Singularities of Algebraic Differential Equations*

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Abstract

There exists a well established differential topological theory of singularities of ordinary differential equations. It has mainly studied scalar equations of low order. We propose an extension of the key concepts to arbitrary systems of ordinary or partial differential equations. Furthermore, we show how a combination of this geometric theory with (differential) algebraic tools allows us to make parts of the theory algorithmic. Our three main results are firstly a proof that even in the case of partial differential equations regular points are generic. Secondly, we present an algorithm for the effective detection of all singularities at a given order or, more precisely, for the determination of a regularity decomposition. Finally, we give a rigorous definition of a regular differential equation, a notoriously difficult notion, ubiquitous in the geometric theory of differential equations, and show that our algorithm extracts from each prime component a regular differential equation. Our main tools are on the one hand the algebraic resp. differential Thomas decomposition and on the other hand the Vessiot theory of differential equations.

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1. Introduction

Many different forms of singular behaviour appear in the context of differential equations and many different views have been developed for them. Most of them are related to singularities of individual solutions of a given differential equation like blow-ups or shocks, i. e. either a solution component or some derivative of it becomes infinite. By contrast, we will be concerned with singularities of the differential equation itself. Using the geometric theory of differential equations [1, 2], (systems of) differential equations are identified with subsets of suitable jet bundles and singularities are special points on these subsets.

* We dedicate this work to the memory of our friend Vladmir P. Gerdt who introduced us to the Thomas decomposition and discussions with whom started this project.

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Within differential topology, singularities of smooth maps between manifolds [3, 4] have been much studied. The *geometric singularities* of differential equations, which are our main topic here, may be viewed as a special case (overviews over some basic results can be found in [5] or [6]). The main emphasis in the literature has been on the classification of singularities (see e. g. [7]) and on the construction of local normal forms for them. Of course, such questions can be reasonably treated only in sufficiently small dimensions and hence most works consider only scalar ordinary differential equations of first or second order. With similar techniques, singularities of solutions of partial differential equations have been studied, e. g. in [8, 9], but as already mentioned this represents a different problem.

By contrast, we are concerned with the effective treatment of general systems of differential equations, i. e. also of under- or overdetermined systems of ordinary or partial differential equations. For this purpose, we extend the needed concepts from differential topology to systems which are not of finite type and we combine them with (differential) algebraic algorithms to make them effective. Such a combination of geometric and algebraic approaches to singularities appeared already in the work of Hubert [10] on scalar first-order ordinary differential equations. However, we cover much more general situations than she did; in particular, we admit systems, equations of arbitrary order and partial differential equations.

We concentrate in this work on the definition and the algorithmic detection of singularities of general differential systems. The analysis of the local solution behaviour around a singularity represents a much harder problem that probably cannot be solved at the same level of generality or effectivity. The algebraic techniques employed by us require that we work over the complex numbers and that we restrict to differential equations with polynomial nonlinearities. From the point of view of applications, the latter restriction is not very serious, as most differential systems arising in applied sciences are polynomial.

Studying fully nonlinear or implicit systems is not at all straightforward and we need to address several challenges. For systems of differential equations, the corresponding subsets of jet bundles are no longer hypersurfaces leading to a much more complicated relation between the given differential system and the subsets defined by it. As a further complication, general systems of differential equations may hide integrability conditions, which must be exhibited explicitly before statements about the existence and uniqueness of solutions can be made. These facts make case distinctions (which are related to the appearance of singularities) unavoidable. Furthermore, in the case of partial differential equations the completion may require to move to higher-order jet bundles, so that a priori it is not even clear at what order any further analysis should be performed.

Our approach proceeds in two steps: a differential one and an algebraic one. In the first step, we use the *differential Thomas decomposition* [11, 12] (see [13, 14, 15, 16, 17] for modern treatments) to split the input system into a finite set of so-called *simple* differential systems. Besides the splitting, the differential step also takes care of the just mentioned problem of hidden integrability conditions, as it includes a completion procedure. Each of the arising simple differential systems is then analysed separately. This decomposition also addresses singular integrals, which are automatically isolated into separate simple systems, whereas the general integral corresponds to other systems. However, we do not claim to detect whether a system corresponds to singular integrals, a difficult question closely connected to the Ritt problem [18, §IV.9]. An alternative to the Thomas decomposition is the Rosenfeld-Gröbner algorithm [19]; the splittings it performs, however, do not in general result in decompositions of the solution set into pairwise disjoint subsets. An elimination method for differential algebra based on splittings analogous to Thomas' ones was developed by Seidenberg [20].

For the algebraic step, we must first choose a suitably high order in which we want to analyse the simple differential system. We associate with the differential system a polynomial radical ideal in the coordinate ring of the jet bundle of the chosen order and introduce this way algebraic jet sets as a geometric model of the differential system (Definition 2.1). Over such sets, we study their Vessiot cones, which are fundamental for defining geometric singularities. Using the *algebraic Thomas decomposition*, we partition algebraic jet sets with respect to the behaviour of the Vessiot cones and show that such a decomposition is equivalent to the identification of all geometric singularities. In order to find algebraic singularities, we augment this procedure with a suitable version of the Jacobian criterion from algebraic geometry.

In the algebraic step, we must study more general situations than usually considered in the differential topological approach to singularities. Hence, we extend this approach in several directions. We provide a more general definition of geometric singularities that can also handle partial differential equations (Definition 4.1). This requires a considerably more involved definition taking into account a whole neighbourhood of the studied point, whereas the classical definitions use pointwise criteria. In the case of systems, one can no longer expect that singularities are isolated points, as it is traditionally done at least for irregular singularities. Therefore, we introduce the novel notion of a *regularity decomposition* of an algebraic jet set (Definition 5.2) as a partitioning into subsets on which the relevant geometric structures (the Vessiot and symbol cones) show a uniform behaviour.

Our first two main results concern these generalisations. Theorem 4.7 proves that the regular points form a Zariski open and dense subset and thus justifies calling the other points singular. In the situations traditionally considered in differential topology or analysis, i. e. for differential equations of finite type, this statement is fairly trivial. As we also include equations which are not of finite type, we must prove the existence of a smooth regular involutive distribution of the right dimension on some neighbourhood of any regular point, which requires the application of advanced results from the geometric theory of differential equations. Our second main result concerns the existence of regularity decompositions for arbitrary differential systems. We provide here a constructive proof by providing an explicit algorithm for the effective construction of such decompositions (Algorithms 5.3 and 5.14) and proving its correctness (Theorem 5.13).

Our third and final main result concerns an old problem in the geometric theory of differential equations. There one usually considers only *regular differential equations*. However, in many cases not even a precise definition of this term is given and an effective test for regularity is still unknown to the best of our knowledge, as it involves considering not only one order, but all orders. Hence, we first provide a rigorous definition of this notion within our framework (Definition 6.1) and then Theorem 6.3 asserts that our algorithm for the construction of a regularity decomposition automatically identifies in each irreducible component a Zariski dense subset that is a regular differential equation.

This article is structured as follows. In Sections 2 and 3 we combine differential algebraic concepts with the geometric theory of differential equations, leading to algebraic jet sets. In Section 4 we extend the classical definition of singularities to arbitrary systems of differential equations, including partial differential equations, and show that regular points are dense. The subsequent Section 5 introduces our concept of a regularity decomposition of a differential system and presents an algorithm to compute this decomposition. Then, Section 6 looks at regular behaviour in prolongations and where it appears in our decomposition. Section 7 treats some examples in detail. Finally, some conclusions are given in Section 8.

2. Connecting Algebra and Geometry

In this section, we lay the groundwork to formalise and effectively prove the theorems of the later sections, by adapting and combining the geometric theory of differential equations and methods from (differential) algebra. For the convenience of the reader, we briefly summarise some basic concepts of the geometric theory in Appendix C and (differential) algebra in the Appendices A and B.

This combination of methods represents a non-trivial task, as the philosophies behind the used geometric and algebraic approaches are very different. In differential algebra, one always considers all orders simultaneously by studying differential ideals. This implies that one has to deal with infinitely many variables. Such an approach is particularly adapted to tackle completion questions, i. e. the construction of hidden integrability conditions, for which it is unclear how geometric approaches could be extended in the presence of singularities.[2] By contrast, in the geometric theory one works typically in a jet bundle of fixed order, which allows to define singularities as points with special properties, whereas the Kolchin topology in differential algebra employs a rather generic notion of points not suitable to describe singularities.

As our algebraic tools require that the underlying field is algebraically closed, we consider throughout complex differential equations, i. e. all variables are assumed to be complex-valued. While the usual starting point of the geometric theory is an arbitrary fibred manifold $\pi : \mathcal{E} \to \mathcal{X}$, we consider only trivial bundles with total space $\mathcal{E} = \mathbb{C}^n \times \mathbb{C}^m$, base space $\mathcal{X} = \mathbb{C}^n$ and π the projection on the first factor (the coordinates $(x^1, \dots, x^n)$ of the base space thus represent the independent variables and the fibre coordinates $(u^1, \dots, u^m)$ of the total space the dependent variables of our differential equations). As all our considerations are of a local nature, this restriction is not serious. But it allows us to identify the total spaces of the jet bundles $J_\ell\pi$ with affine spaces $\mathbb{A}^d_{\mathbb{C}}$ of suitable dimensions d and thus apply standard concepts from algebraic geometry to these spaces. We use two topologies on $J_\ell\pi$, namely the Zariski topology and the standard topology induced by the Euclidean metric. To avoid confusions, we will always explicitly write Zariski, respectively metric, open or closed.

Definition 2.1. An *algebraic jet set* of order ℓ is a locally Zariski closed subset $\mathcal{J}_\ell \subseteq J_\ell\pi$ of a jet bundle of order ℓ (i. e. the difference of two varieties in $J_\ell\pi$). It is an *algebraic differential equation* of order ℓ, if in addition the metric closure of $\pi^\ell(\mathcal{J}_\ell)$ is the whole base $\mathbb{C}^n$. An algebraic jet set or an algebraic differential equation is called *irreducible*, if it is an irreducible locally Zariski closed subset.

Compared with the classical geometric Definition C.2 of a differential equation, varieties are used here instead of manifolds which is simultaneously a generalisation and a restriction. On one side, we permit that the differential equation $\mathcal{J}_\ell$ contains singular points in the sense of algebraic geometry. On the other side, we consider exclusively differential equations which can be globally described as the solution set of an algebraic system on $J_\ell\pi$ with polynomials $p_i, q_j \in \mathcal{D}_\ell$ (see Appendices A and B for notations and definitions).

Definition C.2 furthermore requires that the restriction of the canonical projection $\pi^\ell : J_\ell\pi \to \mathcal{X}$ to the set $\mathcal{J}_\ell$ is a surjective submersion. We are relaxing this requirement in two directions: surjectivity is replaced by a closure condition for the image and we do not impose a maximal

[2] A fundamental problem arises already in the geometric definition of a prolonged equation, if the given equation is not a manifold but only a variety. Thus basic notions like formal integrability or involution are highly non-trivial to generalise to equations admitting singularities, and to our knowledge, nobody has done this so far.

rank condition. The second relaxation is crucial for the definition of geometric singularities. Surjectivity of the restricted projection represents a geometric way of saying that the independent variables are indeed independent, as otherwise our differential equation could imply relations between them. However, this idea is also captured by our condition on the metric closure of its image and for an equation like $xu' = 1$ surjectivity represents too strong a condition. We use the metric closure here instead of the Zariski one, as for the analysis of the local solution behaviour around singularities (which we will not do in this work) it is important that exceptional points may be considered as the limit of a sequence of points in $\pi^\ell(\mathcal{J}_\ell)$.

In applications, the typical starting point is a differential system of the form $S = \{p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0\}$ as introduced in (D) in Appendix B rather than an algebraic differential equation as defined above. Thus we start on the differential algebraic side and discuss now how we can obtain geometric objects (and algebraic descriptions of them). It turns out that this process involves a number of subtleties requiring a careful discussion.

We associate with such a differential system S the *differential* ideal

$$\hat{\mathcal{I}}_{\mathrm{diff}}(S) := \langle p_1, \ldots, p_s \rangle_\Delta \subseteq \mathcal{D}$$

generated by the equations in S. It induces for any order $\ell \in \mathbb{N}_0$ the *algebraic* ideal

$$\hat{\mathcal{I}}_\ell(S) := \hat{\mathcal{I}}_{\mathrm{diff}}(S) \cap \mathcal{D}_\ell \subseteq \mathcal{D}_\ell$$

as the corresponding finite-dimensional truncation. Note that this ideal automatically contains all hidden integrability conditions up to order ℓ. The inequations in the differential system S are also used to define for any order $\ell \in \mathbb{N}_0$ an *algebraic* ideal,[3] however, in a slightly different manner:

$$\mathcal{K}_\ell(S) := \langle \hat{Q}_\ell \rangle_{\mathcal{D}_\ell} \qquad \text{with} \quad \hat{Q}_\ell = \prod_{\substack{j=1 \\ \mathrm{ord}\,(q_j) \leq \ell}}^{t} q_j\,.$$

These ideals lead then to the algebraic jet sets

$$\hat{\mathcal{J}}_\ell(S) := \mathrm{Sol}^{\mathrm{a}}(\hat{\mathcal{I}}_\ell(S)) \setminus \mathrm{Sol}^{\mathrm{a}}(\mathcal{K}_\ell(S)) \subseteq J_\ell\pi \tag{1}$$

consisting of all points of $J_\ell\pi$ satisfying both the equations and the inequations in S, interpreted as algebraic equations in $J_\ell\pi$. Since their definition is based on the differential ideal $\hat{\mathcal{I}}_{\mathrm{diff}}(S)$, these sets satisfy for any $k > 0$ the inclusions $\pi^{\ell+k}_\ell(\hat{\mathcal{J}}_{\ell+k}(S)) \subseteq \hat{\mathcal{J}}_\ell(S)$. In fact, we always have $\pi^{\ell+k}_\ell\big(\mathrm{Sol}^{\mathrm{a}}(\hat{\mathcal{I}}_{\ell+k}(S))\big) = \mathrm{Sol}^{\mathrm{a}}(\hat{\mathcal{I}}_\ell(S))$, but the inequations may lead to a strict inclusion of the above jet sets [21].

Remark 2.2. While it is possible to define the ideals $\hat{\mathcal{I}}_\ell(S)$ and the algebraic jet sets $\hat{\mathcal{J}}_\ell(S)$ for any order $\ell \in \mathbb{N}_0$, these ideals and sets are really meaningful only if no equation p_i in the underlying differential system is of an order greater than ℓ. Assuming that the system S is solvable and the sets $\hat{\mathcal{J}}_\ell(S)$ are algebraic differential equations, their solution sets are otherwise not comparable, as all equations in S of order greater than ℓ are ignored in the construction of $\hat{\mathcal{J}}_\ell(S)$. In particular, for different values of ℓ the corresponding equations $\hat{\mathcal{J}}_\ell(S)$ may have different solution sets. Note that the orders of the inequations in S are irrelevant here, as they should be considered more as conditions on allowed initial data. From now on, we always assume that ℓ is sufficiently large.

[3]Note that it is pointless to introduce a *differential* ideal defined by the inequations, as differentiating an inequation does not lead to a condition that has to be satisfied by any holomorphic or formal solution of the differential system S.

While this construction of the algebraic jet sets $\hat{\mathcal{J}}_\ell(S)$ appears very natural, it faces a number of serious challenges making it inadequate for our purposes:

(i) There may exist differential polynomials that vanish on every solution in $\mathrm{Sol}^{\mathrm{d}}(S)$, but are not contained in the differential ideal $\hat{\mathcal{I}}_{\mathrm{diff}}(S)$.

(ii) It is not so easy to study the algebraic jet sets $\hat{\mathcal{J}}_\ell(S)$, as e. g. the ideals $\hat{\mathcal{I}}_\ell(S)$ are generally not radical—this is a consequence of (i)—and thus not the vanishing ideals of the underlying variety. In particular, it is not immediately obvious whether the algebraic jet sets are non-empty. Furthermore, the algebraic jet sets $\hat{\mathcal{J}}_\ell(S)$ are not necessarily algebraic differential equations, as it is not guaranteed that their projection $\pi^\ell(\hat{\mathcal{J}}_\ell(S))$ satisfies the closure condition of Definition 2.1.

(iii) The effective determination of bases for the algebraic ideals $\hat{\mathcal{I}}_\ell(S)$ is non-trivial, because of the possible existence of hidden integrability conditions.

(iv) The algebraic jet sets $\hat{\mathcal{J}}_\ell(S)$ may be too small, as interpreting differential inequations as algebraic ones leads to a change in their semantics eliminating many "interesting" points. Assume for simplicity that the differential system S contains the (differential) inequation $u_x \neq 0$. It entails that the x-derivative of any solution of S can never be the zero *function*. Nevertheless, it is well possible that the x-derivative of a solution possesses zeros and thus the corresponding jets of this solution have a vanishing *u_x-coordinate*. However, no point on a set $\hat{\mathcal{J}}_\ell(S)$ with $\ell > 0$ can have a vanishing u_x-coordinate, as the algebraic system describing $\hat{\mathcal{J}}_\ell(S)$ contains $u_x \neq 0$ as an (algebraic) inequation [21].

Challenge (i) requires a differential Nullstellensatz for differential systems, i. e. an extension of Theorem B.1 that also includes inequations. [17, Lemma 2.2.62] asserts that the vanishing ideal of $\mathrm{Sol}^{\mathrm{d}}(S)$ is given by the differential ideal

$$\mathcal{I}_{\mathrm{diff}}(S) := \sqrt{\hat{\mathcal{I}}_{\mathrm{diff}}(S) : \hat{Q}^\infty} \subseteq \mathcal{D} \qquad \text{with} \quad \hat{Q} = \prod_{j=1}^{t} q_j \,. \tag{2}$$

Hence, as a first step, we must replace the differential ideal $\hat{\mathcal{I}}_{\mathrm{diff}}(S)$ by this ideal. However, using directly the above definition of $\mathcal{I}_{\mathrm{diff}}(S)$ makes its explicit determination rather expensive because of the required radical computation (so that Challenge (iii) becomes even more pronounced).

Our next step towards overcoming the mentioned difficulties consists of restricting to *simple* differential systems. For any differential system S, a differential Thomas decomposition provides us with simple differential systems $S_1, \dots, S_k$ such that $\mathrm{Sol}^{\mathrm{d}}(S)$ is the disjoint union of the sets $\mathrm{Sol}^{\mathrm{d}}(S_i)$. Hence, using such a decomposition we may analyse instead of the original system S one by one the simple systems $S_1, \dots, S_k$. Recall, however, that such a decomposition is not unique.

So we assume from now on that S is a simple differential system. For simple systems, [17, Prop. 2.2.72] entails that the ideal $\mathcal{I}_{\mathrm{diff}}(S)$ defined in (2) may alternatively be constructed via a simple saturation without an explicit radical computation:

$$\mathcal{I}_{\mathrm{diff}}(S) = \hat{\mathcal{I}}_{\mathrm{diff}}(S) : Q^\infty \qquad \text{with} \quad Q = \prod_{i=1}^{s} \left(\mathrm{init}\,(p_i) \cdot \mathrm{sep}\,(p_i)\right) \,. \tag{3}$$

Note that now we do not saturate with respect to the inequations in S but with respect to the product of the initials and separants of all the equations in the differential system S.[4] As before, we

[4]Given an arbitrary differential system S, let $S_1, \dots, S_k$ be the simple systems of any differential Thomas decompo-

use the differential ideal $\mathcal{I}_{\text{diff}}(S)$ to introduce for any sufficiently large order ℓ (see Remark 2.2) the algebraic ideal

$$\mathcal{I}_\ell(S) := \mathcal{I}_{\text{diff}}(S) \cap \mathcal{D}_\ell \subseteq \mathcal{D}_\ell\,. \tag{4}$$

Since the differential ideal $\mathcal{I}_{\text{diff}}(S)$ is radical, the same is true for all the finite truncations $\mathcal{I}_\ell(S)$, which greatly simplifies the study of their varieties. Our steps so far suggest to consider instead of the sets $\hat{\mathcal{J}}_\ell(S)$ the algebraic jet sets

$$\mathcal{J}_\ell(S) := \mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S)) \setminus \mathrm{Sol}^{\mathrm{a}}(\mathcal{K}_\ell(S)) \subseteq J_\ell\pi\,. \tag{5}$$

Lemma 2.3. *Given a simple differential system S, these algebraic jet sets satisfy $\pi^{k+\ell}_\ell(\mathcal{J}_{\ell+k}(S)) = \mathcal{J}_\ell(S)$ for all prolongation orders $k > 0$.*

Proof. As already mentioned above, the fact that the algebraic ideals $\mathcal{I}_\ell(S)$ stem from a differential ideal entails that $\pi^{k+\ell}_\ell\big(\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell+k}(S))\big) = \mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S))$. Since we are now dealing with a simple differential system, no leader of an inequation is a derivative of a leader of an equation and the leaders of all equations and inequations are pairwise different. Hence, we have $\pi^{k+\ell}_\ell\big(\mathrm{Sol}^{\mathrm{a}}(\mathcal{K}_{\ell+k}(S))\big) = \mathrm{Sol}^{\mathrm{a}}(\mathcal{K}_\ell(S))$. □

Note that this result resembles the definition of formal integrability in the geometric theory of differential equations [2, Def. 2.3.15]. However, many regularity assumptions are made in the geometric theory and given a fibred submanifold $\mathcal{J}_\ell \subseteq J_\ell\pi$, its prolongation $\mathcal{J}_{\ell+k} \subseteq J_{\ell+k}\pi$ is defined via an intrinsic geometric process. Formal integrability is then a special property of some submanifolds $\mathcal{J}_\ell$ encoding the absence of hidden integrability conditions. In our approach, it is an automatic consequence of the use of a differential ideal and the simplicity of the defining differential system.

