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# The inverse syzygy problem in algebraic systems theory

**Werner M. Seiler**[1,*] and **Eva Zerz**[2,**]

[1] [DE]Fachbereich Mathematik und Naturwissenschaften, Universität Kassel, 34132 Kassel, Germany
[2] [DE]Lehrstuhl D für Mathematik, RWTH Aachen, 52062 Aachen, Germany

We present a constructive solution of the inverse syzygy problem over arbitrary coherent rings and show how it can be used to compute certain extension groups.



## 1 The inverse syzygy problem

In algebraic systems theory, linear systems are mathematically modelled by modules over a ring $\mathcal{D}$ (typically a ring of linear differential or difference operators) and control theoretic properties of a system are related to homological properties of the corresponding module. In this note, we are mainly concerned with controllability and show that for a very large class of rings it is equivalent to torsionlessness of the system module. The latter property in turn can be effectively verified via an inverse syzygy problem provided that over $\mathcal{D}$ it is possible to solve effectively the direct syzygy problem.

Due to lack of space, we can only present some results; for proofs and more details we refer to [1]. From a very theoretical point of view, the inverse syzygy problem was already solved by Auslander and Bridger [2]. An effective solution was first presented by Oberst [3] who also noticed the connection to controllability. Like all subsequent works (see e. g. [4]), our results are based on Oberst's algorithm but significantly enlarge the class of admissible rings $\mathcal{D}$. This algorithm can be extended to a construction of certain extension groups of the system module. In [5–7] one can find discussions of various notions of controllability and their relation to the vanishing of extension groups.

In the sequel, we will only assume that $\mathcal{D}$ is a coherent ring, i. e. that any finitely generated left or right ideal of $\mathcal{D}$ can also be finitely presented (in other words, its syzygy module is also finitely generated). In particular, we do not assume that $\mathcal{D}$ is some sort of (possibly non-commutative) polynomial ring and we explicitly allow that $\mathcal{D}$ may contain zero divisors: in contrast to previous works, we do not need the existence of a quotient field of $\mathcal{D}$. For a $\mathcal{D}$-module $\mathcal{M}$, we denote by $\mathcal{M}^* = \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ its dual module (note that $\mathcal{M}^*$ is a right $\mathcal{D}$-module for a left $\mathcal{D}$-module $\mathcal{M}$). All considered modules are assumed to be finitely generated.

Let $\mathcal{M}, \mathcal{N}$ be two free left $\mathcal{D}$-modules and $\beta : \mathcal{M} \to \mathcal{N}$ a module homomorphism. The *direct* syzygy problem consists of finding a free left $\mathcal{D}$-module $\mathcal{P}$ and a homomorphism $\alpha : \mathcal{P} \to \mathcal{M}$ such that $\mathrm{im}\,(\alpha) = \ker\,(\beta)$, whereas in the *inverse* syzygy problem we search for a free left $\mathcal{D}$-module $\mathcal{Q}$ and homomorphism $\gamma : \mathcal{N} \to \mathcal{Q}$ such that $\mathrm{im}\,(\beta) = \ker\,(\gamma)$.

Since we are dealing with homomorphisms between free modules, these can be represented by matrices with entries in $\mathcal{D}$: we write $\beta(\mathbf{P}) = \mathbf{P}B$ with a matrix $B$ of appropriate dimensions and $\beta^*(\mathbf{Q}^*) = B\mathbf{Q}^*$ for the dual map $\beta^* : \mathcal{N}^* \to \mathcal{M}^*$. Solving the direct syzygy problem for $\beta$ corresponds to computing the left syzygies of the rows of $B$. For many classes of rings effective algorithms are known for this task; in particular, Gröbner bases can be used for polynomial rings.

## 2 Oberst's algorithm

Obviously, the inverse syzygy problem corresponds to the direct problem with the arrows reverted and a dualisation reverts arrows. Unfortunately, dualisation is not an exact functor and hence it is a priori not clear whether one can determine this way a solution. In fact, it turns out that opposed to the direct syzygy problem, the inverse problem is not always solvable.

Oberst [3] proposed the following algorithm (originally only for the case that $\mathcal{D}$ is the usual commutative polynomial ring) consisting of five steps:

1. Dualisation: Consider $\beta^* : \mathcal{N}^* \to \mathcal{M}^*$ and $\ker\,(\beta^*) \subseteq \mathcal{N}^*$.
2. Syzygy computation: Let $\gamma^* : \mathcal{Q}^* \to \mathcal{N}^*$ be such that $\mathrm{im}\,(\gamma^*) = \ker\,(\beta^*)$.
3. Dualisation: Consider $\gamma : \mathcal{N} \to \mathcal{Q}$ and $\ker\,(\gamma) \subseteq \mathcal{N}$.
4. Syzygy computation: Let $\hat{\beta} : \hat{\mathcal{M}} \to \mathcal{N}$ be such that $\mathrm{im}\,(\hat{\beta}) = \ker\,(\gamma)$.
5. Check whether $\mathrm{im}\,(\hat{\beta}) = \mathrm{im}\,(\beta)$; if yes, return $\gamma$.

