \,, \\ y^1_{03} + y^1_{11} + y^2_{10} = 0 \,, \\ y^1_{12} + y^3 = 0 \,, \\ y^1_{21} + y^2_{20} - y^3_{01} = 0 \,, \\ -y^1_{22} + y^2_{03} + y^3_{20} - y^3_{02} = 0 \,, \\ y^2_{21} + y^2_{03} + y^3_{20} - 2y^3_{02} = 0 \,, \\ -y^2_{21} + y^3_{02} + y^3_{10} = 0 \,, \\ -y^1_{21} + y^2_{30} + y^2_{02} - y^3_{11} - y^3_{01} = 0 \,, \\ y^3_{40} - y^3_{22} + y^3_{04} + y^3_{30} + y^3_{12} = 0 \,. \end{cases}$$

The weighted principal symbol of the operator $A^{(3)}$ is

$$\sigma_w A^{(3)} = \begin{pmatrix} \xi_1^3 & 0 & 0 \\ \xi_2^3 & 0 & 0 \\ \xi_1\xi_2^2 & 0 & 0 \\ \xi_1^2\xi_2 & 0 & 0 \\ -\xi_1^2\xi_2^2 & 0 & 0 \\ 0 & \xi_1^2\xi_2 + \xi_2^3 & 0 \\ 0 & -\xi_1^2\xi_2 & 0 \\ -\xi_1^2\xi_2 & \xi_1^3 & 0 \\ 0 & 0 & \xi_1^4 - \xi_1^2\xi_2^2 + \xi_2^4 \end{pmatrix}$$

with weights $t_1 = t_2^{(3)} = t_3^{(3)} = 3$, $s_i = 0$, $i = 1, \ldots 4, 6, \ldots 8$, $s_5 = s_9 = 1$. Obviously, this reduced symbol is elliptic. Hence according to Corollary 5.7 we obtain an elliptic system from $A^{(3)}y = 0$ by differentiating some equations of it.

Computing with *MuPAD*[8] [5] an involutive completion of $Ay = 0$, we get

$$\begin{cases} y^1_{30} + y^1_{20} - y^2_{01} + y^3 = 0 \ , \\ y^1_{03} + y^1_{11} + y^2_{10} = 0 \ , \\ y^1_{12} + y^3 = 0 \ , \\ y^1_{21} + y^2_{20} - y^3_{01} = 0 \ , \\ y^2_{21} + y^2_{03} + y^3_{20} - 2y^3_{02} = 0 \ , \\ y^2_{30} + y^2_{20} + y^2_{02} - y^3_{11} - 2y^3_{01} = 0 \ , \\ y^2_{21} - y^3_{02} - y^3_{10} = 0 \ , \\ y^2_{22} - y^3_{03} - y^3_{11} = 0 \ , \\ y^3_{40} - y^3_{22} + y^3_{04} + y^3_{12} + y^3_{30} = 0 \ . \end{cases}$$

One easily verifies that it is equivalent to $A^{(3)}$. $\triangleleft$

Remark 6.16 Let us finally compare our reduction process to the one proposed by Cosner [11]. Consider the Laplace equation in 2 dimensions written as a first order system, i. e. our system (22) with $c = 0$. After adding the integrability condition, we obtained the following elliptic system:

$$\begin{cases} y^1_{10} - y^2 = 0 \ , \\ y^1_{01} - y^3 = 0 \ , \\ y^2_{10} + y^3_{01} = 0 \ , \\ y^2_{01} - y^3_{10} = 0 \ . \end{cases}$$

[8] www.mupad.de

However, Cosner's approach produces the system

$$\begin{cases} y^1_{10} + y^2_{01} = 0 \ , \\ -y^1 + y^3 = 0 \ , \\ -y^2 + y^4 = 0 \ , \\ y_{10} - y^3 = 0 \ , \\ y_{01} - y^4 = 0 \ , \\ y^3_{01} - y^4_{10} = 0 \ , \\ -y^1_{10} + y^3_{10} = 0 \ , \\ -y^1_{01} + y^3_{01} = 0 \ , \\ -y^2_{10} + y^4_{10} = 0 \ , \\ -y^2_{01} + y^4_{01} = 0 \ , \end{cases}$$

which is elliptic in the sense of Definition 4.1, i. e. without weights. Hence, at least for this particular example, our approach produces a much smaller equivalent elliptic system than Cosner's construction. Furthermore, while Cosner is exclusively concerned with the question of ellipticity, we have embedded this problem in the general context of completion which is useful in many other respects, too. As already mentioned in the Introduction, any system of differential equations should be completed to involution before any subsequent analysis and our results show that this automatically takes care of the question of ellipticity. $\triangleleft$

7 Some Reductions for Elliptic Systems

In this section we consider two classical operations with differential systems: the reduction to a lower order system and the reduction to one dependent variable. Our goal is to show explicitly that in both cases ellipticity is preserved.