Remark 2.4. From a geometric point of view, saturations as they appear in (2) and (3), respectively, have the following meaning: $\mathrm{Sol}^{\mathrm{a}}(I : J^\infty)$ is the Zariski closure of the set $\mathrm{Sol}^{\mathrm{a}}(I) \setminus \mathrm{Sol}^{\mathrm{a}}(J)$. Thus, since the same ideal $\mathcal{I}_{\text{diff}}(S)$ appears in (2) and (3), the variety $\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S))$ is the Zariski closure of the set obtained by removing from $\mathrm{Sol}^{\mathrm{a}}(\hat{\mathcal{I}}_\ell(S))$ either all points at which a separant or an initial of an equation in the system S vanishes or $\mathrm{Sol}^{\mathrm{a}}(\mathcal{K}_\ell)$. In both cases, the Zariski closure restores many of the removed points. This is important for us, as most of the singularities we are interested in are actually points of this kind.

However, if a whole irreducible component of $\mathrm{Sol}^{\mathrm{a}}(\hat{\mathcal{I}}_\ell(S))$ consists only of such removed points, then it remains removed. Indeed, there are two possibilities for such a component. Either it does not define an algebraic differential equation on its own. Then trivially it cannot have any solutions and there is no point in looking for singularities. Or if it is an algebraic differential equation, then we analyse it elsewhere. Indeed, recall that we obtained a simple system only by computing a differential Thomas decomposition of our original system and the removed component corresponds to some other simple system arising in this decomposition.

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sition of it. Then [17, Prop. 2.2.72] yields the ideal decomposition

$$\mathcal{I}_{\text{diff}}(S) = \bigcap_{i=1}^{k} \hat{\mathcal{I}}_{\text{diff}}(S_i) : Q_i^\infty\,,$$

where Q_i is the product of the initials and separants of the equations in S_i. This intersection is in general not minimal, but no effective way is known to decide whether or not an ideal in this intersection is superfluous, which is again the so-called Ritt problem [18, §IV.9].

By [22, Thm. 1.94], the ideal $\mathcal{I}_\ell(S)$ is furthermore equidimensional in the sense that all of its associated primes possess the same dimension, which excludes in particular the existence of embedded prime components. This represents a further simplification entailed by the restriction to simple systems.

Remark 2.5. It follows from [22, Cor. 1.96] that the set of equations in any simple differential system forms a regular chain. Hence, the ideals $\mathcal{I}(S)$ and $\mathcal{I}_\ell(S)$ are (differentially resp. algebraically) *characterisable*, i. e. ideals defined by characteristic sets (cf. [23, 24] for a survey of the properties of such ideals and [25] for an application).

Even after this replacement, Challenge (iv) remains open and indicates that we should enlarge the sets $\mathcal{J}_\ell(S)$. However, for a general algebraic differential equation $\mathcal{J}_\ell$ we face another challenge. If we consider the subset of $\mathcal{J}_\ell$ obtained as the union of the images of all prolongations $j_\ell\sigma$ of classical solutions of the equation, then this subset may cover only a small part of $\mathcal{J}_\ell$ (this happens in particular, if hidden integrability conditions exist). As one of the main aspects of singularities is an analysis of the local solution behaviour in their neighbourhood, we only want situations where this subset lies dense in the considered differential algebraic equation. This motivates the following notion.

Definition 2.6. The algebraic differential equation $\mathcal{J}_\ell \subset J_\ell\pi$ is *locally integrable*, if $\mathcal{J}_\ell$ contains a Zariski open and dense subset $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ such that for every point $\rho \in \mathcal{R}_\ell$ at least one classical solution σ exists with $\rho \in \operatorname{im} j_\ell\sigma$.

In general it is difficult to decide whether a given algebraic differential equation $\mathcal{J}_\ell \in J_\ell\pi$ is locally integrable, as this obviously requires an existence theory for solutions. In particular, such a decision cannot be made by a purely geometric analysis of $\mathcal{J}_\ell$, but requires the considerations of higher-order equations, too (large parts of [2] are concerned with this question in the regular case). However, the situation is different under our assumption of a simple differential system, as for such systems the local integrability is essentially part of their definition. More precisely, we obtain the following result which already indicates how the above defined algebraic jet sets $\mathcal{J}_\ell(S)$ can be enlarged without losing this property.

Proposition 2.7. *Let S be a simple differential system with respect to a Riquier ranking and consider for an arbitrary order $\ell \in \mathbb{N}$ the above defined algebraic jet set $\mathcal{J}_\ell(S)$. Then its Zariski closure $\overline{\mathcal{J}_\ell(S)}$ is a locally integrable algebraic differential equation.*

Proof. Obviously, $\mathcal{J}_\ell(S)$ is Zariski dense in $\overline{\mathcal{J}_\ell(S)}$ and it suffices to prove that $\mathcal{J}_\ell(S)$ is a locally integrable algebraic differential equation. The proof of the local integrability essentially boils down to an extension of Remark B.5, where the construction of formal power series solutions is discussed. We consider the Zariski open subset $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell(S)$ consisting of all smooth points at which no separant or initial of an equation in S vanishes. By the considerations in Remark 2.4, $\mathcal{R}_\ell$ is even Zariski dense in $\mathcal{J}_\ell$. As remarked in [16, Cor. 11], one can now straightforwardly adapt the proof of Riquier's Theorem B.2 and conclude that the formal power series constructed in Remark B.5 converges to a holomorphic solution σ defined on some open subset of $\mathbb{C}^n$. □

We are thus lead to consider the Zariski closure $\overline{\mathcal{J}_\ell(S)}$ instead of $\mathcal{J}_\ell(S)$. Since it is a Zariski closed set in $J_\ell\pi$ and thus a variety, we are obviously interested in its vanishing ideal. Since $\mathcal{I}_\ell(S)$ is a radical ideal and we are working over an algebraically closed field, it is a classical result in algebraic geometry that it is given by the quotient ideal $\mathcal{I}_\ell(S) : \hat{Q}_\ell$ (cf. e. g. [26, Chapt. 4, Sect. 4, Thm. 7]). The following lemma shows that in our case this quotient simply means to ignore the inequations in the system.

Lemma 2.8. *For any order $\ell \in \mathbb{N}$ we have $\overline{\mathcal{J}_\ell(S)} = \mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S))$.*

Proof. Our assertion is equivalent to the following equality:

$$\left(\sqrt{\hat{\mathcal{I}}_{\mathrm{diff}}(S) : \hat{Q}^\infty} \cap \mathcal{D}_\ell\right) : \hat{Q}_\ell = \sqrt{\hat{\mathcal{I}}_{\mathrm{diff}}(S) : \hat{Q}^\infty} \cap \mathcal{D}_\ell \,.$$

The inclusion "$\supseteq$" is clear. For the reverse inclusion, we first note that, since $\hat{Q}_\ell$ divides $\hat{Q}$, we have $\hat{Q} = \hat{Q}_\ell \tilde{Q}$ for some $\tilde{Q} \in \mathcal{D}$. Let $P \in \mathcal{D}_\ell$ be such that $(P\hat{Q}_\ell)^k \in \hat{\mathcal{I}}_{\mathrm{diff}}(S) : \hat{Q}^\infty$ for some positive integer k. Then there exists an exponent $r \in \mathbb{N}_0$ such that $P^k \hat{Q}_\ell^k \hat{Q}^r \in \hat{\mathcal{I}}_{\mathrm{diff}}(S)$. Multiplication by $\tilde{Q}^k$ yields that $P^k \hat{Q}^{r+k} \in \hat{\mathcal{I}}_{\mathrm{diff}}(S)$. Hence, $P^k \in \hat{\mathcal{I}}_{\mathrm{diff}}(S) : \hat{Q}^\infty$ and thus P lies in the radical. □

By definition, the equations in a simple differential system define a passive system. This observation allows us to resolve Challenge (ii). Passivity implies consistency, making it impossible that an equation p_i depends only on the independent variables x^j. Hence, for each algebraic jet set $\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S))$ it is clear that its image under the canonical projection π^ℓ satisfies the closure condition of Definition 2.1 and thus that it is an algebraic differential equation. Furthermore, a passive system cannot contain a constant implying via Hilbert's Nullstellensatz that all these sets are non-empty.

Remark 2.9. The passivity of the equations also allows us to solve the remaining Challenge (iii): the explicit construction of generators for the now used algebraic ideals $\mathcal{I}_\ell(S)$. The definition of passivity of a differential system is based on the notion of (non-)multiplicative variables [15, 17]. Consider for any ℓ the set

$$B_{\leq \ell} := \left\{ \delta^\mu p_i \mid 1 \leq i \leq s,\ |\mu| + \mathrm{ord}\,(p_i) \leq \ell, \mu_j = 0 \text{ if } j \text{ not Janet-multiplicative for } p_i \right\} \tag{6}$$

obtained by differentiating each equation in S with respect to its multiplicative variables until the order ℓ is reached. It provides us as a first step with an explicit generating set of the ideal $\hat{\mathcal{I}}_\ell(S)$.

We define an algebraic system $S_{\leq \ell}$ by taking the elements of $B_{\leq \ell}$ as the equations and keeping all inequations of S with order less than or equal to ℓ. Since S is assumed to be a simple differential system, it is easy to see that $S_{\leq \ell}$ is a simple algebraic system (both the initial and the separant of a derivative $\delta_k p_i$ are simply the separant of p_i). In [22, Lemma 1.93] it is shown that $\mathcal{I}_{\mathrm{alg}}(S_{\leq \ell}) = \mathcal{I}_\ell(S)$, where the ideal $\mathcal{I}_{\mathrm{alg}}(S_{\leq \ell})$ is defined in Equation (A.1). Recall from (A.1) that the determination of $\mathcal{I}_{\mathrm{alg}}(S_{\leq \ell})$ requires a saturation as second step. Thus an explicit basis of $\mathcal{I}_\ell(S)$ is obtained by saturating the ideal generated by $B_{\leq \ell}$ by the product of the initials of the elements of $B_{\leq \ell}$. This operation can be done effectively using Gröbner bases. It follows from Remark 2.4 and the definition (A.1) of $\mathcal{I}_{\mathrm{alg}}$ that $\overline{\mathrm{Sol}^{\mathrm{a}}(S_{\leq \ell})} = \overline{\mathcal{J}_\ell(S)}$.

Example 2.10. To demonstrate in particular the effect of the saturation in the definition of the ideal $\mathcal{I}_{\mathrm{diff}}(S)$, we consider the following differential system consisting of two partial differential equations for an unknown function $u(x, y)$:

$$p_1 := uu_x - yu - y^2\,, \qquad p_2 := yu_y - u\,. \tag{7}$$

Adding the inequation $\mathrm{sep}\,(p_1) = u \neq 0$ yields the only simple differential system S appearing in a differential Thomas decomposition of the system (7). If we start with the differential ideal $\hat{\mathcal{I}}_{\mathrm{diff}}(S) = \langle p_1, p_2 \rangle_\Delta$, then the algebraic ideal $\hat{\mathcal{I}}_1(S) = \hat{\mathcal{I}}_{\mathrm{diff}}(S) \cap \mathcal{D}_1$ has the prime decomposition $\hat{\mathcal{I}}_1(S) = \langle p_2, p_3 \rangle \cap \langle u, y \rangle$, where

$$p_3 := u_y u_x - u - y\,, \tag{8}$$

hence $\hat{\mathcal{I}}_{\rm diff}(S)$ cannot be prime either. The saturation by $Q := yu$ used in the definition (3) of $\mathcal{I}_{\rm diff}(S)$ removes the prime component $\langle u, y\rangle$ of $\hat{\mathcal{I}}_1(S)$, more precisely $\mathcal{I}_{\rm diff}(S) = \langle p_2, p_3\rangle_\Delta$ and thus $\mathcal{I}_1 = \langle p_2, p_3\rangle \subset \mathcal{D}_1$ (note that $p_1 = yp_3 - u_x p_2$). Indeed, if we compare for any order $\ell > 0$ the algebraic jet sets $\mathrm{Sol}^{\rm a}(\mathcal{I}_\ell(S)) \subset \mathrm{Sol}^{\rm a}(\hat{\mathcal{I}}_\ell(S)) \subset J_\ell\pi$, then we see that at all removed points the separants of the equations (7) vanish.

In this particular case, the generators of the removed prime component do not define a consistent differential system, as one of them is the independent variable y. Hence, we are not losing any solutions by its removal. In other examples we may remove components defining consistent systems. However, in such cases the properties of the differential Thomas decomposition ensure that the corresponding solutions appear in some other simple differential system.

Remark 2.11. Riquier's Theorem B.2 asserts that a certain initial value problem adapted to the choice of leaders in the equations of the system possesses a unique holomorphic solution (the explicit construction of the corresponding initial conditions is explained in more modern terms in [27]; see also [2, Sect. 9.3]). If the system $S_{\leq \ell}$ is of finite type, then the coordinates of the considered point $\rho \in \mathcal{J}_\ell$ provide all required initial data and in this case the holomorphic solution σ such that $\rho \in \mathrm{im}\, j_\ell\sigma$ is uniquely determined. Otherwise, the coordinates of the considered point $\rho \in \mathcal{J}_\ell$ provide only values for a finite subset of the infinitely many arbitrary Taylor coefficients of the series constructed in Remark B.5. Hence, in this case infinitely many different holomorphic solutions σ exist such that $\rho \in \mathrm{im}\, j_\ell\sigma$, all of which possess the same Taylor expansion up to order ℓ.

3. Vessiot Cones and Generalised Solutions

In Appendix C we recall some basic concepts of Vessiot's approach to a solution theory for differential equations. Again some adaptions are required, as we are now using a more general notion of differential equations. Furthermore, it turns out useful for the study of singularities to introduce a more general concept of solutions than the classical solutions of Definition C.3.

The Vessiot space $\mathcal{V}_\rho[\mathcal{J}_\ell]$ (cf. Definition C.4) at a point ρ on a differential equation $\mathcal{J}_\ell$ consists of the tangential part of the contact distribution at ρ. As an algebraic jet set $\mathcal{J}_\ell$ is a locally Zariski closed subset which may contain non-smooth points, the question arises how this definition should be extended. One could continue to apply it without changes using the tangent space $T_\rho\mathcal{J}_\ell$ in the sense of algebraic geometry. Then one would still obtain linear spaces; however, their dimension would be too high. We therefore prefer another extension. Given a classical solution σ of $\mathcal{J}_\ell$ such that $\rho \in \mathrm{im}\, j_\ell\sigma$, it follows from a well-known characterisation of the *tangent cone* as limit of secants (see e. g. [26, §9.7, Thm. 6]) that actually $T_\rho(\mathrm{im}\, j_\ell\sigma) \subseteq C_\rho\mathcal{J}_\ell$ where $C_\rho\mathcal{J}_\ell$ denotes the tangent cone of $\mathcal{J}_\ell$ at ρ. This observation motivates the following extension of Vessiot spaces.

Definition 3.1. The *Vessiot cone* $\mathcal{V}_\rho[\mathcal{J}_\ell]$ of the algebraic jet set $\mathcal{J}_\ell \subseteq J_\ell\pi$ at a point $\rho \in \mathcal{J}_\ell$ is the set $\mathcal{V}_\rho[\mathcal{J}_\ell] = T_\rho\mathcal{J}_\ell \cap \mathcal{C}_\ell|_\rho$.

We continue to denote the family of all Vessiot cones by $\mathcal{V}[\mathcal{J}_\ell]$. At smooth points, the tangent cone and the tangent space coincide and therefore we still speak of Vessiot spaces at such a point. A Vessiot space can be easily determined by solving a linear system of equations [2, Sect. 9.5]. In the geometric theory, one always assumes that the Vessiot spaces define a smooth regular distribution on $\mathcal{J}_\ell$. By considering the linear system as being parametrised by the points of $\mathcal{J}_\ell$, one obtains in our more general situation by standard genericity arguments the following proposition. As a consequence of it, we still often call $\mathcal{V}[\mathcal{J}_\ell]$ the *Vessiot distribution* of $\mathcal{J}_\ell$ in the sequel, although, strictly speaking, this terminology is not correct, as a cone is generally not a linear space, but only a union of one-dimensional linear spaces.

Proposition 3.2. *Let $\mathcal{J}_\ell$ be an irreducible algebraic jet set. Then the family of Vessiot cones $\mathcal{V}[\mathcal{J}_\ell]$ defines on a Zariski open and dense subset $O_V \subseteq \mathcal{J}_\ell$ a smooth regular distribution.*

Proof. The subset of all smooth points of $\mathcal{J}_\ell$ is Zariski open and dense and defines a connected complex manifold [28, Sect. 0.2]. At any point ρ of this manifold, the tangent space $T_\rho\mathcal{J}_\ell$ and thus the Vessiot space $\mathcal{V}_\rho[\mathcal{J}_\ell]$ can be computed using linear algebra. As a locally Zariski closed set, the algebraic jet set $\mathcal{J}_\ell$ is a Zariski open subset of the zero set of some polynomial functions $\Phi^\tau : J_\ell\pi \to \mathbb{C}$. Since, by definition, the Vessiot spaces are contained in the contact distribution, we make for any vector $\mathbf{V} \in \mathcal{V}_\rho[\mathcal{J}_\ell]$ the ansatz

$$\mathbf{V} = \sum_i a^i C_i^{(\ell)}|_\rho + \sum_{|\mu|=\ell} \sum_\alpha b_\mu^\alpha C_\alpha^\mu|_\rho \tag{9}$$

with yet to be determined coefficients $a^i, b_\mu^\alpha \in \mathbb{C}$. At a smooth point ρ, such a vector is tangential to $\mathcal{J}_\ell$, if and only if it satisfies $d\Phi^\tau|_\rho(\mathbf{V}) = 0$ for all τ. Hence, we obtain a homogeneous linear system for the coefficient vectors $\mathbf{a}$, $\mathbf{b}$,

$$D(\rho)\mathbf{a} + M_\ell(\rho)\mathbf{b} = 0\,, \tag{10}$$

where the entries of the matrices D, M_ℓ are given by[5]

$$D_i^\tau(\rho) = C_i^{(\ell)}(\Phi^\tau)(\rho)\,, \qquad (M_\ell)_\alpha^{\tau\mu}(\rho) = C_\alpha^\mu(\Phi^\tau)(\rho)\,. \tag{11}$$

In general, the behaviour of (10) varies over $\mathcal{J}_\ell$; e. g. the dimension of $\mathcal{V}_\rho[\mathcal{J}_\ell]$ may jump. However, considered as functions of ρ, the solutions of (10) are smooth outside of a Zariski closed set by Cramer's rule and—by potentially enlarging this set—we may even assume that the dimension of the solution remains constant, since dimension is an upper semicontinuous function. Thus, we obtain a smooth regular distribution on a Zariski open and dense set. □

In an analogous way, we extend the notion of a symbol space to that of a symbol cone. Again, it is straightforward to show that on a Zariski open subset of $\mathcal{J}_\ell$ the symbol spaces $\mathcal{N}_\rho[\mathcal{J}_\ell]$ define a smooth regular distribution $\mathcal{N}[\mathcal{J}_\ell]$. At a smooth point $\rho \in \mathcal{J}_\ell$, the symbol space $\mathcal{N}_\rho[\mathcal{J}_\ell]$ consists of those solutions of (10) for which all coefficients $\mathbf{a}$ vanish. Hence, at smooth points we can always decompose the Vessiot space as a direct sum of linear subspaces,

$$\mathcal{V}_\rho[\mathcal{J}_\ell] = \mathcal{N}_\rho[\mathcal{J}_\ell] \oplus \mathcal{H}_\rho\,, \tag{12}$$

with some π^ℓ-transversal complement $\mathcal{H}_\rho$ which is not uniquely determined.

Remark 3.3. If one computes for a differential equation $\mathcal{J}_\ell$ order by order a formal power series solution around some expansion point, then one obtains for the Taylor coefficients of order $\ell + 1$ an inhomogeneous linear system with a matrix and right hand side depending on the lower order coefficients (see [2, Sect. 2.3] for more details). One can show that the linear system (10) is a

[5]The columns of the matrix M_ℓ are labelled by τ and the rows by the pairs (μ, α).

homogenised form of this linear system [2, Rem. 9.5.6]. Let us assume that it is possible to solve (10) in such a way that the coefficients $\mathbf{a}$ remain undetermined (this is actually what we expect to happen generically). Then we can relate the solutions of (10) with the derivatives u^α_ν of order $\ell + 1$ of the power series solution. Indeed, in this case we must find for each value $1 \leq i \leq n$ a solution $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ such that $\bar{a}^j = \delta^j_i$ and $\bar{b}^\alpha_\mu = u^\alpha_{\mu+1_i}$. Conversely, one can see that if no such solution exists for (10) at some point $\rho \in \mathcal{J}_\ell$, then no *smooth* solution σ with $\rho \in \operatorname{im} j_\ell\sigma$ can exist, as at least one derivative of order $\ell + 1$ becomes infinite.

In the decomposition (12), we can choose at any smooth point $\rho \in \mathcal{J}_\ell$ an arbitrary complement $\mathcal{H}_\rho$. A solution σ with $\rho \in \operatorname{im} j_\ell\sigma$ can exist only, if the complement $\mathcal{H}_\rho$ is n-dimensional (cf. Proposition C.5). This raises the question whether it is possible to correlate the choices in the neighbourhood of a point in such a way that the chosen complements form an involutive distribution. If this is possible at all, then for most systems, there are actually infinitely many ways to do this (parametrised by the symbol). Only for a special class of differential equations—comprising in particular most ordinary differential equations—only a unique possibility exists.

Definition 3.4. An algebraic differential equation $\mathcal{J}_\ell$ is *of finite type*, if it contains a Zariski open and dense subset $\mathcal{F}_\ell \subseteq \mathcal{J}_\ell$ such that at all points $\rho \in \mathcal{F}_\ell$ the symbol cone $\mathcal{N}_\rho[\mathcal{J}_\ell]$ vanishes.