* Corresponding author E-mail: seiler@mathematik.uni-kassel.de
** E-mail: eva.zerz@math.rwth-aachen.de

The last two steps verify whether the map $\gamma$ determined in the previous step provides indeed a solution of the inverse syzygy problem (unique up to isomorphism); otherwise no solution exists. The algorithm is illustrated by the following diagram, where the vertical arrows symbolise dualisation:

$$
\begin{tikzcd}
\hat{\mathcal{M}} \arrow[dr, "\hat{\beta}", "\text{step 4}"'] & & \\
& \mathcal{N} \arrow[r, "\gamma", "\text{step 3}"'] \arrow[dd, <->, dotted] & \mathcal{Q} \arrow[dd, <->, dotted] \\
\mathcal{M} \arrow[ur, "\beta"'] \arrow[d, <->, dotted] & & \\
\mathcal{M}^* & \mathcal{N}^* \arrow[l, "\beta^*"', "\text{step 1}"] & \mathcal{Q}^* \arrow[l, "\gamma^*"', "\text{step 2}"]
\end{tikzcd}
\tag{1}
$$

The correctness of the algorithm is guaranteed by the following theorem. Compared with earlier works, a key difference is the use of the notion of *torsionlessness* instead of *torsionfreeness*. Recall that a $\mathcal{D}$-module $\mathcal{M}$ is torsionfree, if it does not contain non-zero torsion elements, i. e. elements $m \in \mathcal{M}$ such that a non zero divisor $d \in \mathcal{D}$ exists with $dm = 0$, whereas $\mathcal{M}$ is torsionless, if the natural homomorphism $\eta : \mathcal{M} \to \mathcal{M}^{**}$ is injective. For many rings $\mathcal{D}$, both notions are equivalent, but in general torsionlessness is a stricter condition (see [1] for a more detailed discussion).

**Theorem 2.1** *Let $\mathcal{D}$ be a coherent ring and $\beta : \mathcal{M} \to \mathcal{N}$ a homomorphism of left $\mathcal{D}$-modules. If the map $\hat{\beta}$ is constructed as outlined above, then the following three statements are equivalent.*

1. *There exists a finite free left $\mathcal{D}$-module $\mathcal{Q}$ and a left $\mathcal{D}$-module homomorphism $\alpha : \mathcal{N} \to \mathcal{Q}$ such that* $\operatorname{im}(\beta) = \ker(\alpha)$*; in other words, the inverse syzygy problem is solvable.*
2. *The left $\mathcal{D}$-module* $\operatorname{coker}(\beta) = \mathcal{N}/\operatorname{im}(\beta)$ *is torsionless.*
3. *The equality* $\operatorname{im}(\beta) = \operatorname{im}(\hat{\beta})$ *holds.*

## 3 Determination of extension groups

One can show that one always has the inclusion $\operatorname{im}(\beta) \subseteq \operatorname{im}(\hat{\beta})$ and that the quotient $\operatorname{im}(\hat{\beta})/\operatorname{im}(\beta)$ is isomorphic to the extension group $\operatorname{Ext}^1_{\mathcal{D}}\big(D(\mathcal{C}), \mathcal{D}\big)$ where $D(\mathcal{C})$ denotes the Auslander-Bridger dual of the module $\mathcal{C} = \operatorname{coker}(\beta)$. Hence Oberst's algorithm provides us with a mean to determine effectively this group. Extending the diagram (1) in an obvious manner, one obtains the following diagram where the bottom row provides a free resolution of $D(\mathcal{C})$ and where there are isomorphisms $\operatorname{Ext}^{i+1}_{\mathcal{D}}\big(D(\mathcal{C}), \mathcal{D}\big) \cong \operatorname{im}(\hat{\gamma}_i)/\operatorname{im}(\gamma_i)$.

$$
\begin{tikzcd}
& & \hat{\mathcal{M}} \arrow[dr, "\hat{\beta}"] & \hat{\mathcal{N}} \arrow[dr, "\hat{\gamma}_1"] & \hat{\mathcal{Q}}_1 \arrow[dr, "\hat{\gamma}_2"] & \hat{\mathcal{Q}}_2 \arrow[dr, "\hat{\gamma}_3"] & \\
& & \mathcal{M} \arrow[r, "\beta"] \arrow[d, <->, dotted] & \mathcal{N} \arrow[r, "\gamma_1"] \arrow[d, <->, dotted] & \mathcal{Q}_1 \arrow[r, "\gamma_2"] \arrow[d, <->, dotted] & \mathcal{Q}_2 \arrow[r, "\gamma_3"] \arrow[d, <->, dotted] & \cdots \\
0 & D(\mathcal{C}) \arrow[l] & \mathcal{M}^* \arrow[l] & \mathcal{N}^* \arrow[l, "\beta^*"'] & \mathcal{Q}_1^* \arrow[l, "\gamma_1^*"'] & \mathcal{Q}_2^* \arrow[l, "\gamma_2^*"'] & \cdots \arrow[l, "\gamma_3^*"']
\end{tikzcd}
\tag{2}
$$

By reverting the role of $\beta$ and its dual map $\beta^*$ one can use the same approach to determine the extension groups $\operatorname{Ext}^i_{\mathcal{D}}(\mathcal{C}, \mathcal{D})$, since $D\big(D(\mathcal{C})\big) = \mathcal{C}$.

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