7.1 Preliminaries

In Section 2.2 we introduced the special differential operator j^q mapping a function $y = (y^1, \dots, y^m)$ to its derivatives up to order q. Let us consider the case $m = 1$. Obviously, j^q is an overdetermined operator with $k = d_q$ and $m = 1$. We will now apply the results of Section 3.3 to it and determine its compatibility operator.[9] It is a trivial exercise to verify that j^q is an involutive operator so that for the construction of the compatibility conditions we must only study the non-multiplicative derivatives of each equation.

Let us write $j^q y = z$. Thus the right hand side z is a vector of dimension d_q and we will denote its components by z^μ where $\mu \in \mathbb{N}_0^n$ runs over all multi indices

[9] An intrinsic description of this compatibility condition, which is sometimes called the *Spencer operator*, is contained in [15,49].

with $0 \leq |\mu| \leq q$. The compatibility conditions (11) take now the form

$$\partial_i z^{\mu} = \begin{cases} z^{\mu+\mathbf{1}_i} & 0 \leq |\mu| < q,\ 1 \leq i \leq n\ , \\ \partial_k z^{\mu-\mathbf{1}_k+\mathbf{1}_i} & |\mu| = q,\ i > \operatorname{cls}\mu,\ k = \operatorname{cls}\mu\ . \end{cases} \tag{23}$$

These equations define a differential operator D_1^q. As both j^q and D_1^q are linear operators with constant coefficients, we may apply the fundamental principle. It tells us that at the level of smooth or distributional solutions $z = j^q y$, *if and only if* $D_1^q z = 0$.

We have described j^q and D_1^q only for the case $m = 1$, but obviously for several unknown functions one must simply take one copy of D_1^q for each component y^j, as there is no interaction between the different components. Thus in the general case the compatibility operator is of the form $D_1^q \otimes I_m$ where I_m is the $m \times m$ identity matrix.

We continue with determining the principal symbols of the operators j^q and D_1^q. Recall that Ξ^j is the vector of all monomials in ξ of degree j; in particular $\Xi^1 = \xi$. Then we find for the principal symbols $\sigma j^q = \begin{pmatrix} 0 \\ \Xi^q \end{pmatrix} \otimes I_m$.

The situation is slightly more complicated for the operator D_1^q. The compatibility systems splits naturally into two parts: the first subsystem corresponds to the first line in (23), i. e. the functions z^{μ} with $|\mu| < q$; the second subsystem to the second line, i. e. the functions z^{μ} with $|\mu| = q$. If we sort the equations by ascending length of μ, then the symbol of the first subsystem is simply $\xi \otimes I_{d_{q-1}}$. We denote the operator defined by the second subsystem as $\bar{D}_1^q$. There does not seem to exist a simple closed-form description of its symbol.

Example 7.1 Let us take $n = q = 2$. The operator j^2 has already been given in (2). If we write $\partial^{\mu} y = z^{\mu}$, then the corresponding compatibility system $D_1^2 z = 0$ consists of the following eight equations:

$$\begin{gathered} \partial_1 z^{00} = z^{10}\ ,\ \ \partial_2 z^{00} = z^{01}\ ,\ \ \partial_1 z^{10} = z^{20}\ ,\ \ \partial_2 z^{10} = z^{11}\ ,\ \ \partial_1 z^{01} = z^{11}\ , \\ \partial_2 z^{01} = z^{02}\ ,\ \ \partial_2 z^{20} - \partial_1 z^{11} = 0\ ,\quad \partial_2 z^{11} - \partial_1 z^{02} = 0\ . \end{gathered}$$

If we sort the unknown functions z^{μ} as z^{00}, z^{10}, z^{01}, z^{20}, z^{11}, z^{02}, then the principal symbol has the form

$$\sigma D_1^2 = \begin{pmatrix} \xi_1 & & & & & \\ \xi_2 & & & & & \\ & \xi_1 & & & & \\ & \xi_2 & & & & \\ & & \xi_1 & & & \\ & & \xi_2 & & & \\ & & & \xi_2 & -\xi_1 & \\ & & & & \xi_2 & -\xi_1 \end{pmatrix}\ .$$

Lemma 7.2 $\ker(\sigma\bar{D}_1^q) = \operatorname{span}(\Xi^q)$ *and consequently* $\ker(\sigma D_1^q) = \operatorname{span}\left(\left(\begin{smallmatrix} 0 \\ \Xi^q \end{smallmatrix}\right)\right)$.