In the literature, one can find many alternative names for equations of finite type. In the theory of linear systems, the term *holonomic* system is very popular. Another common terminology, in particular for partial differential equations, is *maximally overdetermined* system. From a geometric point of view, (regular first-order) equations of finite type correspond to connections over the fibration π (see [2, Remark 2.3.6]).

For the analysis of singularities it turns out to be convenient to introduce more general kinds of solutions directly as geometric objects without reference to a section. The following definition simply relaxes some of the conditions on the subdistribution $\mathcal{H}$ in the second part of Proposition C.5. Note that such a generalised solution lives in the jet bundle $J_\ell\pi$ and not in the total space $\mathcal{E}$ of the fibration π like a section, but it can be projected to $\mathcal{E} = J_0\pi$.

Definition 3.5. Let $\mathcal{J}_\ell \subseteq J_\ell\pi$ be an algebraic differential equation in n independent variables. A *generalised solution* of $\mathcal{J}_\ell$ is an n-dimensional submanifold $\mathcal{N} \subseteq \mathcal{J}_\ell$ such that $T\mathcal{N} \subseteq \mathcal{V}[\mathcal{J}_\ell]|_{\mathcal{N}}$. A *geometric solution* of $\mathcal{J}_\ell$ is the projection $\pi^\ell_0(\mathcal{N}) \subseteq \mathcal{E}$ of a generalised solution $\mathcal{N}$.

If the section $\sigma : \mathcal{X} \to \mathcal{E}$ defines a solution of $\mathcal{J}_\ell$, then $\operatorname{im} j_\ell\sigma$ is automatically a generalised solution with $\operatorname{im}\sigma$ as the corresponding geometric solution; this follows immediately from the definition of the Vessiot distribution. However, if the differential equation $\mathcal{J}_\ell$ has geometric singularities as defined below, then not every geometric solution is the image of a section $\sigma : \mathcal{X} \to \mathcal{E}$ (in fact, generally it is not even a manifold).

4. Singularities of General Differential Equations

In classical analysis, one usually studies singularities like a blow-up or a shock. Thus the singular behaviour refers to an individual solution and consists of either the solution itself or some derivative of it becoming infinite at some finite point $\mathbf{x} \in \mathcal{X}$. By contrast, we study singularities of the differential system S itself: we define singularities as points $\rho \in \mathcal{J}_\ell$ for some sufficiently high order ℓ such that generalised solutions in the sense of Definition 3.5 in the neighbourhood show a "special" behaviour. If $\mathcal{J}_\ell$ is a differential equation of finite type, then we expect that on any sufficiently small neighbourhood of a regular point $\rho \in \mathcal{J}_\ell$ a unique foliation of the neighbourhood by generalised solutions exists and that all generalised solutions are the image of prolonged classical solutions. If the equation is not of finite type, then around regular points still such foliations exist, but they are no longer unique. In fact, infinitely many foliations exist.

Definition 4.1. Let $\mathcal{J}_\ell \subseteq J_\ell\pi$ be a locally integrable algebraic differential equation in n independent variables. A non-smooth point $\rho \in \mathcal{J}_\ell$ is called an *algebraic singularity* of $\mathcal{J}_\ell$. A smooth point $\rho \in \mathcal{J}_\ell$ is called

(i) *regular*, if a metric open neighbourhood $\rho \in \mathcal{U} \subseteq \mathcal{J}_\ell$ exists such that the Vessiot distribution $\mathcal{V}[\mathcal{J}_\ell]$ is regular on $\mathcal{U}$ and can be decomposed as $\mathcal{V}[\mathcal{J}_\ell] = \mathcal{N}[\mathcal{J}_\ell] \oplus \mathcal{H}$ with an n-dimensional, transversal, involutive, smooth distribution $\mathcal{H} \subseteq T\mathcal{U}$;

(ii) *regular singular*, if a metric open neighbourhood $\rho \in \mathcal{U} \subseteq \mathcal{J}_\ell$ exists such that the Vessiot distribution $\mathcal{V}[\mathcal{J}_\ell]$ is regular on $\mathcal{U}$, but at the point ρ no n-dimensional complement to the symbol $\mathcal{N}_\rho[\mathcal{J}_\ell]$ exists, i. e. we have $\dim \mathcal{V}_\rho[\mathcal{J}_\ell] - \dim \mathcal{N}_\rho[\mathcal{J}_\ell] < n$;

(iii) *irregular singular*, if the Vessiot spaces do not form a regular distribution on any metric open neighbourhood $\rho \in \mathcal{U} \subseteq \mathcal{J}_\ell$; i. e. any such neighbourhood contains at least one point $\bar{\rho}$ such that $\dim \mathcal{V}_{\bar{\rho}}[\mathcal{J}_\ell] < \dim \mathcal{V}_\rho[\mathcal{J}_\ell]$.

An irregular singularity $\rho \in \mathcal{J}_\ell$ is called *purely irregular*, if an n-dimensional complement to the symbol space $\mathcal{N}_\rho[\mathcal{J}_\ell]$ exists, i. e. $\dim \mathcal{V}_\rho[\mathcal{J}_\ell] - \dim \mathcal{N}_\rho[\mathcal{J}_\ell] = n$.

This definition of singularities follows the same geometric ideas as the classical one in differential topology (see e. g. [6]), where usually only scalar ordinary differential equations are considered. It extends the definitions given in [2, Def. 9.1.9] for not underdetermined systems of ordinary differential equations and in [29, Def. 3.1] for maximally overdetermined systems of partial differential equations also to systems which are not of finite type.[6]

Example 4.2. As a concrete example where all different types of points appearing in the above definition occur, we consider the following second-order system of semilinear partial differential equations for one unknown function u in two independent variables x, y:

$$\begin{aligned} x^2 u_{xx} + x u_x + (x-1)^2 u &= 0\,, \\ (1-y^2) u_{yy} + 2y u_y + 2u &= 0\,. \end{aligned}$$

If we consider the algebraic differential equation $\mathcal{J}_2 \subset J_2\pi$ defined by it, then one must distinguish seven different cases in the analysis of the linear system defining the Vessiot spaces:

1. *Regular points* on $\mathcal{J}_2$ are characterised by the conditions $x \neq 0$ and $y^2 - 1 \neq 0$. They have a three-dimensional Vessiot space.

2. Points where $x = 0$, $y^2 - 1 \neq 0$ and either $u_x \neq 0$ or $u_y \neq 0$ are *regular singular*. They also possess a three-dimensional Vessiot space. As the coefficients a_1 and a_2 in (9) must satisfy the equation $2u_x a_1 + u_y a_2 = 0$, only a one-dimensional transversal complement exists.

[6] These definitions also assumed that the given differential equation defines a manifold, whereas here we also allow varieties with algebraic singularities.

3. Basically the same holds for points where $y^2 - 1 = 0$, $x \neq 0$ and either $yu_x + u_{xy} \neq 0$ or $u \neq 0$: they are *regular singular* and have a three-dimensional Vessiot space with a one-dimensional transversal complement defined by the equation $(yu_x + u_{xy})a_1 - 2ua_2 = 0$.

4. Points where $x = 0$, $y^2 - 1 = 0$ and either $u_x \neq 0$ or $yu_{xy} + u_x \neq 0$ are *irregular singularities* which are not purely irregular: the Vessiot space is four-dimensional with a one-dimensional transversal complement defined by the condition $a_1 = 0$.

5. Points where $x = 0$, $u_x = 0$, $u_y = 0$ and $y^2 - 1 \neq 0$ are *purely irregular singular* and possess a four-dimensional Vessiot space defined by the equation $(y^2 - 1)b_{02} - 2yu_{xy}a_1 = 0$.

6. The same behaviour is shown by points with $y^2 - 1 = 0$, $u = 0$, $u_y = 0$, $x \neq 0$, but with the Vessiot space defined by the equation $x^2 b_{20} + (x^2 - xy - 2x - 1)u_x a_1 = 0$.

7. Finally, the points where $x = 0$, $y^2 - 1 = 0$, $u_x y = 0$ and $u = 0$ are also *purely irregular singular*, but now with a five-dimensional Vessiot space.

Note that the cases 2, 3 and 4 do not correspond to an algebraic jet set but the union of two such sets, because of the disjunctions in their defining conditions. Hence, if one applies the algorithm we present in the next section to this example, then one obtains actually $10 = 7 + 3$ cases.

Remark 4.3. For differential equations of finite type (and thus, in particular, for all not underdetermined ordinary differential equations), Definition 4.1 can be considerably simplified, as it is no longer necessary to consider neighbourhoods. For a passive equation of finite type, it is a priori clear that the expected dimension of the Vessiot space at a regular point is n. Thus, singularities can be recognised by a simple comparison with this value (see [29] for such a definition of regular and irregular singularities). Our more complicated approach via neighbourhoods is the prize to be paid for the fact that Definition 4.1 is to our knowledge the first attempt to provide a systematic taxonomy of the singularities of arbitrary systems of partial differential equations. We do not claim that our definition already provides a complete taxonomy, however, it appears very natural from the point of view of the geometric theory of differential equations, as it takes all fundamental geometric objects (Vessiot and symbol spaces) into account.

A further effect of the extension beyond equations of finite type is the novel notion of a *purely* irregular singularity. At generic singularities, such a distinction is not necessary: the dimension of the symbol space $\mathcal{N}_\rho[\mathcal{J}_\ell]$ jumps generically at a singularity only by one, and in this case any irregular singularity is automatically purely irregular. At points where no n-dimensional transversal complement to the symbol space exists, not even a formal power series solution can exist. Hence, pure irregularity is important for any kind of solution theory around singularities.

The inclusion of equations which are not of finite type also explains why we impose for regular points the condition that an involutive complement $\mathcal{H}$ must exist on a neighbourhood. It ensures the local existence of at least one foliation by generalised solutions, which is our intuitive picture of regular points, as discussed above. Thus, one may consider this as a solvability condition (see also Prop. C.5, where the assumption of an involutive distribution is crucial).

Remark 4.4. If we ignore the requirement that in the neighbourhood of a regular point the complement $\mathcal{H}$ must be involutive, then the three cases in Definition 4.1 correspond to the analysis of the linear system (10). A necessary condition for a point $\rho \in \mathcal{J}_\ell$ to be regular is that the symbol matrix $M_\ell(\rho)$ has its maximal possible rank and that this rank coincides with the maximal possible rank of the augmented matrix $(D(\rho) \mid M_\ell(\rho))$. At a regular singular point, the augmented matrix has still the maximal possible rank, but the rank of the symbol matrix has dropped. At an irregular singular point even the rank of the augmented matrix has dropped.

In the case of ordinary differential equations, any complement $\mathcal{H}$ can only be one-dimensional and thus is trivially involutive wherever it defines a regular distribution. Hence, for ordinary differential equations Definition 4.1 provides a complete taxonomy of all points on $\mathcal{J}_\ell$. For partial differential equations, it is in general difficult to prove the involutivity of $\mathcal{H}$ around points where the above mentioned necessary condition for a regular point is satisfied.

We always study algebraic jet sets coming from simple differential systems produced by a differential Thomas decomposition. Here it is possible to prove for generic points that they are regular. For the other points satisfying the above necessary condition, two possibilities arise. If they are regular (which we are not able to prove), then there are (prolonged) solutions going through them. By the properties of the differential Thomas decomposition, they must belong to the solution set of another simple differential system arising in the decomposition. Hence, one can argue that they are irrelevant in the analysis of the given simple differential system. If they are not regular, then they fall outside the taxonomy of Definition 4.1. It is unclear whether this case is actually possible; at least we do not know of any concrete example where such points appear. They could be related to novel kinds of singular behaviour that only exist in partial differential equations, but they also could simply be accidentially introduced by taking the Zariski closure.

Remark 4.5. Regular and irregular singularities may be considered as *geometric singularities* in the sense that they represent the critical points of the restriction of the canonical projection $\pi^\ell_{\ell-1} : J_\ell\pi \to J_{\ell-1}\pi$ to the considered subset $\mathcal{J}_\ell$, i. e. of the map $\hat{\pi}^\ell_{\ell-1} : \mathcal{J}_\ell \to \pi^\ell_{\ell-1}(\mathcal{J}_\ell) \subseteq J_{\ell-1}\pi$. In other words, they are the points ρ where the tangent map $T_\rho\hat{\pi}^\ell_{\ell-1}$ is not surjective. Indeed, at smooth points the symbol spaces are the kernels of the restricted projection $\hat{\pi}^\ell_{\ell-1}$. Hence, one may say that geometric singularities are those points where the dimension of the symbol space jumps. This is the classical approach to define singularities of implicit ordinary differential equations, as one can find it e. g. in [5].

Definition 4.1 is really meaningful only, if we can show that the regular points form a Zariski dense subset and thus really represent the "regular" behaviour. The main problem in proving this fact consists in establishing the existence of a smooth distribution $\mathcal{H}$ possessing all the required properties. As this is much easier for systems of finite type, we treat this case separately.

Proposition 4.6. *Let S be a simple differential system, with respect to a Riquier ranking, comprising no equation of an order greater than $\ell \in \mathbb{N}$, for which the associated algebraic differential equation $\mathcal{J}_\ell(S)$ defined in* (5) *is of finite type. Then the regular points in its Zariski closure $\overline{\mathcal{J}_\ell(S)}$ contain a Zariski open and dense subset.*

Proof. By Proposition 2.7, $\overline{\mathcal{J}_\ell(S)}$ is a locally integrable algebraic differential equation. In the proof of that proposition it was shown that every point ρ in a Zariski open and dense subset $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell(S)$ lies in the image of a prolonged classical solution σ. In Remark 2.11 it was discussed that for an equation of finite type this solution σ is uniquely determined by ρ. This implies in particular that for different such solutions the images of their prolongations cannot intersect in a sufficiently small neighbourhood of ρ. Hence, these images define a foliation of such a neighbourhood with n-dimensional leaves and the tangent spaces of the points on the leaves are just the Vessiot spaces there. This observation implies that the Vessiot distribution restricted to this neighbourhood is integrable and hence involutive by the Frobenius Theorem. Therefore, all smooth points $\rho \in \mathcal{R}_\ell$ are regular in the sense of Definition 4.1. ☐

Note that this proof also tells us precisely the local solution behaviour near a regular point: the neighbourhood of the point is foliated by n-dimensional transversal leaves, which are generalised solutions projecting on geometric solutions that are the images of classical solutions. The generalisation to arbitrary systems requires the use of the Vessiot theory of differential equations introduced originally in [30]. A modern presentation relating it to the geometric theory of differential equations can be found in [31] (see also [32] or [2, Sects. 9.5/6]). These references are concerned with the existence of flat Vessiot connections. The horizontal bundle of such a connection is nothing but a smooth distribution $\mathcal{H}$ with all the properties required in the definition of a regular point.

Theorem 4.7. *Let S be a simple differential system, with respect to a Riquier ranking, comprising no equation of an order greater than $\ell \in \mathbb{N}$, and $\mathcal{J}_\ell(S)$ the associated algebraic differential equation defined in* (5)*. Then the regular points in the Zariski closure $\overline{\mathcal{J}_\ell(S)}$ contain a Zariski open and dense subset.*

Proof. As in the proof of Proposition 4.6, we consider again the Zariski open and dense subset $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell(S)$. As a smooth point, any point $p \in \mathcal{R}_\ell$ lies on exactly one irreducible component of $\mathcal{J}_\ell(S)$. The intersection of $\mathcal{R}_\ell$ with this irreducible component is a manifold which, by the proof of Proposition 2.7, defines a formally integrable differential equation in the sense of the geometric theory, since local integrability trivially entails formal integrability. The equations in a simple system form by definition a (differential) Janet basis and it is easy to see that consequently their principal parts (introduced in Appendix C) define at any point $\rho \in \mathcal{R}_\ell$ a (polynomial) Janet basis of the principal symbol module $\mathcal{M}[\rho]$. The maximal degree of a generator in this basis is at most ℓ. By [2, Thm. 5.4.12, Rem. 5.4.13], this Janet basis induces a free resolution of $\mathcal{M}[\rho]$ and the form of this resolution implies that the Castelnuovo-Mumford regularity of $\mathcal{M}[\rho]$ is at most ℓ. By [2, Rem. 6.1.23], this implies that the symbol $\mathcal{N}_\rho[\mathcal{J}_\ell]$ is involutive at any point $\rho \in \mathcal{R}_\ell$, as the order at which a symbol becomes involutive is determined by the regularity of the principal symbol module. Hence, the manifold defines even an involutive differential equation in the sense of the geometric theory. Now [32, Thm. 3] (or equivalently [2, Thm. 9.6.11]) asserts the existence of a smooth distribution $\mathcal{H}$ with the required properties in a neighbourhood of p, so that p is indeed a regular point. □

It should be emphasised that the distribution $\mathcal{H}$ appearing at the end of the proof is never unique for a system which is not of finite type. Again, each particular choice of a distribution $\mathcal{H}$ induces a foliation of a neighbourhood of the regular point with n-dimensional transversal leaves, which are the images of generalised solutions coming from classical solutions. However, for a system not of finite type there always exist infinitely many such choices and hence infinitely many different foliations. Nevertheless, we may still say that regular points are characterised by the existence of at least one such foliation.

Remark 4.8. The rows of the symbol matrix $M_\ell(\rho)$ can be understood as linear generators of the homogeneous component of degree ℓ of the principal symbol module $\mathcal{M}[\rho]$. If an algebraic differential equation $\mathcal{J}_\ell(S)$ is described by a simple algebraic system as in the theorem above, then the symbol matrix $M_\ell(\rho)$ automatically arises in a triangular form and the pivots are the separants of the equations in the system.

Example 4.9. It should be noted that the notions introduced in Definition 4.1 are relative in the sense that they obviously depend on the choice of the differential equation $\mathcal{J}_\ell$. In some situations one may have more than one option and then obtains different results for certain points.

As a simple concrete example, we may consider the Clairaut equation $u = xu' + f(u')$; the corresponding algebraic jet set is shown in Figure 1 in blue. It represents a classical instance of a differential equation with a singular integral. Its general solution is given by the straight lines $u(x) = cx + f(c)$ with a parameter c (some lines are shown in green in the figure). Their envelope is the singular integral given parametrically by $x(\tau) = -f'(\tau)$, $u(\tau) = -\tau f'(\tau) + f(\tau)$ (shown in red). The singular integral is the sole solution of the overdetermined system $u = xu' + f(u')$ and $f'(u') + x = 0$ (the separant of the first equation). If we choose as $\mathcal{J}_1$ the whole blue surface, then all points on the singular integral are irregular singularities, as the Vessiot spaces are two-dimensional there. If we choose instead only the curve defined by the prolonged singular integral (which represents an algebraic differential equation in its own right[7]), then all points on it are regular, as now for the overdetermined system the Vessiot space is always one-dimensional and coincides with the tangent space of the curve. This effect is captured in Definition 4.1 by the use of a metric open neighbourhood of the considered point. Depending on the choice of $\mathcal{J}_\ell$, the dimension of the neighbourhood as a smooth manifold may vary and the neighbourhood decides what is considered as regular and what as singular.

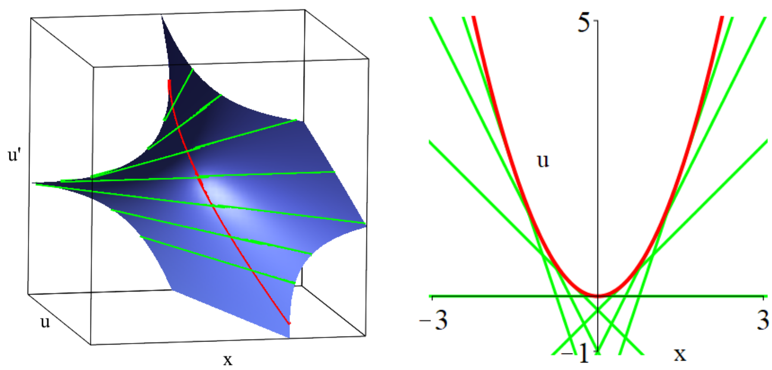

Figure 1: Clairaut equation for $f(s) = -\frac{1}{4}s^2$ with the singular integral in red. Left: generalised solutions in $J_1\pi$. Right: solution graphs in x-u plane.

5. Regularity Decomposition of a Differential System

The geometric theory of differential equations considers usually exclusively "regular" equations, although it is not so easy to provide a rigorous definition of what this regularity should be and even harder to verify effectively whether or not a given equation is regular. Very often, one only finds generic statements that all assertions are valid outside of some (unspecified) hypersurface (see e. g. [33] and references therein). We now define first a rigorous notion of a regular algebraic jet set. For such jet sets, we can extend the pointwise decomposition (12) to a global one: $\mathcal{V}[\mathcal{J}_\ell] = \mathcal{N}[\mathcal{J}_\ell] \oplus \mathcal{H}$ with some smooth vector bundle $\mathcal{H}$. We study the generalisation to a regular differential equation in the sense of the geometric theory of differential equations in the following Section 6.

[7] With the notation from below, the Clairaut equation can be decomposed into two primary components: the general solution and the singular integral. This decomposition can be seen when prolonging to order two.

Definition 5.1. An algebraic jet set $\mathcal{J}_\ell \subseteq J_\ell\pi$ is *regular*, if

(i) it consists only of smooth points, i. e. $\mathcal{J}_\ell$ is a smooth manifold,
(ii) its Vessiot distribution $\mathcal{V}[\mathcal{J}_\ell]$ defines a smooth vector bundle over $\mathcal{J}_\ell$ and
(iii) its symbol $\mathcal{N}[\mathcal{J}_\ell]$ defines a smooth vector bundle over $\mathcal{J}_\ell$.

