Werner M. Seiler

Structure Analysis of Polynomial Modules with Pommaret Bases

We show how Pommaret bases of submodules of free polynomial modules can be used to extract structural information like the dimension, the depth or the Castelnuovo-Mumford regularity. In particular, we discuss the syzygy theory of Pommaret bases and its use for the construction of free resolutions.

1. Involutive Bases

Involutive bases are a comparatively recent concept in commutative algebra. They have been introduced by Gerdt and Blinkov [3] based on ideas from the Janet-Riquier theory of differential equations. A gentle introduction into the basic ideas can be found in [1]; [5] presents the theory in full detail and extends it to a large class of non-commutative algebras including for example rings of linear differential or difference operators or the universal enveloping algebras of finite-dimensional Lie algebras. We report here on some applications of Pommaret bases in the structure analysis of arbitrary polynomial modules. All details and proofs can be found in [6].

Involutive bases are a special kind of Gröbner bases with additional combinatorial properties. They are defined not only with respect to a term order but also with respect to an *involutive division*. The latter induces a restriction of the usual divisibility relation of monomials. Roughly speaking, it assigns each polynomial in the basis a set of *multiplicative variables*. Each polynomial may only be multiplied by polynomials in these variables; but despite this restriction an involutive basis still generates the full submodule. One effect of this restriction is that involutive standard representations are unique in contrast to the situation with ordinary Gröbner bases. A very important kind of involutive bases are *Pommaret bases*. Here the multiplicative variables are determined by a very simple rule. Let k be the smallest index such that x_k divides the leading term of a polynomial. Then we call k the *class* of the polynomial and assign the multiplicative variables $x_1, \dots, x_k$.

There exists a very simple algorithm for the construction of involutive bases. We check for each element of the bases what happens, if we multiply it by one of its non-multiplicative variables. If the involutive normal form of the result does not vanish, it is added to the basis. As soon as all these involutive normal forms vanish, we have reached an involutive basis. It should be mentioned that the question of termination of this algorithm, or more generally of the existence of finite involutive bases, is much more complicated than the corresponding questions for Gröbner bases. Experiments have shown that for many examples this algorithm outperforms Buchberger's algorithm (but naturally there are also examples where the opposite happens).

A very efficient `C` program for the construction of Janet bases is described together with benchmarks in [4]. We have provided a generic implementation for determining involutive bases (with respect to arbitrary involutive divisions) for ideals in polynomial algebras of solvable type within the computer algebra system *MuPAD* (see `www.mupad.de`).

2. Polynomial Modules

Involutive bases are defined for submodules of (finitely generated) free polynomial modules. It is well-known that arbitrary polynomial modules can always be presented as the quotient of a free polynomial module by one of its submodules. In the sequel we only consider graded polynomial modules, i. e. we always assume that we are dealing with a homogeneous submodule.

A *Stanley decomposition* of a polynomial module is a graded vector space isomorphism between the module and a direct sum of free polynomial modules (over polynomial rings with restricted sets of variables). If we are dealing with a submodule of a free module, then any involutive basis trivially induces, by definition, such a combinatorial decomposition. For quotient modules the situation is more complicated. Either one reduces the problem via a Gröbner basis to a monomial one where the construction of a Stanley decomposition is simple [7] or one uses Janet or Pommaret bases which allow for the direct determination of a decomposition of the quotient module.

Any Stanley decomposition yields immediately the Hilbert function and thus the *Krull dimension* of the module. Pommaret bases induce a special kind of decomposition, so-called Rees decompositions, containing further information. If d is the minimal class of a generator in the decomposition, then d is the *depth* of the module. Furthermore, $x_1, \dots, x_d$ is a regular sequence for the module. This result implies a simple criterion for *Cohen-Macaulay modules*: the module is Cohen-Macaulay, if and only if all generators in its Rees decomposition are of the same class.

The maximal degree of a generator in the Pommaret basis of a module is also of considerable interest. One can show that it equals the *Castelnuovo-Mumford regularity* of the leading module (for the used term order). If the Pommaret basis is computed with respect to the degree reverse lexicographic order, then this degree yields in fact the Castelnuovo-Mumford regularity of the full module.

3. Syzygies and Free Resolutions

A classical result in the theory of Gröbner bases is *Schreyer's theorem* on the construction of a Gröbner basis for the first syzygy module from standard representations of the S-polynomials of the elements in the basis. Something similar is possible with involutive bases. The involutive standard representations of the elements obtained by multiplying the generators by their non-multiplicative variables induce a generating set of the syzygy module. For Pommaret bases one obtains a full involutive version of Schreyer's theorem: here this set is again a Pommaret basis for the same term order as in the classical theorem (this is a consequence of Buchberger's second criterion).

Thus for Pommaret bases we can iterate this construction and obtain a syzygy resolution of the given polynomial module. If d is the minimal class of a generator in the basis, then the resolution is of length $n-d$ where n is the number of variables. It follows from our result above on the depth and the Auslander-Buchsbaum theorem that $n-d$ is in fact the *projective dimension* of the module and thus the resolution has minimal length. However, in general, the resolution is not minimal.

In the case of a monomial module, the resolution can be studied in much more detail. It is possible to explicitly describe the corresponding complex including a closed formula for the differential. Furthermore, the complex acquires the structure of a *differential algebra* where the product is again induced by involutive standard representations. Finally, it is possible to characterise those modules for which the resolution is *minimal*. These are the so-called stable modules for which the minimal basis is already the Pommaret basis.

These are generalisations of results of Eliahou and Kervaire [2] who considered only the minimal case. They did not realise that their resolution is in fact a syzygy resolution and had to give a very tedious proof of the exactness of the complex. Using Pommaret bases, most of the results follow immediately; only the proof of the closed formula for the differential is still rather messy.

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Werner M. Seiler, Lehrstuhl für Mathematik I, Universität Mannheim, 68131 Mannheim, Germany
`werner.seiler@math.uni-mannheim.de, http://www.math.uni-mannheim.de/~wms`