Proof By construction, $\bar{D}_1^q$ is the compatibility operator of the differential operator given by all qth order derivatives. As both $\bar{D}_1^q$ and this operator consist only of derivatives of the same order, the respective principal symbols look like the operators. Thus the rows of the matrix $\sigma\bar{D}_1^q$ form a basis of the first syzygy module of the ideal generated by the entries of Ξ^q. But this observation immediately entails the first claim. The second claim is a trivial consequence of the simple form of the symbol of the first subsystem of the compatibility system $D_1^q z = 0$. □

7.2 Reduction to Lower Order System

Let us again consider a differential operator A of order q as in (1). We want to write A as $A = \bar{A}^{(\ell)} \circ j^{q-\ell}$ with an operator $\bar{A}^{(\ell)}$ of order ℓ for $0 < \ell < q$. In the limiting case $\ell = q$ we can set $\bar{A}^{(q)} = A$. The most important case is $\ell = 1$, but the general case has also some interest. Let us further define $A^{(\ell)} z = \bigl(\bar{A}^{(\ell)} z, (D_1^{q-\ell} \otimes I_m) z\bigr)$.

We may construct $\bar{A}^{(\ell)}$ as follows. In the equation $Ay = 0$ all derivatives y_μ^j with $|\mu| \leq q$ may appear. We introduce new dependent variables for all derivatives of order less than or equal to $q - \ell$ and denote them as above by $z^{j,\mu}$ with multi indices $|\mu| \leq q - \ell$. The operator $\bar{A}^{(\ell)}$ is now obtained from A by performing the following substitution in the equation $Ay = 0$:

$$y_\mu^j \longmapsto \begin{cases} z^{j,\mu} & \text{if } |\mu| \leq q - \ell\ , \\ \partial^{\mu_2} z^{j,\mu_1} & \text{if } |\mu| > q - \ell \text{ where } |\mu_1| = q - \ell \text{ and } \mu_1 + \mu_2 = \mu\ . \end{cases} \tag{24}$$

Obviously there are many ways to perform such a substitution, as there are many ways to split the multi index μ into two parts. However, for our purposes any choice is fine.

Lemma 7.3 *The operators A and $A^{(\ell)}$ are equivalent in the smooth category.*

Proof This lemma is a straightforward consequence of the fundamental principle. As already mentioned above, for any order $r > 0$ every smooth solution of the equation $(D_1^r \otimes I_m) z = 0$ is of the form $z = j^r y$ for a smooth function y.

Let y be a solution of $Ay = 0$. The definition of the operator $A^{(\ell)}$ entails immediately that $z = j^{q-\ell} y$ is a solution of $A^{(\ell)} z = 0$ as $\bar{A}^{(\ell)} z = Ay$ and $(D_1^{q-\ell} \otimes I_m) z = 0$ by the definition of the compatibility operator $D_1^{q-\ell} \otimes I_m$. Conversely, let z be a solution of $A^{(\ell)} z = 0$. This implies in particular that $(D_1^{q-\ell} \otimes I_m) z = 0$ and thus by the consideration above that $z = j^{q-\ell} y$. But then $0 = \bar{A}^{(\ell)} z = Ay$ and y is a solution of $Ay = 0$. Hence the operator $j^{q-\ell}$ defines a bijection between the smooth solution spaces of the differential operators A and $A^{(\ell)}$. □

For simplicity, we stated this result in the smooth category. But as already mentioned in Section 3.3, in the case of operators with constant coefficients the fundamental principle remains true for distributional solutions. Thus the equivalence also holds in much larger function spaces and in particular for weak solutions.

For analysing the ellipticity of the operators $A^{(\ell)}$, we need their symbols. To this end let us introduce matrices Ξ_q by the requirement $\Xi^{q+1} = \Xi_q \Xi^q$. Hence we can write

$$\Xi^q = \Xi_{q-1} \cdots \Xi_1 \xi \ . \tag{25}$$

Evidently the choice of the matrices Ξ_q can be done in many ways; however, fixing the rules used in the substitutions in (24) fixes the matrices, and conversely choosing some matrices fixes the substitutions.