Let S be a differential system. As discussed in Section 2, as a first step we compute a differential Thomas decomposition of S into simple differential systems, each of which we then treat separately. Thus we assume from now on that S is already a simple differential system. We choose a sufficiently high order ℓ and consider the associated algebraic jet set $\overline{\mathcal{J}_\ell(S)} \subset J_\ell\pi$. In general it might be a reducible variety. As any point contained in the intersection of two irreducible components of $\overline{\mathcal{J}_\ell(S)}$ is automatically an algebraic singularity, we prefer to study each irreducible component separately. We then want to express each irreducible component as a disjoint union of *regular* algebraic jet sets.

Definition 5.2. Let $S \subset \mathcal{D}$ be a simple differential system and $\overline{\mathcal{J}_\ell(S)} \subset J_\ell\pi$ the associated algebraic jet set in a sufficiently high order ℓ. Let furthermore $\overline{\mathcal{J}_\ell(S)} = \mathcal{J}_{\ell,1} \cup \cdots \cup \mathcal{J}_{\ell,t}$ be its decomposition into irreducible varieties. A *regularity decomposition* of the variety $\mathcal{J}_{\ell,k}$ represents it as a disjoint union of finitely many regular algebraic jet sets $\mathcal{J}^{(1)}_{\ell,k}, \dots, \mathcal{J}^{(r)}_{\ell,k}$, the *regularity components* of $\mathcal{J}_{\ell,k}$, and of the set $\operatorname{ASing}(\overline{\mathcal{J}_\ell(S)})$ of algebraic singularities.

If we classify the points on the irreducible variety $\mathcal{J}_{\ell,k}$ according to Definition 4.1, then if one point on a regularity component $\mathcal{J}^{(i)}_{\ell,k}$ is a regular (irregular) singularity, then all other points on this component are regular (irregular) singularities, too. Indeed, Definition 5.1 implies that at all points on a regular algebraic jet set the symbol and the Vessiot space, respectively, have the same dimension. The situation is more involved for regular points (of partial differential equations), as discussed in Remark 4.4. However, as a consequence of Remark 2.4 and Theorem 4.7, we may conclude that the regular points contain on each prime component $\mathcal{J}_{\ell,k}$ a Zariski open and dense subset. If a point lies in the intersection of several irreducible components, then we classify it separately with respect to each of these components. It is well possible that one obtains here different results (see Examples 4.9 or 7.3 for concrete instances).

We now prove the existence of regularity decompositions by providing an algorithm for their construction. For at least one component of the obtained decomposition (which contains a Zariski open and dense subset), we prove that it consists of regular points. As discussed in Remark 4.4, we cannot exclude the possibility that in some other regularity components the Vessiot and the symbol space have at all points the dimension expected for a regular point, but we are unable to prove the involutivity of the complement $\mathcal{H}$. We say that such points are of *unknown type*.

The first step of our algorithm consists of determining a generating set $\{p_1, \dots, p_s\}$ of the algebraic ideal $\mathcal{I}_\ell(S)$ according to Remark 2.9. As second step, we determine the minimal prime decomposition $\mathcal{I}_\ell(S) = \bigcap_{k=1}^t \mathcal{I}_{\ell,k}$ of the ideal $\mathcal{I}_\ell(S)$, which is radical by definition. According to Lemma 2.8, $\overline{\mathcal{J}_\ell(S)} = \operatorname{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S))$ and, by construction, $\operatorname{Sol}^{\mathrm{a}}(\mathcal{I}_\ell(S)) = \bigcup_{k=1}^t \operatorname{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k})$. We then determine for each irreducible component $\operatorname{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k})$ separately a regularity decomposition.

For the determination of these regularity decompositions, we exploit that our taxonomy of regular and singular points (Definition 4.1) is mainly based on the properties of the linear system (10) determining the Vessiot distribution. If $\{p_{k,1}, \dots, p_{k,s_k}\}$ is a generating set of the prime component $\mathcal{I}_{\ell,k}$, then we use these polynomials for setting up the linear system (10), as it simply encodes a condition of tangency to the irreducible component $\mathcal{J}_{\ell,k} = \operatorname{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k})$.

In addition, we set up a second linear system for the detection of the algebraic singularities defined by

$$\mathbf{J}(p_{k,r}) := \left(\sum_{|\mu|=\ell} \sum_{\alpha} c^{\alpha}_{\mu} \partial_{u^{\alpha}_{\mu}} + \sum_{j} d^{j} \partial_{x^{j}} \right) p_{k,r} = 0, \quad r = 1, \ldots, s_k. \tag{13}$$

The left hand side is obtained by multiplying the Jacobian matrix of $p_{k,1}, \ldots, p_{k,s_k}$ by the vector of auxiliary indeterminates c^{α}_{μ} and d^{j}. The equations in the combined linear system may be considered as elements of the extended polynomial ring $\mathcal{D}^{\mathrm{ex}}_{\ell} = \mathcal{D}_{\ell}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}]$, where we have adjoined the auxiliary indeterminates $\mathbf{a}$, $\mathbf{b}$ of the ansatz (9) and $\mathbf{c}$, $\mathbf{d}$ of (13). Furthermore, we consider this combined linear system only at points on $\mathcal{J}_{\ell,k}$ and thus add the equations $p_{k,1}, \ldots, p_{k,s_k} \in \mathcal{D}_{\ell} \subset \mathcal{D}^{\mathrm{ex}}_{\ell}$. We compute an algebraic Thomas decomposition of the combined system in $\mathcal{D}^{\mathrm{ex}}_{\ell}$ using an ordering satisfying the following conditions: (i) $\mathbf{d} > \mathbf{c} > \mathbf{b} > \mathbf{a} > \mathbf{u} > \mathbf{x}$, (ii) restricted to the variables $\mathbf{u}$ the ordering corresponds to an orderly ranking (cf. Appendix B) and (iii) the variables c^{α}_{μ} and b^{α}_{μ}, respectively, are ordered among themselves in the same way as the derivatives u^{α}_{μ}.

Let $S^{\mathrm{ex}}_{k,i}$ be one of the resulting simple algebraic systems. If $S^{\mathrm{ex}}_{k,i}$ has less than $\operatorname{codim} \mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k})$ equations with leader among the auxiliary indeterminates $\mathbf{c}$, $\mathbf{d}$, we remove all equations with leader among $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and obtain the simple system $S_{k,i}$ over $\mathcal{D}_{\ell}$, which contributes $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$ to $\mathrm{ASing}(\overline{\mathcal{J}_{\ell}(S)})$. Otherwise, again removing all equations with leader among $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, we obtain a simple algebraic system $S_{k,i}$ in $\mathcal{D}_{\ell}$, which contributes the regularity component $\mathcal{J}^{(i)}_{\ell,k} = \mathrm{Sol}^{\mathrm{a}}(S_{k,i})$. In a more formal language, we arrive at Algorithm 5.3.

Algorithm 5.3 (Regularity Decomposition of a Simple Differential System).

Input: a simple differential system S over $K\{U\}$ and a sufficiently high order $\ell \in \mathbb{N}$

Output: a regularity decomposition for each prime component $\mathcal{I}_{\ell,k}(S)$ of the ideal $\mathcal{I}_{\ell}(S) \subset \mathcal{D}_{\ell}$

Algorithm:

compute a generating set $\{p_1, \ldots, p_s\}$ of the radical ideal $\mathcal{I}_{\ell}(S)$ according to Remark 2.9

compute a prime decomposition $\mathcal{I}_{\ell}(S) = \mathcal{I}_{\ell,1}(S) \cap \ldots \cap \mathcal{I}_{\ell,t}(S)$ of $\mathcal{I}_{\ell}(S)$ and a generating set $\{p_{k,1}, \ldots, p_{k,s_k}\}$ for each prime component $\mathcal{I}_{\ell,k}(S)$

for $k \in \{1, \ldots, t\}$ **do**

  compute an algebraic Thomas decomposition $S^{\mathrm{ex}}_{k,1}, \ldots, S^{\mathrm{ex}}_{k,r_k}$ with respect to a total order $\mathbf{d} > \mathbf{c} > \mathbf{b} > \mathbf{a} > \mathbf{u} > \mathbf{x}$ satisfying the above mentioned conditions of the algebraic system

$$\left\{ \begin{array}{rcl} \mathbf{J}(p_{k,j}) & = & 0\,, \\ \mathbf{V}(p_{k,j}) & = & 0\,, \\ p_{k,j} & = & 0\,, \end{array} \right\} \qquad j = 1, \ldots, s_k \tag{14}$$

  defined over $\mathcal{D}^{\mathrm{ex}}_{\ell}$, where

$$\mathbf{V} = \sum_{i} a^{i}\, C^{(\ell)}_{i} + \sum_{|\mu|=\ell} \sum_{\alpha} b^{\alpha}_{\mu}\, C^{\mu}_{\alpha} \quad \text{and} \quad \mathbf{J} = \sum_{\mu} \sum_{\alpha} c^{\alpha}_{\mu} \partial_{u^{\alpha}_{\mu}} + \sum_{i} d^{i} \partial_{x^{i}}$$

od

return the systems $S_{k,i}$ consisting of those equations $p = 0$ and inequations $q \neq 0$ in $S^{\mathrm{ex}}_{k,i}$ with $p \in \mathcal{D}_{\ell}$ and $q \in \mathcal{D}_{\ell}$

The remainder of this section is dedicated to explaining this algorithm and proving its correctness.

Remark 5.4. Algorithm 5.3 is in principle not yet completely specified, as we say nothing about how the algebraic Thomas decomposition in Step 4 is computed. In fact, the correctness of the algorithm depends on whether this Thomas decomposition has been computed in a "good" way (this is made precise in Proposition 5.11). As any reasonable implementation automatically satisfies this condition, we have not mentioned it in the algorithm. It also should be noted that the output of our algorithm depends not only on details of the implementation of the Thomas decomposition, but strongly on the used ranking. In a system with several independent variables $\mathbf{x}$ or several differential unknowns $\mathbf{u}$, very different results can be obtained for different orderings inside each of the blocks $\mathbf{x}$ and $\mathbf{u}$, respectively. In particular, the obtained regularity decomposition is often overly complicated, i. e. consists of too many different components, as in Step 4 we implicitly compute a Thomas decomposition of the variety $\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k}(S))$, which might entail many case distinctions that are not necessary for our purposes. In a post-processing step these unnecessary case distinctions could be conflated by comparing for all cases with smooth points the linear part corresponding to the system $\mathbf{V}(p_{k,j}) = 0$ and combining all cases where equivalent equations have been obtained.

In Remark 2.9 it was already mentioned that the ideal $\mathcal{I}_\ell(S)$ can be generated by a simple algebraic system. Now [22, Thm. 1.94] entails that this ideal is equidimensional in the strong sense that all its associated primes have the same dimension.[8] In particular, no embedded primes can exist. Because of the assumptions that S is a simple differential system and that ℓ is sufficiently large, the prime decomposition of $\mathcal{I}_\ell(S)$ computed in the second step of our algorithm induces also a prime decomposition of the differential ideal $\mathcal{I}_{\mathrm{diff}}(S)$, as we show below. Note, however, that even if the prime decomposition of $\mathcal{I}_\ell(S)$ is minimal, there is no guarantee that also the differential prime decomposition is minimal. Here we encounter again the well-known Ritt problem [18, §IV.9] in differential algebra: no algorithm is known to decide whether one differential prime ideal is contained in another one.

For the next proof it is important to discuss the relationship between the notion of a simple differential system as defined in Definition B.3 and the notion of a regular differential system often used in differential algebra—see e. g. [24, Def. 4.7]. The following lemma and its proof entail that we may always assume without loss of generality that a simple differential system is also regular, as the only difference between these two notions is the extent to which autoreduction has been performed. [24, Def. 4.7] uses partial reductions, i. e. only reductions using derived equations are performed, but no purely algebraic reductions. However, it is always assumed that the whole differential polynomial is reduced. By contrast, the conditions imposed in Definitions A.1 and B.3 require only head reductions, but algebraic reductions are also performed. From a theoretical point of view, it is irrelevant whether or not tail reductions are performed. From a computational point of view, they are often expensive and thus it is better to omit them.

Lemma 5.5. *Let S be a simple differential system as in* (D). *Then S is equivalent to a regular differential system in the sense that some tail (pseudo) reductions turn S into a regular system with the same leaders and the same saturated multiplicatively closed set generated by the initials and separants.*

Proof. From S we collect the left hand sides of the equations and inequations, respectively, in the two sets P and Q. The first two properties in Definition B.3 entail that, modulo tail reduction,

[8] Some authors call such ideals unmixed dimensional and speak of equidimensional ideals already when all minimal primes have the same dimensions.

P is a differential triangular set. The second property also ensures that all Δ-polynomials that can be formed with elements of P reduce to zero modulo P. The first and third property imply that, modulo tail reduction, each inequation is partially reduced with respect to P. Finally, denote by Q^∞ the smallest subset of $\mathcal{D}$ that contains 1 and Q and has the property that $q, \tilde{q} \in Q^\infty$ is equivalent to $q\tilde{q} \in Q^\infty$, i. e. the saturated multiplicatively closed set generated by Q. Note that tail (pseudo) reduction amongst elements of P might change their initials and separants, by multiplying them by the initial or separants of the reducing polynomial. Finally, any separant of an equation in P lies in Q^∞, up to reduction by P. Thus all conditions in the definition [24, Def. 4.7] of a regular differential system are satisfied. □

Proposition 5.6. *Assume that S does not contain any equation or inequation of order greater than ℓ and denote by q the product of all separants of the equations in S. Then the differential ideals $\langle p_{k,1}, \ldots, p_{k,s_k} \rangle_\Delta : q^\infty$ (in the notation of Algorithm 5.3) for $1 \le k \le t$ represent all essential prime components of the differential ideal $\mathcal{I}_{\text{diff}}(S)$.*

Proof. By Lemma 5.5, we may assume that S is a regular differential system. The statement follows then immediately from [24, Thm. 4.13]. □

Remark 5.7. When setting up the linear equations describing the Vessiot spaces in Step 4, it suffices to consider only those generators $p_{k,j}$ that depend on some jet variables of order ℓ, as all other generators would only contribute the trivial equation $0 = 0$. Indeed, if p is a generator of lower order, then we have trivially $C^\mu_\alpha(p) = 0$, and it follows from (C.2) that $C^{(\ell)}_i(p) = D_i p$. Since by Proposition 5.6 the ideal $\mathcal{I}_{\ell,k}(S)$ is the truncation of a differential ideal, the formal derivative $D_i p$ (defined in Appendix C) can be written as a linear combination of the generators $p_{k,j}$. Hence, $\mathbf{V}(p)$ vanishes modulo $\mathcal{I}_{\ell,k}(S)$, i. e. it is zero at all considered points.

As preparation for showing the correctness of Algorithm 5.3, we prove some results about the simple algebraic systems $S^{\text{ex}}_{k,i}$ produced by the algorithm relating them to algebraic singularities and the Vessiot and symbol spaces. Furthermore, we provide a technical proposition needed for the correctness proof.

Proposition 5.8. *Given any subset $\mathrm{Sol}^{\mathrm{a}}(S_{k,i}) \subseteq V_{\ell,k} = \mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k})$, either all points contained in it are smooth in $V_{\ell,k}$ or all are algebraic singularities of $V_{\ell,k}$.*

Proof. By substituting the coordinates of a point $\rho \in V_{\ell,k}$ into (13), we obtain a system of linear equations in $\mathbf{c}$, $\mathbf{d}$ whose solution space is the tangent space to $V_{\ell,k}$ at ρ. The point ρ is smooth in $V_{\ell,k}$, if and only if this tangent space has dimension $\dim V_{\ell,k}$, and singular otherwise. The algebraic system (14) consists only of equations, and the equations which involve the indeterminates $\mathbf{c}$, $\mathbf{d}$ are homogeneous of degree one in these indeterminates. Since $\mathbf{c}$, $\mathbf{d}$ are ranked higher than the indeterminates $\mathbf{u}$, $\mathbf{x}$, we conclude that the simple algebraic system $S^{\text{ex}}_{k,i}$ obtained in Step 4 of Algorithm 5.3 contains no inequations with a leader among $\mathbf{c}$, $\mathbf{d}$, and every equation which involves the indeterminates $\mathbf{c}$, $\mathbf{d}$ is homogeneous of degree one in these indeterminates. Consider now those equations with leader among $\mathbf{c}$, $\mathbf{d}$ in $S^{\text{ex}}_{k,i}$. Due to the linearity and the triangularity of the system, the number of these equations is equal to the codimension of the tangent space and this codimension is independent of the choice of $\rho \in \mathrm{Sol}^{\mathrm{a}}(S_{k,i})$, because $S^{\text{ex}}_{k,i}$ is simple. Hence, $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$ consists entirely of smooth points, if and only if the number of equations with leader among $\mathbf{c}$, $\mathbf{d}$ is equal to the codimension of $V_{\ell,k}$, and entirely of singular points otherwise. □

Remark 5.9. No equation in the system (14) contains simultaneously indeterminates from $\mathbf{a}$, $\mathbf{b}$ and from $\mathbf{c}$, $\mathbf{d}$. Since the algebraic Thomas decomposition method does not apply polynomial division to a pair of equations involving different sets of indeterminates from $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{d}$, respectively, the correctness does not depend on the choice of how $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are ordered.

Proposition 5.10. *Let $S^{\mathrm{ex}}_{k,i}$ be a simple algebraic system obtained in Step 4 of Algorithm 5.3 such that $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$ consists entirely of smooth points. Denote by $N_{\mathbf{a}}$ and $N_{\mathbf{b}}$ the number of equations with a leader among the variables $\mathbf{a}$ and $\mathbf{b}$, respectively. Then at any point $\rho \in \mathrm{Sol}^{\mathrm{a}}(S_{k,i}) \subseteq \mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k})$ the dimension of the symbol space and of the Vessiot space, resp., is given by*

$$\dim \mathcal{V}_\rho[\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k}(S))] = m\binom{\ell+n-1}{\ell} + n - N_{\mathbf{b}} - N_{\mathbf{a}}\,, \tag{15}$$

$$\dim \mathcal{N}_\rho[\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k}(S))] = m\binom{\ell+n-1}{\ell} - N_{\mathbf{b}}\,. \tag{16}$$

Furthermore, an n-dimensional complement $\mathcal{H}_\rho$ to the symbol space $\mathcal{N}_\rho[\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k}(S))]$ exists in the Vessiot space $\mathcal{V}_\rho[\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k}(S))]$, if and only if $N_{\mathbf{a}} = 0$. Finally, the set $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$ is a regular algebraic jet set.

Proof. The algebraic system (14) consists only of equations, and the equations which involve the indeterminates $\mathbf{a}$ and $\mathbf{b}$ are homogeneous of degree one in these indeterminates and independent of the indeterminates $\mathbf{c}$ and $\mathbf{d}$. Thus, for notational simplicity, we may ignore in the sequel the equations containing $\mathbf{c}$ and $\mathbf{d}$. Since $\mathbf{a}$ and $\mathbf{b}$ are ranked higher than the indeterminates $\mathbf{u}$, $\mathbf{x}$, the simple algebraic system $S^{\mathrm{ex}}_{k,i}$ cannot contain inequations with a leader in $\mathbf{a}$ or $\mathbf{b}$, and every equation which involves the indeterminates $\mathbf{a}$ and $\mathbf{b}$ is still homogeneous of degree one in these indeterminates. The triangularity of $S^{\mathrm{ex}}_{k,i}$ means that these equations correspond to a reduced row echelon form of the determining equations of the Vessiot distribution. This row echelon form is preserved for any choice of the point $\rho \in \mathrm{Sol}^{\mathrm{a}}(S_{k,i})$. Since the dimensions of the vectors $\mathbf{b}$ and $\mathbf{a}$ are $m\binom{\ell+n-1}{\ell}$ and n, respectively, the claimed expression for the dimension of the Vessiot spaces follows immediately from the linearity of the equations.

We consider next the symbol spaces. Let $\rho = (\mathbf{u}, \mathbf{x})$ be a point on $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$. The symbol space $\mathcal{N}_\rho[\mathrm{Sol}^{\mathrm{a}}(\mathcal{I}_{\ell,k}(S))]$ consists of all solutions of $S^{\mathrm{ex}}_{k,i}$ of the form $(\mathbf{b}, \mathbf{0}, \mathbf{u}, \mathbf{x})$. We rank $\mathbf{b}$ higher than $\mathbf{a}$. Hence, any equation with a leader in $\mathbf{a}$ is independent of the indeterminates $\mathbf{b}$ and can be ignored when the symbol is computed, as it is automatically satisfied by homogeneity. This observation entails the claimed expression for the dimension of the symbol space.

The dimension of any complement $\mathcal{H}_\rho$ is trivially the difference of the dimensions of the Vessiot and the symbol space. Hence, by the just derived expressions for these dimensions, it is given by $n - N_{\mathbf{a}}$, which proves the last assertion. Finally, we note that by the above mentioned independence of the pivots in the row echelon form of the chosen point $\rho \in \mathrm{Sol}^{\mathrm{a}}(S_{k,i})$, the dimensions of the Vessiot and the symbol spaces are constant over $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$. Hence, this set is a regular algebraic jet set. □

One key point in proving the correctness of Algorithm 5.3 concerns the last step when we move from the systems $S^{\mathrm{ex}}_{k,j}$ including the indeterminates $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ to the projected systems $S_{k,j}$. The next proposition asserts that the disjointness is preserved by this operation. We consider the following generalisation of our set-up. Let $R = \mathbb{C}[y_1, \ldots, y_m][z_1, \ldots, z_n]$ be a polynomial ring equipped with the ranking $z_1 > z_2 > \ldots > z_n > y_1 > y_2 > \ldots > y_m$. Let S be a (not necessarily simple) algebraic system, defined over R, which does not contain any inequation with a leader in $\{z_1, \ldots, z_n\}$ and whose equations with a leader in $\{z_1, \ldots, z_n\}$ are homogeneous of degree one as polynomials in z_1, …, z_n. Applying any judicious algorithm computing a

Thomas decomposition—e. g. the one from [13]—to S computes an output of this form (for the necessity of this form see Remark 5.12 below), since both the initials and the discriminants of the homogeneous polynomials of degree one are polynomials in the variables $\mathbf{y}$ and hence no case distinction with respect to any polynomial in the variables $\mathbf{z}$ is necessary. Moreover, let $S_1, \ldots, S_r$ be an algebraic Thomas decomposition of S with respect to a ranking $>$ such that no S_i contains an inequation with a leader in $\{z_1, \ldots, z_n\}$. Our situation is recovered by identifying the variables $\mathbf{z}$ with the parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and the variables $\mathbf{y}$ with the appearing jet variables. In the sequel, we denote by $(S_i)_{<z_j}$ the algebraic subsystem consisting of all equations and inequations in the simple system S_i with a leader less than z_j. Thus $(S_i)_{<z_n}$ corresponds to the projected system without any of the variables $\mathbf{z}$.