We can represent the symbols $\sigma\bar{A}^{(\ell)}$ as matrices of size $k \times md_{q-\ell}$ which are of the form $\sigma\bar{A}^{(\ell)} = \bigl(0, \sigma\bar{B}^{(\ell)}\bigr)$ and $\sigma\bar{B}^{(\ell)}$ is of size $k \times mn_{q-\ell}$. In the limiting cases we have $\sigma\bar{B}^{(0)} = M_q$, i. e. the geometric symbol, and $\sigma\bar{A}^{(q)} = \sigma A$, i. e. the principal symbol of the original operator.

Lemma 7.4 *The principal symbols of the operators $A^{(\ell)}$ are given by*

$$\sigma_r A^{(\ell)} = \begin{pmatrix} 0 & \sigma\bar{B}^{(\ell)} \\ \xi \otimes I_{md_{q-\ell-1}} & 0 \\ 0 & \sigma(\bar{D}_1^{q-\ell} \otimes I_m) \end{pmatrix}$$

where $\sigma\bar{B}^{(\ell)} = M_q(\Xi_{q-1} \cdots \Xi_{q-\ell} \otimes I_m)$.

Proof We prove only the formula for $\sigma\bar{B}^{(\ell)}$. Since $A = \bar{A}^{(\ell)} \circ j^{q-\ell}$, the symbols satisfy $\sigma A = \sigma\bar{A}^{(\ell)} \cdot \sigma j^{q-\ell}$ and as mentioned above $\sigma j^{q-\ell} = \begin{pmatrix} 0 \\ \Xi^{q-\ell} \end{pmatrix} \otimes I_m$. Then we obtain using (3) and (25)

$$\begin{aligned}\sigma A =& M_q(\Xi^q \otimes I_m) = M_q(\Xi_{q-1}\Xi^{q-1} \otimes I_m) = M_q(\Xi_{q-1} \otimes I_m)(\Xi^{q-1} \otimes I_m) \\ =& M_q(\Xi_{q-1} \otimes I_m) \cdots (\Xi_{q-\ell} \otimes I_m)(\Xi^{q-\ell} \otimes I_m) \\ =& M_q(\Xi_{q-1} \cdots \Xi_{q-\ell} \otimes I_m)(\Xi^{q-\ell} \otimes I_m) \\ =& \bigl(0, M_q(\Xi_{q-1} \cdots \Xi_{q-\ell} \otimes I_m)\bigr)\sigma j^{q-\ell}\end{aligned}$$

implying our claim. □

Let us set $\sigma_r\bar{C}^{(\ell)} = \begin{pmatrix} \sigma\bar{B}^{(\ell)} \\ \sigma(\bar{D}_1^{q-\ell} \otimes I_m) \end{pmatrix}$. It is evident that ellipticity of the symbol $\sigma_r A^{(\ell)}$ is equivalent to ellipticity of the symbol $\sigma_r\bar{C}^{(\ell)}$.

Example 7.5 Let us take the Laplacian in $\mathbb{R}^3$. Then

$$M_2 = \bigl(1\ 0\ 1\ 0\ 0\ 1\bigr)\ , \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}\ ,$$

$$\Xi_1 = \begin{pmatrix} 0 & 0 & \xi_3 \\ 0 & \xi_3 & 0 \\ 0 & \xi_2 & 0 \\ \xi_3 & 0 & 0 \\ \xi_2 & 0 & 0 \\ \xi_1 & 0 & 0 \end{pmatrix}\ , \qquad \sigma_r\bar{C}^{(1)} = \begin{pmatrix} M_2\Xi_1 \\ \sigma\bar{D}_1^1 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}\ ,$$

$$\sigma A = M_2\Xi_1\xi = \xi_1^2 + \xi_2^2 + \xi_3^2\ .$$

Note that the first row of $\sigma_r \bar{C}^{(1)}$ corresponds to the divergence and the remaining ones to the curl of a vector field. Hence from the Laplacian in three dimensions we get canonically the curl-div system when we rewrite it as a first order system. $\lhd$

Theorem 7.6 *If the operator $A^{(\ell)}$ is elliptic for some ℓ, then the operators $A^{(\ell)}$ are elliptic for all $0 < \ell \leq q$.*

Proof Now suppose that A is not elliptic. Then there is a vector $v \neq 0$ such that $\sigma A v = 0$. Let $v^{(\ell)} = (\Xi^{q-\ell} \otimes I_m) v = \Xi^{q-\ell} \otimes v$. Obviously $v^{(\ell)} \neq 0$ for $\xi \neq 0$, and $\sigma(\bar{D}_1^{q-\ell} \otimes I_m) v^{(\ell)} = 0$ by Lemma 7.2. But then

$$\begin{aligned}\sigma \bar{B}^{(\ell)} v^{(\ell)} &= M_q(\Xi_{q-1} \cdots \Xi_{q-\ell} \otimes I_m)(\Xi^{q-\ell} \otimes v) \\ &= M_q(\Xi^q \otimes v) = M_q(\Xi^q \otimes I_m) v = \sigma A v = 0 \ .\end{aligned}$$

This implies that $\sigma_r \bar{C}^{(\ell)} v^{(\ell)} = 0$ and hence $A^{(\ell)}$ is not elliptic.

On the other hand suppose that $A^{(\ell)}$ is not elliptic. Hence there is a $v^{(\ell)} \neq 0$ such that $\sigma_r \bar{C}^{(\ell)} v^{(\ell)} = 0$. In particular then $\sigma(\bar{D}_1^{q-\ell} \otimes I_m) v^{(\ell)} = 0$. But then by Lemma 7.2 $v^{(\ell)} = (\Xi^{q-\ell} \otimes I_m) v$ for some $v \neq 0$ and we get

$$\begin{aligned}\sigma A v =& M_q(\Xi_{q-1} \dots \Xi_1 \xi \otimes I_m) v = M_q(\Xi_{q-1} \dots \Xi_{q-\ell} \Xi^{q-\ell} \otimes v) \\ =& M_q(\Xi_{q-1} \dots \Xi_{q-\ell} \otimes I_m)(\Xi^{q-\ell} \otimes v) = \sigma \bar{B}^{(\ell)} v^{(\ell)} = 0 \ .\end{aligned}$$

Hence the operator A is not elliptic either. $\square$

As a consequence we have the following

Theorem 7.7 *Any P–elliptic system is equivalent to an elliptic system.*

Proof Consider the system (20) where each operator A_l is elliptic. Let us set $\ell = t_m \geq 1$. By Theorem 7.6 we may replace A_l by $A_l^{(\ell)}$ which is also elliptic. The reduced symbol of the resulting operator $A^{(\ell)}$ is

$$\sigma_r A^{(\ell)} = \begin{pmatrix} \sigma \bar{A}_1^{(\ell)} & \sigma \bar{A}_2^{(\ell)} & \dots & \sigma \bar{A}_{b-1}^{(\ell)} & \sigma A_b \\ \sigma(D_1^{t_{J_1}-\ell} \otimes I_{J_1}) & 0 & \dots & 0 & 0 \\ 0 & \sigma(D_1^{t_{J_2}-\ell} \otimes I_{J_2}) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma(D_1^{t_{J_{b-1}}-\ell} \otimes I_{J_{b-1}}) & 0 \end{pmatrix}$$

This is clearly elliptic. $\square$

7.3 Reduction to One Unknown Function

With the help of a little trick apparently due to Drach [13], one may rewrite any system of differential equations in several unknown functions as a system in only one unknown function. It requires the introduction of one new independent variable for each unknown function and raises the order of the system by one.

Assume that the original linear qth order system $Ay = f$ contains as usual the independent variables $x^1, \dots, x^n$ and the dependent variables $y^1, \dots, y^m$. Then we introduce m additional independent variables $\hat{x}_j$ and one new dependent variable $\hat{y}$ related to the old ones by the relation

$$\hat{y} = \hat{x}_1 y^1 + \cdots + \hat{x}_m y^m \ .$$

This allows us to represent any derivative $\partial^{|\mu|} y^j / \partial x^\mu$ as $\partial^{|\mu|+1} \hat{y} / \partial x^\mu \partial \hat{x}_j$. If we perform the corresponding substitutions in our system and if we add the equation

$$\frac{\partial^2 \hat{y}}{\partial \hat{x}_j \partial \hat{x}_k} = 0 \ , \qquad j, k = 1, \dots, m \ , \tag{26}$$

then we obtain a new system of order $q + 1$ in only one dependent variable; the new operator thus obtained will be denoted by $\hat{A}$. Any solution of it has the form

$$\hat{y}(x, \hat{x}) = \hat{x}_1 y^1(x) + \cdots + \hat{x}_m y^m(x) + \Lambda(x) \tag{27}$$

where $\Lambda(x)$ is an arbitrary function and $y(x)$ a solution of the original system. One may consider the appearance of the function Λ as a kind of "gauge symmetry". It is not difficult to show that the new system is involutive, if and only if the original system is involutive [45, App. A.3].