Proposition 5.11. *In the situation described above, the solution sets* $\mathrm{Sol}^{\mathrm{a}}((S_1)_{<z_n})$, $\mathrm{Sol}^{\mathrm{a}}((S_2)_{<z_n})$, $\ldots$, $\mathrm{Sol}^{\mathrm{a}}((S_r)_{<z_n})$ *of the projected systems are also pairwise disjoint.*

Proof. We first note that any subsystem $(S_i)_{<z_j}$ is also simple. By the properties of simple algebraic systems [17, Subsect. 2.2.1], every solution

$$(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \mathbb{C}^{(n+m)-j}$$

of the subsystem $(S_i)_{<z_j}$ can be extended to a solution

$$(\alpha_j, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \mathbb{C}^{(n+m)-(j-1)}$$

of the larger subsystem $(S_i)_{<z_{j-1}}$. Indeed, the subsystem $(S_i)_{<z_{j-1}}$ can differ from $(S_i)_{<z_j}$ by at most one additional equation or inequation with leader z_{j-1} which then restricts the possible values for α_j. In the case of $j = 1$, we set $(S_i)_{<z_0} := S_i$.

For $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$, we define the intersections

$$V_\beta \quad := \quad \mathrm{Sol}^{\mathrm{a}}(S) \cap \Big\{ (z_1, \ldots, z_n, \beta_1, \ldots, \beta_m) \mid z_1, \ldots, z_n \in \mathbb{C} \Big\},$$

and for $i \in \{1, \ldots, n\}$ and $(\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n) \in \mathbb{C}^{n-i}$ we set

$$V_{\alpha,\beta} := \Big\{ (\alpha_i, \alpha_{i+1}, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \mid \alpha_i \in \mathbb{C} \text{ and } \\ \exists\, \alpha_1, \ldots, \alpha_{i-1} \in \mathbb{C} \,:\, (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \mathrm{Sol}^{\mathrm{a}}(S) \Big\}.$$

Since the equations in S with a leader in $\{z_1, \ldots, z_n\}$ are homogeneous of degree one as polynomials in $z_1, \ldots, z_n$, for each $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$ the set V_β is either empty or an affine subspace of $\mathbb{C}^{n+m}$. For the same reason, for each $i \in \{1, \ldots, n\}$ and $\alpha = (\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n) \in \mathbb{C}^{n-i}$ the set $V_{\alpha,\beta}$ is also either empty or an affine subspace of $\mathbb{C}^{(n+m)-(i-1)}$.

Assume that $\mathrm{Sol}^{\mathrm{a}}((S_{i_1})_{<z_n})$ and $\mathrm{Sol}^{\mathrm{a}}((S_{i_2})_{<z_n})$ are not disjoint for $i_1 \neq i_2$. Let $(\beta_1, \ldots, \beta_m) \in \mathrm{Sol}^{\mathrm{a}}((S_{i_1})_{<z_n}) \cap \mathrm{Sol}^{\mathrm{a}}((S_{i_2})_{<z_n})$. Since both S_{i_1} and S_{i_2} are simple algebraic systems, the point $(\beta_1, \ldots, \beta_m)$ can be extended to solutions $\rho = (\alpha_1^{(i_1)}, \alpha_2^{(i_1)}, \ldots, \alpha_n^{(i_1)}, \beta_1, \ldots, \beta_m)$ and $(\alpha_1^{(i_2)}, \alpha_2^{(i_2)}, \ldots, \alpha_n^{(i_2)}, \beta_1, \ldots, \beta_m)$ of S_{i_1} and S_{i_2}, respectively. The disjointness of the solution sets $\mathrm{Sol}^{\mathrm{a}}(S_{i_1})$ and $\mathrm{Sol}^{\mathrm{a}}(S_{i_2})$ implies that there exists $k \in \{1, \ldots, n\}$ such that $\alpha_k^{(i_1)} \neq \alpha_k^{(i_2)}$. Let k be maximal with that property. Hence, $(\alpha_k^{(i_1)}, \ldots, \alpha_n^{(i_1)}, \beta_1, \ldots, \beta_m)$ and $(\alpha_k^{(i_2)}, \ldots, \alpha_n^{(i_2)}, \beta_1, \ldots, \beta_m)$ are two distinct elements of the affine subspace $V_{\alpha,\beta}$ of $\mathbb{C}^{(n+m)-(k-1)}$, where $\alpha = (\alpha_{k+1}^{(i_1)}, \alpha_{k+2}^{(i_1)}, \ldots, \alpha_n^{(i_1)}) = (\alpha_{k+1}^{(i_2)}, \alpha_{k+2}^{(i_2)}, \ldots, \alpha_n^{(i_2)})$. Therefore, $V_{\alpha,\beta}$ is not finite.

We introduce the index set

$$I(\rho,k)=\left\{\, i\in\{1,\dots,r\} \mid (\alpha^{(i_1)}_{k+1},\dots,\alpha^{(i_1)}_{n},\beta_1,\dots,\beta_m)\in \mathrm{Sol}^{\mathrm{a}}((S_i)_{<z_k}) \right\}.$$

Then we have i_1, $i_2 \in I(\rho,k)$ and

$$V_{\alpha,\beta}=\bigcup_{i\in I(\rho,k)} \mathrm{Sol}^{\mathrm{a}}((S_i)_{<z_{k-1}}).$$

Since the affine subspace $V_{\alpha,\beta}$ of $\mathbb{C}^{(n+m)-(k-1)}$ is not finite, but $I(\rho,k)$ is finite, there exists $j_1 \in I(\rho,k)$ such that $\mathrm{Sol}^{\mathrm{a}}((S_{j_1})_{<z_{k-1}})$ is infinite. Hence, S_{j_1} contains no equation with a leader z_k. However, by assumption, S_{j_1} contains no inequation with a leader z_k either. By exchanging the roles of i_1 and i_2 if necessary, we may assume without loss of generality that $j_1 \neq i_1$. We conclude that $(S_{i_1})_{<z_{k-1}}$ and $(S_{j_1})_{<z_{k-1}}$ have the common solution $(\alpha^{(i_1)}_{k},\dots,\alpha^{(i_1)}_{n},\beta_1,\dots,\beta_m)$.

If $k=1$, this observation contradicts the disjointness of $\mathrm{Sol}^{\mathrm{a}}(S_{i_1})$ and $\mathrm{Sol}^{\mathrm{a}}(S_{j_1})$. Otherwise, the thus obtained common solution can be extended to a solution $(\alpha^{(j_1)}_{1},\alpha^{(j_1)}_{2},\dots,\alpha^{(j_1)}_{k-1},\alpha^{(i_1)}_{k},\dots,\alpha^{(i_1)}_{n},\beta_1,\dots,\beta_m)$ of S_{j_1}. By a similar reasoning as above, the disjointness of the solution sets $\mathrm{Sol}^{\mathrm{a}}(S_{i_1})$ and $\mathrm{Sol}^{\mathrm{a}}(S_{j_1})$ implies that there exists $l\in\{1,\dots,k-1\}$ such that $\alpha^{(i_1)}_{l}\neq\alpha^{(j_1)}_{l}$. Let l be maximal with that property. Then $V_{\alpha',\beta}$ is not finite, where $\alpha'=(\alpha^{(i_1)}_{l+1},\alpha^{(i_1)}_{l+2},\dots,\alpha^{(i_1)}_{n})$. Hence, there exists $j_2\in I(\rho,l)$ such that S_{j_2} neither contains an equation with a leader z_l nor an inequation with a leader z_l. Without loss of generality, we may assume $j_2\neq i_1$. Then $(\alpha^{(i_1)}_{l},\dots,\alpha^{(i_1)}_{n},\beta_1,\dots,\beta_m)$ is a common solution of $(S_{i_1})_{<z_{l-1}}$ and $(S_{j_2})_{<z_{l-1}}$. If $l=1$, this is a contradiction. Otherwise, this argument can be repeated to obtain a contradiction. Hence, the sets $\mathrm{Sol}^{\mathrm{a}}((S_1)_{<z_n})$, $\mathrm{Sol}^{\mathrm{a}}((S_2)_{<z_n})$, $\dots$, $\mathrm{Sol}^{\mathrm{a}}((S_r)_{<z_n})$ are pairwise disjoint. □

Remark 5.12. The assumption in Proposition 5.11 about the absence of inequations with leader in $\{z_1,\dots,z_n\}$ cannot be omitted. For example, let $R=\mathbb{C}[y][z_1,z_2]$ and $z_1>z_2>y$ and consider the system $S=\{z_1=0\}$. Then the systems

$$S_1:\ \left\{\begin{array}{rcl} z_1 &=& 0\\ z_2 &=& 0\end{array}\right. \qquad S_2:\ \left\{\begin{array}{rcl} z_1 &=& 0\\ z_2 &\neq& 0\\ y &=& 0\end{array}\right. \qquad S_3:\ \left\{\begin{array}{rcl} z_1 &=& 0\\ z_2 &\neq& 0\\ y &\neq& 0\end{array}\right.$$

provide a Thomas decomposition of S with respect to $>$, where $\mathrm{Sol}^{\mathrm{a}}((S_1)_{<z_2})$ and $\mathrm{Sol}^{\mathrm{a}}((S_2)_{<z_2})$ are not disjoint and $\mathrm{Sol}^{\mathrm{a}}((S_1)_{<z_2})$ and $\mathrm{Sol}^{\mathrm{a}}((S_3)_{<z_2})$ are not disjoint either. Note, however, that this Thomas decomposition involves case distinctions which would not occur in an application of the Thomas algorithm to S. In fact, the original algebraic system S is already simple. The assumption of Proposition 5.11 is automatically satisfied by any *comprehensive* Thomas decomposition, which can also be computed algorithmically [34, Alg. 3.80].

Theorem 5.13. *Algorithm 5.3 terminates and is correct.*

Proof. The termination is obvious, as only terminating subalgorithms are used. For the correctness, it is sufficient to show that the output is correct for any prime component $\mathcal{I}_{\ell,k}(S)$ of $\mathcal{I}(S_\ell)$. Let $k\in\{1,\dots,t\}$. We argue first that the output systems $S_{k,1},\dots,S_{k,r_k}$ form a Thomas decomposition. Since $S_{k,i}$ is obtained from the simple algebraic system $S^{\mathrm{ex}}_{k,i}$ by omitting the equations and inequations with a leader among $\mathbf{d}$, $\mathbf{c}$, $\mathbf{b}$, $\mathbf{a}$, the algebraic system $S_{k,i}$ is simple. In the proofs of Proposition 5.8 and 5.10 it was shown that we are in a situation where Proposition 5.11 is applicable to the Thomas decomposition $S^{\mathrm{ex}}_{k,1},\dots,S^{\mathrm{ex}}_{k,r_k}$. Hence, the output systems $S_{k,1},\dots,S_{k,r_k}$ have pairwise disjoint solution sets $\mathrm{Sol}^{\mathrm{a}}(S_{k,i})$ which either consist entirely of algebraic singularities by Proposition 5.8 or are regular algebraic jet sets by Proposition 5.10. □

Finally, we describe how one determines a regularity decomposition of a general differential system S in some order $\ell \in \mathbb{N}$. The first step is to compute a differential Thomas decomposition of S into simple differential systems $S_1, \ldots, S_r$. For each simple system one needs to check that the order ℓ we have chosen for the regularity decomposition is sufficiently high. This means that we need to guarantee that no equation or inequation in a simple differential system S_i is cut off when going from S_i to the algebraic ideal $\mathcal{I}_\ell(S_i)$. If the simple differential systems $S_1, \ldots, S_r$ do contain an equation or inequation of order greater than ℓ, then a regularity decomposition in this order is not possible. In this case one needs to adjust the order ℓ. If the order is high enough, then one computes in a last step a regularity decomposition of each simple differential system S_i in order ℓ with Algorithm 5.3. A formal summary of this process is Algorithm 5.14 below.

Algorithm 5.14 (Regularity Decomposition of a General Differential System).

Input: a differential system S, defined over $K\{u\}$, and order $\ell \in \mathbb{N}$

Output: regularity decompositions in order ℓ of the irreducible components of the algebraic jet sets of the simple systems in a differential Thomas decomposition of S

Algorithm:

compute a differential Thomas decomposition $S_1, \ldots, S_r$ of the differential system S

if one of the systems S_i has an equation or an inequation of order greater than ℓ **then**

  error: order ℓ too small.

fi

return the regularity decompositions in order ℓ of the simple differential systems S_i determined by Algorithm 5.3.

6. Regular Differential Equations

A basic assumption in most of the geometric theory of differential equations is that one is dealing with a *regular* equation. This means that not only the given differential equation $\mathcal{J}_\ell \subset J_\ell\pi$ but also all its prolongations to higher order are smooth manifolds on which symbol and Vessiot spaces define vector bundles. For nonlinear systems, it is generally very hard to verify these infinitely many conditions and no effective method is known. We now provide a definition of regular differential equation adapted to our framework and prove that we can identify in the output of Algorithm 5.3 one unique regular equation for each irreducible component and that this equation lies dense in the irreducible component.

The key problem encountered here is that the definition of a regular differential equation requires to look at prolongations. So far we could avoid prolongations, as we assumed throughout that we start with a differential system S and then associate with it at any order ℓ an algebraic jet set defined via the differential ideal $\mathcal{I}_{\mathrm{diff}}(S)$. The problem of computing prolongations then corresponds to explicitly constructing the polynomial ideals $\mathcal{I}_\ell(S)$, a question which has been settled above. By contrast, we assume now that we start with an algebraic differential equation $\mathcal{J}_\ell \subset J_\ell\pi$ which is a regular algebraic jet set in the sense of Definition 5.1. The geometric theory describes an intrinsic prolongation process which, however, assumes that one is dealing with a fibred submanifold. In our framework, this assumption is not necessarily satisfied and thus we must develop another approach.

As a locally Zariski closed subset of $J_\ell\pi$, we may consider $\mathcal{J}_\ell$ as the solution set of an algebraic system S in the jet variables up to order ℓ. Identifying the jet variables with the derivatives of the dependent variables, we can also interpret S as a differential system which we associate with $\mathcal{J}_\ell$. Forming the differential ideal $\mathcal{I}_{\text{diff}}(S)$ corresponds to adding all differential consequences of the equations describing $\mathcal{J}_\ell$. Obviously, this construction is independent of the choice of the algebraic system S.

It may happen that $1 \in \mathcal{I}_{\text{diff}}(S)$. In this case, the system S is differentially inconsistent and any further differential analysis is pointless. Otherwise, we consider for any $k \geq 0$ the algebraic jet sets $\mathcal{J}_{\ell+k}(S)$. It may happen that $\mathcal{J}_\ell(S) \subsetneq \mathcal{J}_\ell$, namely if some of the differential consequences are of an order less than or equal to ℓ (i. e. if hidden integrability conditions exist in S). In this case, it is again pointless to analyse $\mathcal{J}_\ell$: one should study $\mathcal{J}_\ell(S)$ instead. Otherwise, we call the algebraic jet set $\mathcal{J}_{\ell+k} = \mathcal{J}_{\ell+k}(S)$ the *k-th prolongation* of $\mathcal{J}_\ell$.

Definition 6.1. The algebraic differential equation $\mathcal{J}_\ell \subset J_\ell\pi$ is called *regular*, if the differential system S associated with it satisfies

(i) $\mathcal{I}_{\text{diff}}(S)$ is a prime differential ideal,
(ii) $\mathcal{J}_\ell(S) = \mathcal{J}_\ell$ and
(iii) for all $k \geq 0$ the algebraic jet sets $\mathcal{J}_{\ell+k}(S)$ are regular and algebraic differential equations.

Given an algebraic differential equation $\mathcal{J}_\ell \subset J_\ell\pi$, it is not obvious how one can effectively verify that it is regular, since the above definition comprises infinitely many conditions, as in the geometric theory. We now show that Algorithm 5.3 solves this problem to some extent, as one can always identify regular differential equations in its output.

Proposition 6.2. *For each prime component $\mathcal{I}_{\ell,k}(S)$ arising in Algorithm 5.3, there exists among the simple systems $S_{k,i}$ in the output a unique distinguished system $S^{\text{gen}}_{\ell,k}$ such that* $\mathrm{Sol}^{\mathrm{a}}(S^{\text{gen}}_{\ell,k})$ *is Zariski dense in $\mathcal{J}_{\ell,k}$.*

Proof. System (14) comprises the equations $p_{k,1} = 0, \dots, p_{k,s_k} = 0$ defining the irreducible variety $\mathcal{J}_{\ell,k}$ and linear equations in the auxiliary indeterminates $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. Hence, the variety defined by (14) is trivially fibred over $\mathcal{J}_{\ell,k}$ and therefore irreducible. By [17, Cor. 2.2.66], any Thomas decomposition for an irreducible variety contains a unique simple system whose solution set is dense in that variety. Therefore, there exists a unique index i such that $\mathrm{Sol}^{\mathrm{a}}(S^{\text{ex}}_{k,i})$ is a dense subset of the variety defined by (14). Since $S^{\text{ex}}_{k,i}$ contains no inequations with leader among the $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and the equations involving $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are homogeneous of degree one, the projected system $S_{k,i}$ has the claimed property. □

Theorem 6.3. *In the notation of Proposition 6.2, the set* $\mathrm{Sol}^{\mathrm{a}}(S^{\text{gen}}_{\ell,k})$ *is a regular differential equation.*

Proof. Assume for notational simplicity that already $\mathcal{I}_{\text{diff}}(S)$ is a prime differential ideal so that we can drop the index k. In this case, the ideal $\mathcal{I}_\ell(S)$ is generated by the triangular set $B_{\leq \ell}$ defined in (6) followed by a saturation with respect to the inequations in S (cf. Equations (2) and (3)). Since our ordering of the variables $\mathbf{c}$ and $\mathbf{b}$, respectively, is linked to an orderly ranking of the derivatives $\mathbf{u}$, the two linear subsystems of (14) arise now immediately in a row echelon form[9] and its pivots are separants of the equations in $B_{\leq \ell}$. Furthermore, in the generic system

[9] Recall from Remark 2.4 that the saturation only eliminates unwanted points. Hence, at the remaining points we can use for the construction of the tangent space the equations in the triangular set $B_{\leq \ell}$ instead of some ideal generators obtained after the saturation.

S_ℓ^{gen} of the algebraic Thomas decomposition the separants and initials of the equations in $B_{\leq\ell}$ are implied being non-zero. It is now trivial to see that $\mathrm{Sol}^{\mathrm{a}}(S_{\ell,k}^{\text{gen}})$ is a regular algebraic jet set.

We now show that the same holds for $\mathrm{Sol}^{\mathrm{a}}(S_{\ell+1}^{\text{gen}})$, the generic branch obtained by applying our algorithm at the next order. By induction, we obtain that the generic branch defines at any prolongation order a regular algebraic jet set and thus our claim. By the same arguments as above, the ideal $\mathcal{I}_{\ell+1}(S)$ is generated by the triangular set $B_{\leq\ell+1}$ followed by a saturation. Since we assume that S contains no equations or inequations of an order greater than ℓ, $B_{\leq\ell+1}$ is obtained by augmenting $B_{\leq\ell}$ by certain formal derivatives (defined in Appendix C) of its elements of order ℓ. By the properties of the formal derivative, the new elements are linear in their leaders and their initials (and thus also their separants) are the separants of the elements of $B_{\leq\ell}$ from which they are derived. This implies that no new separants or initials arise during the prolongation. Again, these separants and initials are implied to be non-zero by the algebraic Thomas decomposition. Since again the linear subsystems of (14) arise immediately in triangular form with separants as pivots, the made observation about the separants entails trivially that $\mathrm{Sol}^{\mathrm{a}}(S_{\ell+1}^{\text{gen}})$ is a regular algebraic jet set, too.

For the general case, we exploit again that, by Lemma 5.5, we may assume that S is a regular differential system. [24, Thm. 4.13] asserts that any characteristic set C describing a prime component of $\mathcal{I}_{\text{diff}}(S)$ has the same leaders as the differential system S. In Algorithm 5.3, we first compute in Step 2 some basis for each prime component $\mathcal{I}_{\ell,k}(S)$ and then in Step 4 perform an algebraic Thomas decomposition. The generic branch of this decomposition determines a characteristic set $C_{\ell,k}$ describing $\mathcal{I}_{\ell,k}(S)$, namely the equations in $S_{\ell,k}^{\text{gen}}$. Furthermore, among the inequations in $S_{\ell,k}^{\text{gen}}$ we must find the initials and separants of C. As in the proof of Proposition 5.6, [24, Thm. 4.13] allows us to interpret C also as a differential characteristic set. By definition of a simple differential system, S is passive for the Janet division. This implies that C must also be passive for the Janet division. Indeed, otherwise C would induce integrability conditions and any characteristic set of the ideal induced by C would require additional leaders which contradicts [24, Thm. 4.13]. But now we can apply to C exactly the same reasoning as in the special case above and conclude that $\mathrm{Sol}^{\mathrm{a}}(S_{\ell,k}^{\text{gen}})$ is a regular differential equation. □

Corollary 6.4. *For any index k, the set* $\mathrm{Sol}^{\mathrm{a}}(S_{\ell,k}^{\text{gen}})$ *consists entirely of regular points of the algebraic differential equation* $\mathcal{J}_{\ell,k}$.

Proof. By the considerations in the proof of Theorem 6.3, the equations in $S_{\ell,k}^{\text{gen}}$ are passive for the Janet division. Since $S_{\ell,k}^{\text{gen}}$ arises from an algebraic Thomas decomposition, it is a simple algebraic system. No leader of an inequation is the derivative of the leader of an equation, as all (suitable, cf. Equation (6)) derivatives of the differential equations have been added as algebraic equations. Hence, $S_{\ell,k}^{\text{gen}}$ is also simple as a differential system. It follows now from Theorem 4.7 that the regular points form a Zariski dense subset of $\mathcal{J}_{\ell,k}$. Since $\mathrm{Sol}^{\mathrm{a}}(S_{\ell,k}^{\text{gen}})$ is also Zariski dense in $\mathcal{J}_{\ell,k}$ by Proposition 6.2, it contains regular points. By Proposition 5.10, this means that at all of its points the Vessiot and symbol spaces have the right dimensions. Furthermore, we have seen above that at the points in $\mathrm{Sol}^{\mathrm{a}}(S_{\ell,k}^{\text{gen}})$ no initial or separant vanishes. Hence, we can conclude as in the proof of Theorem 4.7 that $\mathrm{Sol}^{\mathrm{a}}(S_{\ell,k}^{\text{gen}})$ is actually an involutive differential equation and thus that around each point the required involutive complement to the symbol spaces exists. □

7. Examples

Example 7.1. We continue Example 2.10. There it was already mentioned that a differential Thomas decomposition of the differential system defined by the partial differential equations $p_1 = 0$ and $p_2 = 0$ with p_1 and p_2 given by (7) yields only one simple differential system comprising, besides the two given equations, the inequation $\operatorname{sep}(p_1) = u \neq 0$. Now we want to apply Algorithm 5.3 for the determination of the geometric singularities of this simple differential system in order $\ell = 1$, or, more precisely, a regularity decomposition of $\mathcal{J}_1(S)$. All different types of singularities introduced in Definition 4.1 appear in this example.

The first step of Algorithm 5.3 requires the saturation already discussed in Example 2.10, which leads to the addition of a third generator p_3 given by (8). The algebraic ideal $\mathcal{I}_1(S)$ generated by these three generators is prime, so that nothing is to be done in the second step. It was already mentioned above that p_1 is a linear combination of p_2 and p_3 and thus can in principle be omitted. As the equations $p_2 = 0$ and $p_3 = 0$ can be solved for u and y, respectively, the variety $\mathcal{J}_1(S)$ is a graph and thus no algebraic singularities occur here. Therefore, we ignore in the sequel the equations $\mathbf{J}(p_k) = 0$. In general, such a redundancy is not easy to recognise and we therefore do not exploit it any more in the following computations. The linear part of the system (14) defining the Vessiot spaces takes here the form

$$\begin{pmatrix} u & 0 & u_x(u_x - y) & -2y - u + u_y(u_x - y) \\ 0 & y & -u_x & 0 \\ u_y & u_x & -u_x & -1 - u_y \end{pmatrix} \cdot \begin{pmatrix} b_{10} \\ b_{01} \\ a^1 \\ a^2 \end{pmatrix} = 0\,. \tag{17}$$

The nonlinear part is given by $p_1 = p_2 = p_3 = 0$. The algebraic Thomas decomposition of this system performed in Step 4 yields after the projection in the last step the following four systems

$$\begin{aligned} S_1 &:= \{p_1 = 0, p_2 = 0, u \neq 0, y \neq 0\}\,, \\ S_2 &:= \{u_x = 0, u_y \neq 0, u = 0, y = 0\}\,, \\ S_3 &:= \{u_x \neq 0, u_y = 0, u = 0, y = 0\}\,, \\ S_4 &:= \{u_x = 0, u_y = 0, u = 0, y = 0\}\,. \end{aligned}$$

We now show that the corresponding algebraic jet sets $\mathcal{J}_1(S_i)$ are all regular and thus define a regularity decomposition of our system in order 1. Obviously, $\mathcal{J}_1(S_1)$ is a Zariski open subset of a three-dimensional variety in $J_1\pi$. $\mathcal{J}_1(S_2)$ and $\mathcal{J}_1(S_3)$ are disjoint Zariski open subsets of two-dimensional varieties lying in the Zariski closure of $\mathcal{J}_1(S_1)$. Finally, $\mathcal{J}_1(S_4)$ is a curve lying in the intersection of the Zariski closures of all the other systems. Of the four jet sets, only $\mathcal{J}_1(S_1)$ is an algebraic differential equation, as for the other three systems the projections $\pi^1(\mathcal{J}_1(S_i))$ violate the closure condition of Definition 2.1 because of the equation $y = 0$.

We finally discuss the Vessiot spaces for the points on these algebraic jet sets so that we can classify them according to the taxonomy of Definition 4.1. Linear systems for them are part of the extended systems S_i^{ex} obtained in Step 4, namely those (homogeneous linear) equations that depend on $\mathbf{a}$ and $\mathbf{b}$. For better readability, we describe them in terms of their coefficient matrices. Essentially, these matrices arise by using the equations in the corresponding projected system S_i to simplify the entries of the matrix in (17). The Thomas decomposition uses pseudo-divisions in the simplification process to obtain pivots which are provable non-vanishing on the whole considered component which sometimes makes some entries not immediately obvious.

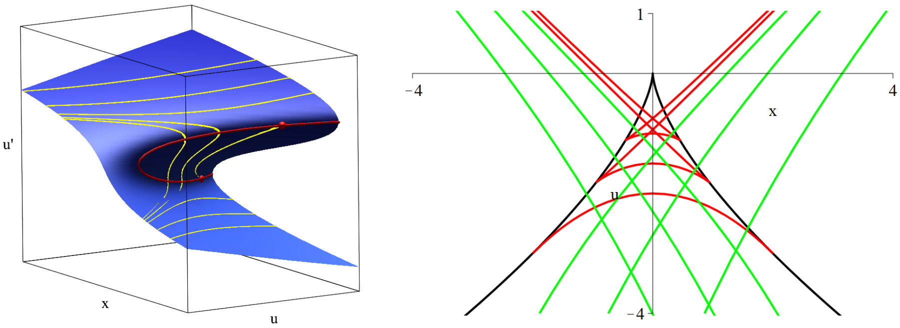

Figure 2: Hyperbolic gather. Left: Surface with singularities in jet space. Right: solution graphs—note how the red curves “go backwards” after meeting the black curve, a generic behaviour at regular singularities.

For points on the first algebraic jet set $\mathcal{J}_1(S_1)$, we get the matrix

$$\begin{pmatrix} u^3 & 0 & y^3(u+y) & -u^2(y+u) \\ 0 & u & -u-y & 0 \end{pmatrix}.$$

The two corresponding equations express b_{10} and b_{01} in terms of the unconstrained variables a^1 and a^2. Thus all Vessiot spaces are two-dimensional and all symbol spaces vanish, so that the Vessiot spaces are transversal. Hence, all the points on the algebraic jet set $\mathcal{J}_1(S_1)$ are regular points of the differential equation $\mathcal{J}_1(S)$. By Theorem 6.3, $\mathcal{J}_1(S_1)$ is a regular differential equation, as obviously S_1 is the generic branch in the algebraic Thomas decomposition. Thus our findings are consistent with Corollary 6.4.

An analogous comparison of the dimensions of Vessiot and symbol spaces, respectively, determines the singular character of the points on $\mathcal{J}_1(S_2)$, $\mathcal{J}_1(S_3)$ and $\mathcal{J}_1(S_4)$. The respective Vessiot spaces are the kernels of the three matrices:

$$\begin{pmatrix} u_y & 0 & 0 & -1-u_y \end{pmatrix}, \qquad \begin{pmatrix} 0 & u_x & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

All points on $\mathcal{J}_1(S_2)$ are purely irregular singular, as their Vessiot spaces are three-dimensional, but still contain a two-dimensional transversal part. At points on $\mathcal{J}_1(S_3)$, the dimension of the Vessiot spaces is still two; however, the transversal part is only one-dimensional. Hence, they are regular singular. Finally, the Vessiot spaces at points on $\mathcal{J}_1(S_4)$ are three-dimensional with only one-dimensional transversal complements to the symbol spaces. Thus, these points are irregular singular. These considerations also prove that all sets $\mathcal{J}_1(S_i)$ are regular algebraic jet sets and hence the four sets together define a regularity decomposition of $\mathcal{J}_1(S)$ in order 1.

Example 7.2. The hyperbolic gather is a classical example from catastrophe theory and is defined by the differential polynomial $p := (u')^3 + uu' - x$. A real picture of the corresponding algebraic differential equation is given by the blue surface on the left in Figure 2 with its fold line shown in red. All singularities lie on this fold line. On the right, Figure 2 shows some (real) solution graphs and one can see how solutions reach a forward or backward impasse when they hit the projection of the fold line shown in black.

The hyperbolic gather represents probably one of the simplest examples to demonstrate the artifacts that the algebraic Thomas decomposition may introduce in the output of Algorithm 5.3

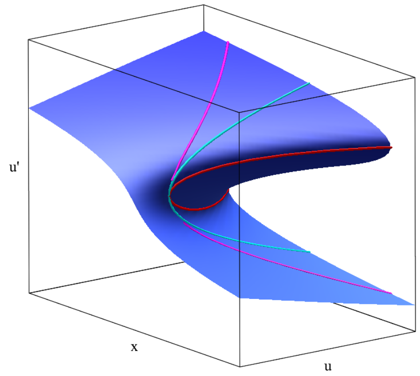

Figure 3: Hyperbolic gather with redundant case distinctions.

(compare Remark 5.4). Using the implementation presented in [13], one obtains a regularity decomposition consisting of seven components (all composed of smooth points). One of them consists of the two irregular singularities shown as distinguished points on the fold line in Figure 2; three other components describe the remainder of the fold line (one of them singles out the "tip" of the fold line, one contains only complex points not visible in the real picture).

The remaining three components contain the regular points. The corresponding extended algebraic systems are given by

$$\begin{aligned}
S_1^{\mathrm{ex}} &:= \{p = 0,\ 4u^3 + 27x^2 \neq 0,\ x \neq 0,\ (3(u')^2 + u)\,b + ((u')^2 - 1)\,a = 0\}\,,\\
S_2^{\mathrm{ex}} &:= \{(u')^3 + uu' = 0,\ x = 0,\ u \neq 0,\ (3(u')^2 + u)\,b + ((u')^2 - 1)\,a = 0\}\,,\\
S_3^{\mathrm{ex}} &:= \{uu' + 3x = 0,\ 4u^3 + 27x^2 = 0,\ x \neq 0,\ 81x^2\,b + (36x^2 - 4u^2)\,a = 0\}\,,
\end{aligned}$$

where we omitted the equations corresponding to the Jacobian criterion. Here S_1 is obviously the unique distinguished system of Proposition 6.2 defining a regular differential equation. However, if one takes the respective equations for the Vessiot space into account, then one sees[10] that the distinction between the three systems has no meaning for our analysis of singularities.

The appearance of these unnecessary case distinctions can be easily explained from the geometry of the corresponding algebraic jet set $\mathcal{J}_1$ shown once more in Figure 3 over the real numbers. Again, the red curve shows all geometric singularities of $\mathcal{J}_1$. The set $\mathrm{Sol}^{\mathrm{a}}(S_3)$ consists of those points of $\mathcal{J}_1$ which lie either above or below the fold line and is shown in magenta. This set must be singled out by any algebraic Thomas decomposition for the ordering $u' > u > x$, as at its projection to the x-u plane the fibre cardinality changes (this statement remains true over the complex numbers, as the hyperbolic gather simply depicts the solutions of the reduced cubic equation in u' with coefficients u and $-x$). Finally, the set $\mathrm{Sol}^{\mathrm{a}}(S_2)$ (shown in cyan) contains those points where the discriminant of the discriminant of p, i.e. x, vanishes. This set has a geometric relevance only at its intersection with the fold line, as it singles out the point where the fold line itself folds (respectively where the underlying cubic equation has a triple zero). Because of the inner working of the algorithm used to compute an algebraic Thomas decomposition, this condition leads to a separate case.

[10] In the case of S_3, this requires that one takes the coefficients as they appear in S_1 and S_2 and rewrites them modulo the equations in S_3.

Example 7.3. We consider now a situation where one is dealing with a reducible variety, so that the second step of Algorithm 5.3 becomes non-trivial. Its treatment demonstrates why we prefer to consider only irreducible varieties. The starting point is the differential system consisting of only one equation in factored form,

$$p := (u' - c)((u')^2 + u^2 + x^2 - 1) = 0\,, \tag{18}$$

where $c \in [-1, 1]$ is a real constant, and no inequation. A differential Thomas decomposition yields a single simple differential system S which contains besides the equation $p = 0$ only the inequation $\operatorname{sep}(p) \neq 0$.

Algorithm 5.3 (for $\ell = 1$) computes in the first step the algebraic ideal $\mathcal{I}_1(S)$, which is here simply generated by p, as the saturation has no effect. Its prime decomposition yields two prime ideals generated by the two factors of p: $p_1 = u' - c$ and $p_2 = (u')^2 + u^2 + x^2 - 1$. Considered over the reals, we are dealing here with a sphere and a horizontal plane intersecting it. Obviously, both irreducible varieties are without algebraic singularities, so that we ignore in the sequel the equations $\mathbf{J}(p_k) = 0$. It is trivial to see that a regularity decomposition of $\mathcal{J}_1(p_1)$ yields only one regularity component, namely $\mathcal{J}_1(p_1)$ itself, and all points on it are regular. In particular, $\mathcal{J}_1(p_1)$ is trivially a regular differential equation.

The second irreducible component was already analysed in [2, Ex. 9.1.12]. The linear equation for the Vessiot spaces is $2u'b + (2uu' + 2x)a = 0$. For the ranking $b > a > u' > u > x$, the implementation presented in [13] determines an algebraic Thomas decomposition consisting of five simple algebraic systems (since no algebraic singularities exist on this component, we ignore again the part stemming from the Jacobian criterion):

$$S_1^{\mathrm{ex}}: \left\{\begin{array}{rcl} u'\,b + (uu' + x)\,a &=& 0,\\ (u')^2 + u^2 + x^2 - 1 &=& 0,\\ u^2 + x^2 - 1 &\neq& 0,\\ x^2 - 1 &\neq& 0 \end{array}\right. \qquad S_2^{\mathrm{ex}}: \left\{\begin{array}{rcl} u'\,b + (uu' + x)\,a &=& 0,\\ (u')^2 + u^2 &=& 0,\\ u &\neq& 0,\\ x^2 - 1 &=& 0 \end{array}\right.$$

$$S_3^{\mathrm{ex}}: \left\{\begin{array}{rcl} a &=& 0,\\ u' &=& 0,\\ u^2 + x^2 - 1 &=& 0,\\ x^3 - x &\neq& 0 \end{array}\right. \quad S_4^{\mathrm{ex}}: \left\{\begin{array}{rcl} a &=& 0,\\ u' &=& 0,\\ u &=& 0,\\ x^2 - 1 &=& 0 \end{array}\right. \quad S_5^{\mathrm{ex}}: \left\{\begin{array}{rcl} u' &=& 0,\\ u^2 - 1 &=& 0,\\ x &=& 0 \end{array}\right.$$

The reduced systems corresponding to the first two systems S_1^{ex} and S_2^{ex} can be combined into one simple algebraic system leading to the following subset of the differential equation $\mathcal{J}_1(p_2)$:

$$\mathcal{R}_1 = \mathrm{Sol}^{\mathrm{a}}(S_1) \cup \mathrm{Sol}^{\mathrm{a}}(S_2) = \mathrm{Sol}^{\mathrm{a}}\Big(\{p_2 = 0,\ u' \neq 0\}\Big)\,. \tag{19}$$

Such a combination is also possible for the third and the fourth system and yields another subset of $\mathcal{J}_1(p_2)$ disjoint of $\mathcal{R}_1$:

$$\mathcal{R}_2 = \mathrm{Sol}^{\mathrm{a}}(S_3) \cup \mathrm{Sol}^{\mathrm{a}}(S_4) = \mathrm{Sol}^{\mathrm{a}}\Big(\{p_2 = 0,\ u' = 0, u^2 - 1 \neq 0,\ x \neq 0\}\Big)\,. \tag{20}$$

We have thus constructed a regularity decomposition of $\mathcal{J}_1(p_2)$ with three regularity components: $\mathcal{R}_1$ and $\mathcal{R}_2$ as defined above and $\mathcal{R}_3 = \mathrm{Sol}^{\mathrm{a}}(S_5)$.

We now classify the points on these three components according to the taxonomy of Definition 4.1. By Proposition 5.10, we have $\dim \mathcal{V}_\rho[\mathcal{J}_1(p_2)] = 1$ for all points $\rho \in \mathcal{R}_1$. Moreover, for these points we have $u' \neq 0$. Since u' is the initial of the equation with leader b, the assumption $a = 0$ implies $b = 0$ and hence the symbol space $\mathcal{N}_\rho[\mathcal{J}_1(p_2)]$ is trivial. We conclude that all points in $\mathcal{R}_1$ are regular. It follows again from Proposition 5.10 that $\dim \mathcal{V}_\rho[\mathcal{J}_1(p_2)] = 1$ also for all points $\rho \in \mathcal{R}_2$. Since the condition $a = 0$ belongs to the equations describing $\mathcal{R}_2$, all these Vessiot spaces are vertical, i. e. $\mathcal{V}_\rho[\mathcal{J}_1(p_2)] = \mathcal{N}_\rho[\mathcal{J}_1(p_2)]$ everywhere on $\mathcal{R}_2$. Thus all these points are regular singular. As the system S_5 defining $\mathcal{R}_3$ contains no equations depending on a or b, everywhere on $\mathcal{R}_3$ the Vessiot spaces are two-dimensional and hence all points there are irregular singular.

In this example, it is not difficult to verify that $\mathcal{R}_1$ is a regular differential equation, although Theorem 6.3 guarantees this only for the dense subset $\mathrm{Sol}^{\mathrm{a}}(S_1)$. The inequation $x^2 - 1 \neq 0$ is irrelevant for the initials and separants of S_1 and the systems S_1^{ex} and S_2^{ex} contain exactly the same equation for the coefficients a and b in our ansatz for the Vessiot space.

We can now compare the results for $\mathcal{J}_1(p_1)$ and $\mathcal{J}_1(p_2)$ for the points on their intersection, i. e. at points which are algebraic singularities of the original reducible variety $\mathcal{J}_1(S)$. If $c \neq 0$, then the points on $\mathcal{J}_1(p_1) \cap \mathcal{J}_1(p_2)$ have been classified as regular for both irreducible components. However, for $c = 0$ the points on the intersection are still regular with respect to $\mathcal{J}_1(p_1)$, but regular singular with respect to $\mathcal{J}_1(p_2)$. This exemplifies again the statement made in the beginning of Example 4.9 that the taxonomy of Definition 4.1 is relative and strongly depends on the considered algebraic jet set.

A natural question in such a situation is whether generalised solutions exist which lie on both components. Let us assume for simplicity that $c \neq 0$. Then on each component there exists a unique generalised solution going through ρ. Over the complex numbers, the identity theorem for holomorphic functions excludes the possibility to combine pieces of these to new solutions. Over the real numbers, solutions of lower regularity are admitted even if we restrict to classical solutions. In our case, we can construct additional solutions through ρ by approaching ρ on one of these two solutions and by then "switching" to the other one. As the resulting curve in $J_1\pi$ is still continuous, it corresponds to the prolongation of a function which is at least C^1 at the value x where the switching occurs.

As the direction of the tangent of a generalised solution encodes the value of the second derivative, a necessary and sufficient condition for the thus constructed solution to be even C^2 at x is that at the intersection point the Vessiot spaces with respect to the two irreducible components are identical. In our case, all Vessiot spaces $\mathcal{V}_\rho[\mathcal{J}_1(p_1)]$ are spanned by the vector $\partial_x + c\partial_u$, whereas a basis of the Vessiot space $\mathcal{V}_\rho[\mathcal{J}_1(p_2)]$ at any point $\rho = (\bar{x}, \bar{u}, \bar{p}) \notin \mathcal{R}_3$ is given by the vector $\bar{p}(\partial_x + \bar{p}\partial_u) - (\bar{x} + \bar{u}\bar{p})\partial_p$. If we assume that we are on the intersection, i. e. that $\bar{p} = c$ and $\bar{u}^2 + \bar{x}^2 = 1 - c^2$, then it is easy to see that the Vessiot spaces can be identical only for $c \neq 0$ and then this happens only at the two points

$$\rho_\pm = \left(\mp c\sqrt{\frac{1-c^2}{1+c^2}}, \pm\sqrt{\frac{1-c^2}{1+c^2}}, c\right).$$

By analysing the next prolongation of our equation, it is not difficult to show that the "switching" solutions are exactly C^2, as the value of the second derivative jumps at the switching point.