Let us analyse the symbol of the transformed system $\hat{A}$. Denote the dual variables for the new independent variables by $\hat{\xi}$. Then it is not difficult to see that

$$\sigma_r \hat{A} = \begin{pmatrix} \sigma A \hat{\xi} \\ \hat{\Xi}^2 \end{pmatrix} \ .$$

Obviously, $\hat{A}$ is not elliptic, because the symbol vanishes for any pair $(\xi, \hat{\xi})$ with $\hat{\xi} = 0$. Obviously, this is due to the appearance of the arbitrary function $\Lambda(x)$ in (27) and hence removing this arbitrariness should lead to an elliptic system. A simple possibility consists of adding the "gauge fixing" condition

$$\sum_{j=1}^{m} \hat{x}_j \hat{y}_{\hat{x}_j} - \hat{y} = 0 \ . \tag{28}$$

It follows trivially from (27) that this equation is compatible with the operator $\hat{A}$, as its sole effect is to require $\Lambda = 0$. Furthermore, the augmented system is equivalent to our original system, as now the solutions are in a one-to-one correspondence.

The addition of (28) still does not make the system elliptic. But as the augmented system is no longer involutive, we must complete it. We show now that the completed system is elliptic. It is convenient to state the problem in ideal theoretic terms. Let $\mathbb{R}[\xi, \hat{\xi}]$ be the polynomial ring and introduce the polynomials

$$p_0 = \langle \hat{x}, \hat{\xi} \rangle - 1 \ , \quad p_j = \left(\sigma A \hat{\xi} \right)_j \ , \quad p_{ij} = \hat{\xi}_i \hat{\xi}_j \ .$$

The first polynomial is the *full* symbol of the "gauge fixing" condition (28); the remaining ones are the entries of the reduced principal symbol $\sigma_r \hat{A}$. The analysis of the ideal $\mathcal{I}$ generated by these polynomials yields some information about the principal symbol of the augmented system.

Lemma 7.8 *If the principal symbol σA is a square matrix, then the ideal $\mathcal{I}$ contains the polynomial* $\det(\sigma A) \in \mathbb{R}[\xi]$.

Proof Let p denote the vector with the entries p_j and $\mathrm{adj}(\sigma A)$ the adjoint matrix of the principal symbol σA. Then $\langle \hat{x}, \mathrm{adj}(\sigma A)p\rangle = \langle \hat{x}, \mathrm{adj}(\sigma A)\sigma A\hat{\xi}\rangle = \det(\sigma A)\langle \hat{x}, \hat{\xi}\rangle$ by definition of the adjoint. Thus $\det(\sigma A) = \langle \hat{x}, \mathrm{adj}(\sigma A)\, p\rangle - \det(\sigma A)\, p_0$ implying our claim. □

Theorem 7.9 *If the linear operator A is elliptic, then the involutive completion of the linear system consisting of the Drach transformed operator $\hat{A}$ and the equation* (28) *is elliptic, too.*

Proof Let us first consider the case that A is a square operator. By the previous lemma, we know that during the completion of the transformed system an integrability condition arises the principal part of which is given by $\det(\sigma A)$. The principal symbol of the corresponding system is elliptic, as by assumption $\det(\sigma A) \neq 0$ for all $\xi \neq 0$ and $\hat{\Xi}^2 \neq 0$ for all $\hat{\xi} \neq 0$. As ellipticity is preserved during the completion, we are done.