Thus we can conclude that over the real numbers we have through each point on the intersection four solutions: two analytic functions with prolongations staying completely on one component and two C^1 functions switching between components. For the points $\rho_\pm$ the latter two solutions are even C^2; a higher regularity is not possible for "switching" solutions. Figure 4 provides a graphical presentation of the situation over the reals for the choice $c = -\frac{3}{4}$. The red

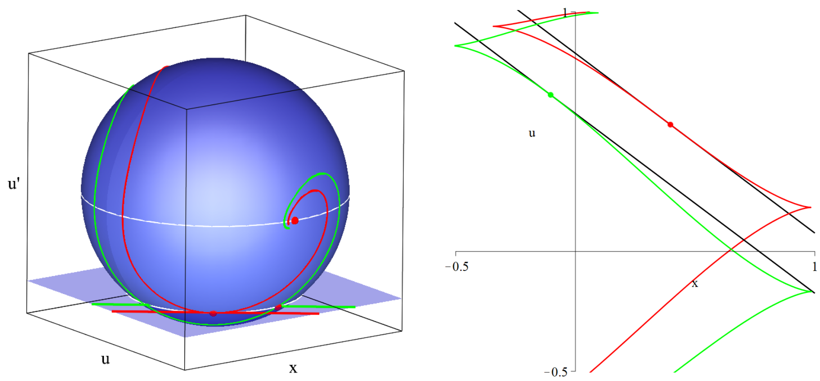

Figure 4: First-order differential equation with two irreducible components. Left: generalised solutions in $J_1\pi$. Right: solution graphs in x-u plane.

curves intersect at ρ_+; the green curves at some point different from $\rho_\pm$. Geometrically, $\rho_\pm$ are distinguished by the fact that the value of u' at these points represents a local extremum along the generalised solution of $\mathcal{J}_1(p_2)$ going through it. This in turn means that the graph of the corresponding classical solution has an inflection point there. This can be seen in the right part of the picture where the black lines correspond to the solutions of $u' = c$ and the red and green curves to solutions of $(u')^2 + u^2 + x^2 = 1$. Obviously, the black lines are tangent to the coloured curves at the marked intersection points. But the red curve crosses the black line, whereas the green curve stays on one side.

Example 7.4. To conclude this section, we study equations with "intrinsic" algebraic singularities, i. e. singularities that are not solely due to the intersection of irreducible components. Some classical examples can already be found in the work of Ritt. He studied for instance the equation $(u')^2 - 4u^3 = 0$ [35, II.§19]. Here, all points $(x, 0, 0)$ are algebraic singularities, whereas all other points on the corresponding algebraic differential equation $\mathcal{J}_1$ are regular. As the differential Thomas decomposition applied in the first step of Algorithm 5.14 shows, a singular integral, namely the solution $u(x) = 0$, exists here besides the generic component. Obviously, our algebraic singularities just form the graph of the first prolongation of this solution. When we apply Algorithm 5.3 to the generic component, then it uses the inequations in the entered differential system only for the saturation; otherwise they are ignored. Hence the analysed algebraic differential equation $\mathcal{J}_1$ is the full variety corresponding to the given equation. In particular, $\mathcal{J}_1$ contains all the algebraic singularities, but Algorithm 5.3 recovers them and puts them again into a separate regularity component. The singular integral represents here a kind of limit towards which all the other solutions tend asymptotically.

As a second example, we consider the cone in the first-order jet bundle, i. e. we study the scalar differential equation $\mathcal{J}_1$ given by

$$(u')^2 - u^2 - x^2 = 0$$

which obviously possesses an isolated algebraic singularity at the origin. The regularity decomposition of $\mathcal{J}_1$ determined with our algorithm yields two components: one consisting solely of this algebraic singularity and one containing all other points which are regular.

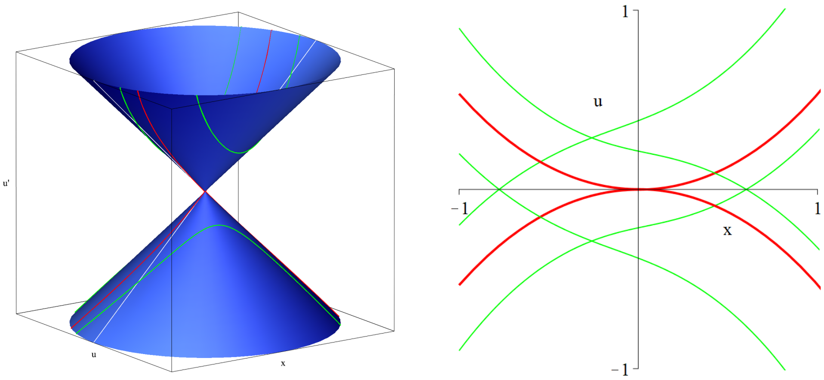

Figure 5: Generalised solutions going through an algebraic singularity of a real first-order differential equation. Left: situation in $J_1\pi$. Right: projection to x-u plane.

It is of obvious interest to study the local solution behaviour around this algebraic singularity and again we find a much wider range of possibilities over the real numbers. In our case, a real analysis can be performed with a simple ad hoc approach. Around any regular point $(\bar{x}, \bar{u}, \bar{u}') \in \mathcal{J}_1$, the Vessiot distribution is generated by the vector field $X = u'\partial_x + (x^2 + u^2)\partial_u + (x - uu')\partial_{u'}$. Note that X vanishes when one approaches the origin. By restricting to either the lower or the upper half cone, we can express u' by x and u and project to the x-u plane obtaining the vector field $Y = \pm\sqrt{x^2 + u^2}\partial_x + (x^2 + u^2)\partial_u$. It can trivially be continued to the origin where it vanishes. However, it is not differentiable at this point. Therefore, its behaviour at this stationary point cannot be decided using the Jacobian matrix. Transforming to polar coordinates (i. e. performing a blow up) shows that there is a unique invariant manifold going through the algebraic singularity which corresponds to the graph of a (prolonged) solution. We obtain one such solution from each half cone (see the red curves in Figure 5). As the graphs of both solutions possess a horizontal tangent at the origin, it is possible to "switch" at the singularity from one to the other. Hence, we find that our equation possesses exactly four C^1 solutions for the initial condition $u(0) = 0$ and $u'(0) = 0$. By analysing the prolongations of our equation, it is not difficult to verify that the solutions that stay inside of one half cone are even smooth, whereas the "switching" solutions are only C^1, as their second derivative jumps from 1 to -1 or vice versa at $x = 0$. Figure 5 also shows in white the Vessiot cone at the algebraic singularity which consists of two intersecting lines. One sees that they are indeed the tangents to the prolonged solutions through the singularity.

8. Conclusions

We developed a framework for the detection of all singularities of an arbitrary differential system with polynomial nonlinearities at a fixed order. It is based on the notion of an algebraic jet set (Definition 2.1) and covers both ordinary and partial differential equations. Our framework merges concepts from differential topology with tools from differential algebra and algebraic geometry. In particular for partial differential equations, it provides the first general and rigorous definition of singularities. While we could not prove that the taxonomy of Definition 4.1 is complete for systems which are not of finite type, our first main result, Theorem 4.7, shows that the definition is meaningful in the basic sense that regular points represent the generic case.

We augmented the classical theory of singularities of differential equations by the novel notion of a regularity decomposition (Definition 5.2), which is based on the concept of a regular algebraic jet set and in particular allows for a rigorous handling of situations where singularities are not isolated. A regularity decomposition essentially decomposes an algebraic jet set into subsets on which all relevant geometric structures show a uniform behaviour. Our second main result, Theorem 5.13, provides an algorithmic proof for the existence of regularity decompositions for arbitrary simple differential systems.

Finally, we solved a long standing problem in the geometric theory of differential equations: the construction of effectively provably regular equations. Most results in the geometric theory assume that one is dealing with a regular differential equation. However, to the best of our knowledge, nobody has so far provided an effective criterion for checking whether or not a given differential equation is regular. The basic problem is that such a criterion must take into account the infinitely many prolongations of the considered differential equation. Our third main result, Theorem 6.3, shows that the regularity decomposition determined by our algorithm contains in each prime component of the given system a unique regularity component which defines a regular differential equation.

Our approach is based on both the differential and the algebraic Thomas decomposition and is therefore fully algorithmic. An algebraic Thomas decomposition is crucial for the detection of all singularities. However, as discussed in Example 7.2, such a decomposition yields in general more than we really need, as it also takes into account the geometry of the embedding of the given algebraic differential equation into the ambient jet bundle. From a theoretical point of view, these unnecessary case distinctions are ugly but harmless. From a computational point of view, they considerably increase the computational costs and thus it would be useful to find a way to avoid them. Based on the existing implementation of these decompositions in Maple [13] and the built-in Maple procedure for prime decomposition, it is straightforward to implement our Algorithms 5.3 and 5.14 for constructing a regularity decomposition in Maple. Indeed, one of the authors (MLH) provided such an implementation and all examples in this work have been computed with it.

Our results lead immediately to a number of new questions. The most obvious one concerns the *local solution behaviour* around singularities, in particular the existence of solutions connecting two or more regularity components. Its investigation requires first an analysis of the "neighbourhood relationships" of the found components, i. e. does a certain component lie in the Zariski closure of another component? Such information can be straightforwardly obtained by classical Gröbner bases techniques (cf. e. g. [26]). A deeper study of the local solution behaviour requires additional methods which are beyond the scope of this work. Furthermore, such a study can most probably not be done at the same level of generality as this work; one has to specialise to more specific classes of systems.

For geometric singularities of ordinary differential equations considered *over the real numbers* much is already known from the works in the context of differential topology cited in the Introduction. Typical questions here are existence, (non)uniqueness and regularity of one- and two-sided solutions. At regular singularities the situation is fairly simple: they are generically either the initial or the terminal point of two classical solutions (thus generically only one-sided solutions exist at such points). A precise formulation covering also non-generic situations can e. g. be found in [29, Thm. 4.1].

For the analysis of irregular singularities, one can employ dynamical systems theory, as usually the Vessiot distribution is generated outside of an irregular singularity by a vector field which vanishes at the singularity. Generalised solutions through the singularity can then be constructed as one-dimensional invariant manifolds and typically several (possibly even infinitely many) solutions intersect at such a singularity. In low-dimensional situation, it is useful to be able to actually see the singularities and solutions through and around them. In [36], a MATLAB toolbox for producing corresponding 2D and 3D plots is presented. A detailed analysis of a specific class of scalar quasilinear[11] second-order ordinary differential equations along these lines can be found in [39]. In particular, it is shown there how regularity questions can be answered geometrically by studying prolongations.

For linear ordinary differential equations, the analysis of singularities *over the complex numbers* has a long tradition going back at least to the classical works of Fuchs and Frobenius which is nowadays often considered as part of differential Galois theory (cf. [40] and references therein). Note that in this context the terminology regular and irregular singularity is often used with a different meaning than in this work. In a complex setting, the regularity of solutions is of course no issue. Instead one studies questions like the monodromy of multivalued solutions or the Stokes phenomenon (cf. e. g. [41] or [42]) which are both from a theoretical and an algorithmic point of view still far from being solved.

We mentioned already in Remark 4.4 that for partial differential equations the taxonomy of Definition 4.1 might be incomplete. The deeper problem behind this question is to define precisely what in this case the *regular behaviour* should be. For equations of finite type, the prolonged solutions lead to a foliation of the differential equation around any regular point, as in this case the vanishing of the symbol space implies that the Vessiot distribution itself is the unique complement to the symbol space and its integral manifolds form the leaves of a (unique) foliation by the Frobenius theorem. If the differential equation is not of finite type, infinitely many possible complements exist and each of them leads to a different foliation by its integral manifolds. Here it is still unclear whether our definition is already sufficient to avoid any possible kind of singular behaviour. For regular differential equations, the different complements can be constructed by solving a combined algebraic-differential system which arises out of the structure equations of the Vessiot distribution (see the discussion in [32]). It has not yet been studied how this construction is affected by singularities and whether further kinds of singularities may be hidden in the structure equations.

The study of solutions around *algebraic singularities* has not found much attention yet. Within differential topology, they simply do not occur, as it is always assumed that one is dealing with a manifold. Recently, Falkensteiner and Sendra [43] used the classical theory of algebraic curves to study formal power series solutions of autonomous algebraic ordinary differential equations of first order by relating them to places. However, an extension of their approach to higher dimensional situations appears to be highly nontrivial. Our analysis in the non-autonomous Example 7.4 corresponding to an algebraic surface was performed in a rather ad hoc manner, but the principal idea should be extendable to more complicated situations, as the definition of a simple algebraic system means that each equation in the system is solvable for its leader. Thus one can at least in principle obtain an explicit expression for a vector field generating the Vessiot distribution (for ordinary differential equations), as we used it in the example.

In the case of a reducible algebraic differential equation, additional algebraic singularities appear at the intersection of irreducible components. As over the complex numbers only meromorphic solutions are relevant, the local solution behaviour can be determined on each compo-

[11] It should be noted that quasilinear differential equations possess a special geometry, as here the Vessiot distribution is projectable [37], leading to phenomena not arising in the fully nonlinear equations usually studied in differential topology. Using classical analytical techniques, such equations have been analysed in some detail, e. g. in [38].

nent separately. Over the real numbers, Example 7.3 demonstrates that solutions exist which "switch" from one component to another one, although generally such solutions will only be of finite regularity at the singularity. In this example, the points on the intersection were regular for each component, so that on each component only one solution through them exists leading to a total of four solutions through the singularity. It is trivial to construct examples where an intersection point is an irregular singularity for both involved components. If we assume furthermore that in both cases this singularity is a folded node, then we find already on each component infinitely many solutions going through the singularity. Now each solution on the first component can be combined with any solution on the second component leading to "∞^2" many "switching" solutions.

In this article, it was always assumed that the given differential system is studied in a fixed, sufficiently high order ℓ. This obviously raises the questions how this order should be chosen and how the obtained results depend on the chosen order. More generally, one may ask how do singularities behave under *prolongations*, i. e. how are regularity decompositions at two different orders related. It is easy to see that no singularities can arise above a regular point and that at a regular singularity no power series solutions can exist, as the fibre above it is always empty. The fibre over an irregular singularity consists entirely of singularities, but it is not clear of which type. For algebraic singularities, Tuomela [44] provided an example where a prolongation leads to a resolution of the singularity (in the sense of algebraic geometry), but it is unclear when this happens. A power series solution can exist at an irregular singularity, if and only if at each order of prolongation the corresponding fibre contains again at least one irregular singularity. Thus we meet anew the problem of checking infinitely many conditions and a deeper study is beyond the scope of this article.

Such questions are related to classical decidability questions for power series solutions as e. g. studied by Denef and Lipshitz [45]. As is often the case, an undecidable question for power series solutions is decidable for many relevant special cases. One of the theorems in [45] asserts that one can algorithmically decide for an algebraic *ordinary* differential equation whether such a solution exists at a fixed point $x \in \mathcal{X}$. From our geometric point of view, this result is easy in many cases: whenever at least one regular point lies above x such a solution exists. It becomes highly non-trivial when *all* points above x are singular (semi-linear equations typically show this behaviour at certain points x), as it implies that the above mentioned infinitely many conditions can be reduced to finitely many. Denef and Lipshitz [45] also proved a number of undecidability results (including the existence of power series solutions for *partial* differential equations) by presenting concrete counterexamples where the construction of formal power series solutions leads to classical undecidable diophantine problems. All these counterexamples are linear differential equations with geometric singularities in our sense. Our work sheds additional light on the many cases of regular points above x where the existence of power series solutions is decidable.

From an analytical point of view, one is not interested in arbitrary solutions around some given point x, but one studies concrete initial value problems prescribing further differential variables (in particular, for initial data corresponding to a singularity like in [39]). The theorems presented in [45] do not cover this situation. However, in one of their proofs [45, Thm. 3.1], they admit such additional conditions (assuming only that one is outside of any singular integral) so that actually key results of them can be applied to initial value problems.

For studying both the local solution behaviour and prolongations, it appears very interesting to combine our approach with the construction of various forms of formal power series (including Laurent and Puiseux series) via Newton polygons or polyhedra [46, 47, 48], as in particular the leading term of such series solutions may hint at the "right" order. We note that in the recent manuscript [49] such techniques are applied for proving existence and uniqueness results for formal power series also at singularities (see also [50]).

The algorithms behind the algebraic and the differential Thomas decomposition require that the base field is algebraically closed. For this reason, we considered in this work exclusively differential equations over the complex numbers. From an application point of view, it is of great interest to have a similar theory as developed in this work for *real* differential equations. A first step in this direction can be found in [51] for ordinary differential equations. There, the algebraic Thomas decomposition is replaced by a parametric Gauss algorithm followed by a quantifier elimination. This process represents a suitable alternative for the effective detection of real singularities and as a by-product avoids to some extent the above mentioned problem that the algebraic Thomas decomposition leads to unnecessary case distinctions because of the geometry of the embedding of the differential equation. As demonstrated in [51], an analysis of Example 7.2 leads now to no redundant cases.

A. Algebraic Systems and the Algebraic Thomas Decomposition

We fix a total ordering (or *ranking*) on the variables of the polynomial ring $\mathcal{P} = \mathbb{C}[x^1, \dots, x^n]$ by setting $x^i < x^j$ for $i < j$. The greatest variable with respect to $<$ appearing in a non-constant polynomial $p \in \mathcal{P}$ is called the *leader* of p and denoted by $\operatorname{ld}(p)$; for $p \in \mathbb{C}$ we set $\operatorname{ld}(p) = 1$. We regard every polynomial $p \in \mathcal{P} \setminus \mathbb{C}$ as a univariate polynomial in the indeterminate $x^k := \operatorname{ld}(p)$. Then the coefficients of p as a polynomial in x^k are contained in $\mathbb{C}[x^i \mid 1 \leq i < k]$. The coefficient of the highest power of $\operatorname{ld}(p)$ in p is called the *initial* of p and denoted by $\operatorname{init}(p)$. Finally, we introduce the *separant* of p as $\operatorname{sep}(p) := \partial p / \partial x^k$.

An *algebraic system* S is a finite set of polynomial equations and inequations

$$S = \{\, p_1 = 0, \, \dots, \, p_s = 0, \, q_1 \neq 0, \, \dots, \, q_t \neq 0 \,\} \tag{A}$$

with polynomials $p_i, q_j \in \mathcal{P}$ and $s, t \in \mathbb{N}_0$. Its *solution set* is defined as

$$\operatorname{Sol}^{\mathrm{a}}(S) := \{\, a \in \mathbb{C}^n \mid p_i(a) = 0, \, q_j(a) \neq 0 \text{ for all } i, j \,\} \, .$$

Obviously, $\operatorname{Sol}^{\mathrm{a}}(S)$ is a locally Zariski closed set, namely the difference of the two varieties $\operatorname{Sol}^{\mathrm{a}}(\{p_1 = 0, \dots, p_s = 0\})$ and $\operatorname{Sol}^{\mathrm{a}}(\{q_1 \cdots q_t = 0\})$.

Definition A.1. An algebraic system S as in (A) is said to be *simple* (with respect to a given ranking $<$), if the following three conditions hold:

(i) All equations and inequations have pairwise different leaders, i. e. we have

$$\bigl|\{\, \operatorname{ld}(p_1), \dots, \operatorname{ld}(p_s), \operatorname{ld}(q_1), \dots, \operatorname{ld}(q_t) \,\} \setminus \{1\}\bigr| = s + t$$

(*triangularity*).

(ii) For every $r \in \{p_1, \dots, p_s, q_1, \dots, q_t\}$, the equation $\operatorname{init}(r) = 0$ has no solution in $\operatorname{Sol}^{\mathrm{a}}(S)$ (*non-vanishing initials*).

(iii) For every $r \in \{p_1, \dots, p_s, q_1, \dots, q_t\}$, the equation $\operatorname{sep}(r) = 0$ has no solution in $\operatorname{Sol}^{\mathrm{a}}(S)$ (*square-freeness*).

We associate with the simple algebraic system S the saturated ideal

$$\mathcal{I}_{\mathrm{alg}}(S) := \langle p_1, \dots, p_s \rangle : q^{\infty} \subset \mathcal{P} \qquad \text{where } q = \operatorname{init}(p_1) \cdots \operatorname{init}(p_s) \, . \tag{A.1}$$

According to [17, Prop. 2.2.7], it represents the vanishing ideal of the Zariski closure of $\mathrm{Sol}^{\mathrm{a}}(S)$, i. e. the ideal of all polynomials in $\mathcal{P}$ which vanish on $\mathrm{Sol}^{\mathrm{a}}(S)$. In particular, $\mathcal{I}_{\mathrm{alg}}(S)$ is always a radical ideal.

Simple systems are a special class of algebraic systems for which the solution set can be obtained iteratively by finding zeros of univariate polynomials. First observe that triangularity implies that the simple system S contains either at most one equation $p(x^1) = 0$ with leader x^1 or at most one inequation $q(x^1) \neq 0$ with leader x^1. The number of zeros of p (of q) in $\mathbb{C}$ is equal to the degree of p (of q, respectively) due to square-freeness. In the former case, any zero $a^1 \in \mathbb{C}$ of p can be chosen for the coordinate x^1 of a solution of S. In the latter case, all elements of $\mathbb{C}$ except the zeros of q can be chosen instead. If S does not contain any equation or inequation with leader x^1, then a^1 is arbitrary. We substitute a^1 for x^1 in the equation or inequation with leader x^2 in S leading to a univariate polynomial in x^2. The degree of this polynomial is independent of the choice of a_1 due to the non-vanishing initial. Again because of square-freeness, the number of zeros of this polynomial is equal to its degree. By iterating this process, we obtain a solution $(a^1, a^2, \ldots, a^n) \in \mathbb{C}^n$ of S and every solution of S can be obtained in this way. This process makes use of the fact that the projections from the solution set of S onto the subspace with coordinates $x^1, x^2, \ldots, x^k$ have uniform fibre cardinality [52].

Definition A.2. Let S be an algebraic system as in (A). A *Thomas decomposition* of it consists of finitely many simple algebraic systems $S_1, \ldots, S_k$ such that the solution set $\mathrm{Sol}^{\mathrm{a}}(S)$ is the disjoint union of $\mathrm{Sol}^{\mathrm{a}}(S_1), \ldots, \mathrm{Sol}^{\mathrm{a}}(S_k)$.

Thomas [11, 12] proved that any algebraic system admits a (non-unique) Thomas decomposition. Using subresultants and case distinctions, it can be algorithmically determined [13]. An implementation in Maple is described in [14].