If the operator A is not square, then its ellipticity implies that we may choose for each vector $\xi \neq 0$ a square subsystem A' such $\det(\sigma A') \neq 0$. It follows now by the same argument as in the proof of the lemma above, that during the completion of the transformed system an integrability condition arises the principal part of which is $\det(\sigma A')$. As this argument holds for all vectors $\xi \neq 0$, the completion must lead to an elliptic symbol. □

Example 7.10 Consider the modified Cauchy-Riemann system

$$\begin{cases} y^1_{10} - y^2_{01} + y^1 = 0\ , \\ y^1_{01} + y^2_{10} + y^2 = 0\ . \end{cases}$$

The Drach transformation with gauge fixing yields the second order system

$$\begin{cases} \hat{y}_{1010} - \hat{y}_{0101} + \hat{y}_{0010} = 0\ , \\ \hat{y}_{0110} + \hat{y}_{1001} + \hat{y}_{0001} = 0\ , \\ \hat{x}_1\hat{y}_{0010} + \hat{x}_2\hat{y}_{0001} - \hat{y} = 0\ , \\ \hat{y}_{0020} = \hat{y}_{0011} = \hat{y}_{0002} = 0\ . \end{cases}$$

Note that we have now four-dimensional multi indices where the first two entries correspond to derivatives with respect to x^1, x^2 and the last two entries to derivatives with respect to $\hat{x}_1, \hat{x}_2$. We have

$$p_0 = \hat{x}_1\hat{\xi}_1 + \hat{x}_2\hat{\xi}_2 - 1\ , \quad p_1 = \xi_1\hat{\xi}_1 - \xi_2\hat{\xi}_2\ , \quad p_2 = \xi_2\hat{\xi}_1 + \xi_1\hat{\xi}_2\ .$$

Then we compute

$$\hat{x}_1(\xi_1 p_1 + \xi_2 p_2) + \hat{x}_2(-\xi_2 p_1 + \xi_1 p_2) - |\xi|^2 p_0 = |\xi|^2$$

indicating that the completion of the transformed system is elliptic. Indeed, if we perform the same computation with the full differential equations, we obtain the following result. Denote the equation (28) by f_0 and the equations of the system by f_1 and f_2. Then we find

$$\hat{x}_1(\partial_{x^1} f_1 + \partial_{x^2} f_2) + \hat{x}_2(-\partial_{x^2} f_1 + \partial_{x^1} f_2) - \left(\partial^2_{x^1} + \partial^2_{x^2}\right) f_0 =$$
$$\hat{y}_{2000} + \hat{y}_{0200} + \hat{x}_1\left(\hat{y}_{1010} + \hat{y}_{0101}\right) + \hat{x}_2\left(\hat{y}_{1001} - \hat{y}_{0110}\right) =$$
$$\hat{y}_{2000} + \hat{y}_{0200} + 2\hat{y}_{1000} + \hat{y} = 0$$

The symbol of this integrability condition is clearly $|\xi|^2$ which is the determinant of the principal symbol of our modified Cauchy-Riemann system. $\lhd$

8 Conclusions

Agmon [1, pp. 63–67] developed a regularity theory for overdetermined elliptic systems in one dependent variable. As shown in Section 7.3, we may rewrite any overdetermined system in an arbitrary number of dependent variables as an equivalent one in one dependent variable. Furthermore, ellipticity is preserved by this operation, if we perform the mentioned “gauge fixing”. Thus we may extend Agmon’s results to arbitrary elliptic systems. Of course one can formulate such results directly without Drach’s transformation. In [15] and [50] one can find some relevant a priori estimates in terms of Sobolev space norms which show precisely the regularity of the solution in terms of the data. In fact in these estimates the weights needed in DN–elliptic symbols get a rather natural interpretation. Evidently to get the relevant estimates one should also specify correct boundary conditions. It turns out in addition of ellipticity of the operator the boundary operators should satisfy the *Shapiro–Lopatinskij* condition. Discussing this condition is beyond the scope of the present paper and we just refer to [4, 15] for definitions.

Anyway we have shown that in general it is necessary first to transform the given system to involutive form before one can decide whether or not it is elliptic. This is consistent with the observation that whatever property of the system one is interested in, it is in general necessary to compute the involutive form before the analysis. Of course in some situations the full involutive form may not be necessary but on the other hand there are situations where it is rather clear that the problems encountered are only due to the fact that the given system is not involutive. As an example we might cite the problems in the numerical solution of DAEs [52, 53] and the spurious solutions in computational electromagnetics [26].

Moreover we have seen that the notion of DN–ellipticity, while perhaps convenient in certain situations, does not define a larger class of systems than elliptic ones. Its apparent generality is only a consequence of restricting attention to square systems. Square systems are convenient in many ways, but the property of “squareness” is in no way intrinsic, so this restriction is conceptually rather artificial.

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