B. Differential Systems and the Differential Thomas Decomposition

We proceed to the *differential polynomial ring* (see [35, 53, 54] for more information on the for us relevant parts of differential algebra). Let $K = \mathbb{C}(x^1, \ldots, x^n)$ be the field of rational functions on $\mathbb{C}^n$ and δ_i the derivation $\partial/\partial x^i$. Given a set of *differential indeterminates* $U = \{u^1, \ldots, u^m\}$, we define the ring of differential polynomials as the polynomial ring $K\{U\} := K[\, u^\alpha_\mu \mid 1 \leq \alpha \leq m,\, \mu \in \mathbb{N}_0^n \,]$ in the infinitely many variables u^α_μ. The derivations $\delta_i : K \to K$ extend to derivations $\delta_i : K\{U\} \to K\{U\}$ via $\delta_i(u^\alpha_\mu) := u^\alpha_{\mu+1_i}$, additivity, and the Leibniz rule. Here 1_i is the multi-index of length n whose entries are 0 except for the i-th entry which is 1. We define $\delta^\mu := \delta_1^{\mu_1} \ldots \delta_n^{\mu_n}$ and $|\mu| := \mu_1 + \ldots + \mu_n$, the length of any multi-index $\mu \in \mathbb{N}_0^n$. Given differential polynomials $p_1, \ldots, p_s \in K\{U\}$, we distinguish between the algebraic ideal $\langle p_1, \ldots, p_s \rangle$ consisting of all linear combinations of them and the differential ideal $\langle p_1, \ldots, p_s \rangle_\Delta$ containing in addition all differential consequences $\delta^\mu p$ of any element p of it.

We introduce the subring $\mathcal{D} \subset K\{U\}$ of those differential polynomials where also the coefficients are polynomials in the variables x^i. Moreover, for any $\ell \in \mathbb{N}_0$ we consider the subalgebra

$$\mathcal{D}_\ell = \mathbb{C}[\, x^i,\, u^\alpha_\mu \mid 1 \leq i \leq n,\ 1 \leq \alpha \leq m,\ |\mu| \leq \ell \,],$$

which is the coordinate ring of the affine space $\mathbb{A}^d_{\mathbb{C}}$ where $d = n + m\binom{n+\ell}{\ell}$. Later, we identify the jet bundle $J_\ell \pi$ of the geometric theory (see Appendix C) with the affine space $\mathbb{A}^d_{\mathbb{C}}$ and consider $\mathcal{D}_\ell$ as its coordinate ring. Consequently, we call the variables u^α_μ of the polynomial ring $K\{U\}$ *jet variables*.

A *ranking* on the differential polynomial ring $K\{U\}$ is a total ordering $<$ on the set of jet variables u^α_μ such that $u^\alpha < \delta_i u^\alpha$ for all i and α, and such that $u^\alpha_\mu < u^{\alpha'}_{\mu'}$ implies $\delta_i u^\alpha_\mu < \delta_i u^{\alpha'}_{\mu'}$ for all i, α, α', μ, μ'. A ranking $<$ is *orderly*, if $|\mu_1| < |\mu_2|$ implies $u^{\alpha_1}_{\mu_1} < u^{\alpha_2}_{\mu_2}$ for all α_1, α_2, μ_1, μ_2. A *Riquier ranking* satisfies the following property: if the relation $u^\alpha_\mu < u^\alpha_{\mu'}$ holds for one value of the index α, then it must hold for all values of α (the meaning of this condition is discussed in [2, p. 428]). The definitions of leader, initial and separant given above can be extended straightforwardly.

A *differential system* S is given by a finite set of differential polynomial equations and inequations

$$S = \{\, p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0 \,\} \tag{D}$$

with $p_i, q_j \in K\{U\}$ and $s, t \in \mathbb{N}_0$. Note that by clearing denominators we may (and will) always assume that actually $p_i, q_j \in \mathcal{D}$.

As always for differential equations, the issue arises what kind of functions are permitted as solutions. We use here mainly local holomorphic functions $f : \mathcal{U} \to \mathbb{C}$ defined on some metric open domain $\mathcal{U} \subseteq \mathbb{C}^n$. However, in our approach the actual nature of the considered functions is not so important and we could equally well work with formal power series or meromorphic functions. In the sequel, we simply assume that some set of functions admissible as solutions has been fixed and we denote by $\mathrm{Sol}^{\mathrm{d}}(S)$ the set of solutions in this set. We further assume that a differential Nullstellensatz holds for this set. This is needed to establish a one-to-one correspondence between the solution sets of differential systems and the radical differential ideals of the differential polynomial ring. For a system of differential equations in $K\{U\}$ with our choice of K, a differential Nullstellensatz holds for local holomorphic functions (see e. g. [53, 55]).

Theorem B.1 (Nullstellensatz for Holomorphic Functions)**.** *Let $p_1, \ldots, p_s \in K\{U\}$ be differential polynomials and $I = \langle p_1, \ldots, p_s \rangle_\Delta$ the differential ideal generated by them. Moreover, let $q \in K\{U\}$ be a differential polynomial which vanishes for all local holomorphic solutions of I. Then some power of q is an element of I.*

The concept of *passivity* introduced by Riquier [56] and Janet [57] represents a differential algebraic version of completeness or formal integrability. For lack of space, we cannot recall here all the required definitions, but refer to [15] for a modern presentation of the form in which it is used here. Riquier [56, Chapt. VII, §115] showed how one can formulate for a passive system an initial value problem (see [2, Sect. 9.3] for a modern formulation of this construction) admitting an existence and uniqueness theorem for holomorphic solutions.

Theorem B.2 (Riquier's Theorem)**.** *Let $<$ be an orderly Riquier ranking. Then for a system of holomorphic differential equations which is orthonomic and passive with respect to $<$ the corresponding initial value problem possesses for holomorphic initial data locally a unique holomorphic solution.*

The assumption of passivity allows for the algorithmic construction of formal power series solution for any ranking (see Remark B.5 below). In the case of an orderly Riquier ranking, one can then prove the convergence of this series obtaining the above theorem. Orthonomic means that each equation can be solved in a unique manner for its leader. Obviously, a general implicit differential equation does not satisfy this condition. For this reason, we need as in the algebraic case the notion of a simple system permitting us the use of Riquier's Theorem.

Definition B.3. A differential system S as in (D) is *simple* (with respect to a given ranking $<$), if the following conditions hold:

(i) S is simple as an algebraic system (in the finitely many jet variables u^α_μ which actually occur in S ordered according to $<$).[12]
(ii) $\{p_1, \ldots, p_s\}$ is a passive system (for the Janet division).
(iii) No leader of an inequation q_j is an (iterated) derivative of the leader of an equation p_k.

Definition B.4. A *Thomas decomposition* of a differential system S consists of finitely many simple differential systems $S_1, \ldots, S_k$ such that $\mathrm{Sol}^{\mathrm{d}}(S)$ is the disjoint union of the solution sets $\mathrm{Sol}^{\mathrm{d}}(S_1), \ldots, \mathrm{Sol}^{\mathrm{d}}(S_k)$.

Thomas [11, 12] proved also in the differential case the existence of such decompositions. Again, it is possible to construct them algorithmically by interweaving algebraic Thomas decompositions and the Janet-Riquier theory [13]. The resulting algorithm is implemented in MAPLE [14, 16].

Remark B.5. For a *simple* differential system S it is possible to construct systematically formal power series solutions. Let ℓ be the maximal order of an equation or an inequation in S and add to S all partial derivatives of order at most ℓ of the equations in S. Now we choose an expansion point $x_0 = (x_0^1, \ldots, x_0^n) \in \mathbb{C}^n$ such that all equations and inequations in S are defined at $x = x_0$ and no initial and no separant vanishes for $x = x_0$. Hence, a formal power series solution is of the form $u^\alpha = \sum_{\mu \in \mathbb{N}_0^n} c^\alpha_\mu \frac{(x-x_0)^\mu}{\mu!}$. We choose $c^\alpha_\mu \in \mathbb{C}$ for all derivatives u^α_μ up to order ℓ. These choices must be performed in such a manner that after substituting x by x_0 and all u^α_μ by the corresponding constants c^α_μ no initial or separant of an equation or inequation vanishes and all equations and inequations are satisfied. If u^α_μ is the leader of an equation, then only finitely many values are possible for c^α_μ. If it is the leader of the derivative of an equation, then there is no freedom in choosing c^α_μ, as any differentiated equation is linear in its leader. If u^α_μ is the leader of an inequation, then all but finitely many values are possible for c^α_μ. For all other jet variables u^α_μ up to order ℓ, the constants c^α_μ can be chosen completely freely.

The jet variables u^α_μ of an order greater than ℓ can be partitioned into two disjoint sets. For those which are not the derivative of the leader of an equation in S, the corresponding constant c^α_μ can be chosen arbitrarily. For all remaining ones the constants c^α_μ are uniquely determined by some derived equations, which are quasi-linear. The properties of a simple differential system (in particular, the passivity) ensure that now the formal power series $u^\alpha = \sum_{\mu \in \mathbb{N}_0^n} c^\alpha_\mu \frac{(x-x_0)^\mu}{\mu!}$ with $1 \leq \alpha \leq m$ define a solution of S around x_0.

Of course, this construction does not necessarily produce all power series solutions of a simple differential system, c. f. [21, Example 4.8,4.9] or [22, §2].

C. The Geometry of Differential Equations

Since the algebraic tools used in the algorithms developed in this work require an algebraically closed field, we concentrate on complex differential equations. Thus, in the following all manifolds[13] are complex and all variables are to be understood as complex-valued. Restricting to holomorphic sections, one can define jet bundles in the familiar way and there are no changes with respect to the real theory outlined in standard references like [1, 2, 58, 59].

[12]We consider here the independent variables x^i as part of the coefficient field. One should also note that if only the inclusion of these variables yielded an algebraically simple system, then S would be differentially inconsistent.

[13]For us, manifolds have the same local dimension at every point and thus look locally like an open subset of some $\mathbb{C}^d$ with a fixed d.

The basic geometric setting is a fibred manifold $\pi : \mathcal{E} \to \mathcal{X}$ (i. e. π is a surjective submersion). The coordinates on the base space $\mathcal{X}$ are the independent variables $x^1, \dots, x^n$; the fibre coordinates $u^1, \dots, u^m$ represent the dependent variables or unknown functions. The ℓth order *jet bundle* $J_\ell\pi$ consists of all Taylor polynomials of degree ℓ. Naturally induced coordinates on it are thus in addition all derivatives of the u^i up to order ℓ; we use for them the usual multi-index notation u^α_μ where $\mu \in \mathbb{N}_0^n$ is a multi-index of length n. In the sequel, these natural coordinates are called *jet variables*. For convenience, we identify $\mathcal{E} = J_0\pi$.

Functions are replaced in the geometric framework by (local) sections:[14] maps $\sigma : \mathcal{U} \subseteq \mathcal{X} \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{U}}$. Locally, any section can be written in the form $\sigma(x) = (x, s(x))$ with a local holomorphic function $s : \mathbb{C}^n \to \mathbb{C}^m$. Given a section $\sigma : \mathcal{X} \to \mathcal{E}$, *prolongation* to order ℓ yields the section of the jet bundle $j_\ell\sigma : \mathcal{X} \to J_\ell\pi$ which is locally defined by $j_\ell\sigma(x) = (x, s(x), s_x(x), \dots, s_{x\cdots x}(x))$, i. e. we simply add all partial derivatives of the function s up to order ℓ.

The jet bundles of different orders form a natural hierarchy of fibrations via the canonical projections $\pi^{\ell+k}_\ell : J_{\ell+k}\pi \to J_\ell\pi$ which “forget” the higher order Taylor coefficients. Of particular interest are the projections $\pi^\ell_{\ell-1}$ by just one order, as $J_\ell\pi$ is an affine bundle over $J_{\ell-1}\pi$ modelled on the vector bundle $S_\ell(T^*\mathcal{X}) \otimes V\pi$ [2, Prop. 2.2.6]. The *fundamental identification* provides an isomorphism between this vector bundle and the vertical bundle $V\pi^\ell_{\ell-1} = \ker T\pi^\ell_{\ell-1}$. In addition, every jet bundle is fibred over the base space by the canonical projection $\pi^\ell : J_\ell\pi \to \mathcal{X}$ mapping each Taylor polynomial to its expansion point. This last projection is very important in our context: whenever we speak without further details of a transversal or a vertical vector field, it refers to this fibration π^ℓ.

A crucial geometric structure on the jet bundle $J_\ell\pi$ is the *contact distribution* $\mathcal{C}_\ell \subset T(J_\ell\pi)$. In local jet coordinates, it is generated by the following vector fields:

$$\begin{aligned} C^{(\ell)}_i &= \partial_{x^i} + \sum_\alpha u^\alpha_i \partial_{u^\alpha} + \sum_{0<|\mu|<\ell} \sum_\alpha u^\alpha_{\mu+1_i} \partial_{u^\alpha_\mu} \qquad (1 \le i \le n) \,, \\ C^\mu_\alpha &= \partial_{u^\alpha_\mu} \qquad\qquad (|\mu| = \ell, 1 \le \alpha \le m) \,, \end{aligned} \tag{C.1}$$

where $\mu + 1_i$ denotes the multiindex obtained by raising the i-th entry of μ by one. The first n fields are transversal to the fibration π^ℓ and encode the chain rule, whereas the remaining fields span the vertical bundle $V\pi^\ell_{\ell-1}$. Intuitively, the contact distribution encodes the different roles played by the three different kinds of coordinates: independent variables, dependent variables, and derivatives. One way to express this intuition formally is given by the following well-known characterisation of prolongations.

Proposition C.1. *A section $\gamma : \mathcal{X} \to J_\ell\pi$ of the ℓth jet bundle is a prolongation, i. e. of the form $\gamma = j_\ell\sigma$ for a section $\sigma : \mathcal{X} \to \mathcal{E}$, if and only if $T(\operatorname{im}\gamma) \subseteq \mathcal{C}_\ell$.*

The following intrinsic definition of a differential equation as a submanifold does not distinguish between scalar equations and systems and it is independent of any concrete way to describe the submanifold. In particular, the definition does not assume that the submanifold is (a part of) the solution set of polynomial equations. While it is in principle a global definition, many results in the geometric theory are only of a local nature. The imposed condition automatically excludes the appearance of singularities as studied in this article.

[14]For notational simplicity, we almost always omit the domain of definition $\mathcal{U}$ and use a seemingly global notation. However, all statements in this work are of a local nature.

Definition C.2. A *differential equation* of order ℓ is a fibred submanifold $\mathcal{J}_\ell \subseteq J_\ell\pi$ such that the restriction of the canonical projection $\pi^\ell : J_\ell\pi \to \mathcal{X}$ to the set $\mathcal{J}_\ell$ is a surjective submersion.

The notion of a solution can be easily expressed in an intrinsic manner, too. Note that the above definition of a differential equation does not yet entail the existence of solutions, as it does not exclude hidden integrability conditions which may lead to an inconsistency.

Definition C.3. A *(classical or strong) solution* of the differential equation $\mathcal{J}_\ell \subseteq J_\ell\pi$ is a section $\sigma : \mathcal{X} \to \mathcal{E}$ such that its prolongation satisfies $\operatorname{im} j_\ell\sigma \subseteq \mathcal{J}_\ell$.

Let $\sigma : \mathcal{X} \to \mathcal{E}$ be a classical solution of the differential equation $\mathcal{J}_\ell \subseteq J_\ell\pi$. Then, by definition, $\operatorname{im} j_\ell\sigma \subseteq \mathcal{J}_\ell$ is a smooth submanifold. Hence, we find at any point $\rho \in \operatorname{im} j_\ell\sigma$ that $T_\rho(\operatorname{im} j_\ell\sigma) \subseteq T_\rho\mathcal{J}_\ell$. Furthermore, for any prolonged section $T_\rho(\operatorname{im} j_\ell\sigma) \subseteq C_\ell|_\rho$ by Proposition C.1. Thus the tangential part of the contact distribution restricted to $\mathcal{J}_\ell$ may be considered as the space of all infinitesimal solutions (or integral elements).

Definition C.4. The *Vessiot space*[15] $\mathcal{V}_\rho[\mathcal{J}_\ell]$ of the differential equation $\mathcal{J}_\ell \subseteq J_\ell\pi$ at a point $\rho \in \mathcal{J}_\ell$ is the set $\mathcal{V}_\rho[\mathcal{J}_\ell] = T_\rho\mathcal{J}_\ell \cap C_\ell|_\rho$. The family of all Vessiot spaces on $\mathcal{J}_\ell$ is briefly denoted by $\mathcal{V}[\mathcal{J}_\ell]$.

One should note that generally the family $\mathcal{V}[\mathcal{J}_\ell]$ does *not* define a smooth regular distribution, as the dimension of the Vessiot spaces may differ at different points on $\mathcal{J}_\ell$. It is a standard assumption in the geometric theory (related to the notion of a *regular* differential equation) that this should not happen.

The fibration $\pi^\ell_{\ell-1} : J_\ell\pi \to J_{\ell-1}\pi$ allows us to define at any point $\rho \in J_\ell\pi$ the vertical space $V_\rho\pi^\ell_{\ell-1} = \ker T_\rho\pi^\ell_{\ell-1}$. We call the vertical part of the Vessiot space at a point $\rho \in \mathcal{J}_\ell$ the *symbol space* $\mathcal{N}_\rho[\mathcal{J}_\ell] = \mathcal{V}_\rho[\mathcal{J}_\ell] \cap V_\rho\pi^\ell_{\ell-1}$. It is not difficult to show that the Vessiot space can be decomposed as a direct sum of linear subspaces, $\mathcal{V}_\rho[\mathcal{J}_\ell] = \mathcal{N}_\rho[\mathcal{J}_\ell] \oplus \mathcal{H}_\rho$, with some π^ℓ-transversal complement $\mathcal{H}_\rho$ which is not uniquely determined.

The relationship between solutions and the Vessiot distribution is recalled in the following well-known assertion (see e. g. [2, Prop. 9.5.7]). One may say that the basic idea of Vessiot's approach to differential equations consists of studying certain subdistributions of the Vessiot distribution—which can to a large extent be analysed by elementary linear algebra—instead of solutions themselves (in [32] these subdistributions are called *Vessiot connections*).

Proposition C.5. *Let the section $\sigma : \mathcal{X} \to \mathcal{E}$ be a solution of the differential equation $\mathcal{J}_\ell \subseteq J_\ell\pi$. Then $T(\operatorname{im} j_\ell\sigma)$ is an n-dimensional, π^ℓ-transversal, involutive, smooth subdistribution of $\mathcal{V}[\mathcal{J}_\ell]|_{\operatorname{im} j_\ell\sigma}$. Conversely, let $\mathcal{H} \subseteq \mathcal{V}[\mathcal{J}_\ell]$ be an n-dimensional, transversal, involutive, smooth subdistribution defined on some open subset of $\mathcal{J}_\ell$. Then any n-dimensional integral manifold of $\mathcal{H}$ (and such manifolds always exist by the Frobenius Theorem [2, Thm. C.3.3]) is locally of the form* $\operatorname{im} j_\ell\sigma$ *for a solution σ of $\mathcal{J}_\ell$.*

Given a smooth function $\Phi : J_\ell\pi \to \mathbb{C}$, its *formal derivative* with respect to the independent variable x^i yields a function $D_i\Phi : J_{\ell+1}\pi \to \mathbb{C}$ which can be conveniently defined via the above

[15] In particular in the Russian literature, the terminology *Cartan space* is more common. We follow here the argumentation of Fackerell [60] that Vessiot put a much stronger emphasis on the vector field side, whereas Cartan prefered to work with differential forms.

introduced contact fields (C.1):

$$D_i\Phi = C_i^{(\ell)}(\Phi) + \sum_{|\mu|=\ell}\sum_{\alpha=1}^{m} C_\alpha^\mu(\Phi)u_{\mu+1_i}^\alpha \tag{C.2}$$

where $\mu + 1_i$ denotes the multi-index obtained by raising the ith entry of μ by one.

Assume that Φ depends on some jet variables other than only the independent variables x^i and that $k \geq 0$ is the maximal order of these jet variables. Then $D_i\Phi$ depends on jet variables up to order $k + 1$ and is always linear in those of the maximal order (and thus quasi-linear). Let $\mathcal{P} = \mathbb{C}[\xi^1, \dots, \xi^n]$ be a polynomial ring in $n = \dim \mathcal{X}$ variables and $\rho \in J_\ell\pi$ an arbitrary point. We define the *principal part* of Φ at the point ρ as the polynomial vector

$$\mathrm{pp}_\rho\,\Phi = \sum_{|\mu|=k}\sum_{\alpha=1}^{m} \frac{\partial\Phi}{\partial u_\mu^\alpha}(\rho)\,\xi^\mu \mathbf{e}_\alpha \in \mathcal{P}^m \tag{C.3}$$

where $\mathbf{e}_\alpha$ denotes the standard basis vectors in the free module $\mathcal{P}^m$ over the polynomial ring $\mathcal{P} = \mathbb{C}[x^1, \dots, x^n]$ whose rank is the fibre dimension m of $\mathcal{E}$. Note that the entries of $\mathrm{pp}_\rho\,\Phi$ are homogeneous polynomials of degree k .

Locally, the differential equation $\mathcal{J}_\ell$ may be considered as the zero set of some functions $\Phi_i : J_\ell\pi \to \mathbb{C}$. We choose a point $\rho \in \mathcal{J}_\ell$ and let $\ell_i \leq \ell$ be the maximal order of jet variables effectively appearing in Φ_i and $\mathbf{f}_i = \mathrm{pp}_\rho\,\Phi_i \in \mathcal{P}^m$ its principal part at ρ. The *(reduced) principal symbol module* at the point ρ is now the $\mathcal{P}$-module $\mathcal{M}[\rho] = \langle \mathbf{f}_1, \dots, \mathbf{f}_s \rangle$ spanned by all the principal parts. The degree ℓ component of this module can be identified with the annihilator of the symbol space $\mathcal{N}_\rho[\mathcal{J}_\ell]$ (see [2, Rem. 7.1.18]).

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