An Involutive GVW Algorithm and the Computation of Pommaret Bases

Amir Hashemi, Thomas Izgin, Daniel Robertz and Werner M. Seiler

Abstract. The GVW algorithm computes simultaneously Gröbner bases of a given ideal and of the syzygy module of the given generating set. In this work, we discuss an extension of it to involutive bases. Pommaret bases play here a special role in several respects. We distinguish between a fully involutive GVW algorithm which determines involutive bases for both the given ideal and the syzygy module and a semi-involutive version which computes for the syzygy module only an ordinary Gröbner basis. A prototype implementation of the developed algorithms in MAPLE is described.

1. Introduction

Gröbner bases provide a powerful tool for a wide variety of problems in commutative algebra, algebraic geometry and many other areas of science and engineering. For example, it can be interpreted as a generalization of the Gaussian elimination to the polynomial case [13]. In 1965, Buchberger introduced the theory of Gröbner bases together with an algorithm to compute them [2]. Later, he presented two criteria [3] to improve his algorithm by detecting superfluous reductions a priori. Since then, many mathematicians like Lazard, Gebauer, Möller, Mora, Traverso, Faugère and Gao steadily worked on finding more such criteria or new methods to compute Gröbner bases more efficiently. In this direction, Lazard used techniques from linear algebra [13]. Gebauer and Möller [7] used syzygies to find superfluous reductions by applying Buchberger's criteria in an effective way. Moreover, Möller, Mora and Traverso [16] described the first signature-based algorithm to compute Gröbner bases. Faugère has found a signature-based algorithm, so-called F5, which is more efficient than the previous algorithms [4]. Since then, several papers have been published trying to simplify the F5 algorithm. The goal, of course, was also to develop an algorithm which is faster than F5 on benchmark systems. Indeed, Gao, Guan, Volny invented the so-called G2V algorithm that seems to be (two to ten times) faster than F5 on benchmark systems according to [5]. Based on G2V, the GVW algorithm was created by Gao, Volny and Wang which again seems to be faster than G2V [6]. It is worth noting that GVW not only computes a Gröbner basis of a given ideal but also one for the corresponding syzygy module. Thus, it may be also applicable for computing free resolutions. Moreover, there are various papers about adapting the GVW algorithm to different mathematical applications as well as to make it more efficient. For instance, in [14], the authors are interested in adapting the GVW algorithm to principal ideal domains. For efficiency, in [15], the authors use an approach from linear algebra to implement the GVW algorithm with the help of matrix operations. There, they address one major weakness of the GVW algorithm: all performed reductions must obey a certain restricting rule which leads to the fact that some elements that may be reduced according to other theories are not allowed to be reduced anymore. Thus, the algorithm becomes more inefficient as these elements will lead to more multiples of generators that need to be considered. However, the authors in [15] suggest a substitution method to create sparser matrices for signature-based algorithms by storing equivalent but sparser polynomials. They also demonstrate the efficiency of their algorithm. There are also so-called Hilbert-driven signature-based algorithms which use the Hilbert function to make the algorithm more efficient [19]. Moreover, there are approaches to deal with inhomogeneous ideals, too, introducing the concept of *mutant pairs* [20].

In this paper, we are interested in presenting a variant of the GVW algorithm for computing involutive bases. These bases are indeed Gröbner bases with additional combinatorial properties. They originate from the works of Janet on the algebraic analysis of partial differential equations [12]. Zharkov and Blinkov introduced the notion of involutive polynomial bases using related works of Pommaret [21]. Later, Gerdt and Blinkov introduced involutive divisions [9]. Of special interest are Pommaret bases as one can read off many invariants of the ideals they generate like dimension, depth and Castelnuovo-Mumford regularity [18]. These invariants remain unchanged after coordinate transformations, which is very important theoretically as well as from a computational point of view as Pommaret bases do not always exist [18, 10]. However, in [17] it is shown that a finite Pommaret basis of a *homogeneous ideal* for the *degree reverse lexicographical order* always exists after finitely many linear coordinate transformations of a certain type (see also [10] for an extensive discussion of this and related facts). Gerdt pointed out the special relationship between the Janet and Pommaret divisions in [8]. From further works on the relationship, we know that a Janet basis is also a Pommaret basis if the latter one exists (see [18, Thm. 4.3.15]). Thus, Seiler presented two approaches for computing a Pommaret basis of homogeneous ideals for the degree reverse lexicographical order: One can compute a Janet basis which always exists as the Janet division is Noetherian [18, Lem. 3.1.19]. If a Pommaret basis exists, we already have computed it. Otherwise, he suggested to perform a coordinate transformation and compute a Janet basis of the transformed system and iterate this procedure. The second approach is to compute a Pommaret basis in a direct way and check during the algorithm whether a finite Pommaret basis exists, i.e. if the ideal is in *quasi-stable position*. If it is not in quasi-stable position, one may interrupt the algorithm, perform a coordinate transformation and start over again [18, p. 130].

Binaei et al. described in [1] a *semi-involutive*[1] version of the GVW algorithm and proved the termination by relating it to Gerdt's algorithm [1, Thm. 6]. However, the proof is only given for Noetherian divisions, and thus, not for the Pommaret division. Also, their claim in [1, Thm. 5] itself has flaws which we will point out in this work.

Moreover, we will develop a semi-involutive version of the GVW algorithm, but also a *fully involutive variant*, where we will compute a (weak) Pommaret basis of the syzygy module. Thus, we will distinguish a "fully involutive" and a "semi-involutive" variant in the sequel. For both variants, we will give a proof of correctness for the Pommaret division and Janet division. In the case of the Pommaret division, we also give a proof of termination using coordinate transformations and a bound for the regularity of the ideal; and for the Janet division, we refer to results in [1]. Therefore, we can present two ways to compute a Pommaret basis of a homogeneous ideal: In both strategies, we start using the Janet version of the involutive GVW algorithm. From there we get an upper bound q for the regularity of the ideal [18, Cor 5.5.18]. Next, we check if the output is already a Pommaret basis. If not, we perform a coordinate transformation. As transformed syzygies are still syzygies, we can use them to make the algorithm in the next run more efficient because syzygies can be used for detecting superfluous reductions. Nevertheless, we can proceed in two different ways from there. First, we could iterate the Janet version. Secondly, we can use one of our Pommaret versions of the GVW algorithm (going only at most to the degree bound $q+2$). We will introduce criteria where the algorithm may stop earlier with an error message that the ideal (or, in the fully involutive case, its

[1] By this we mean that they aimed to compute an involutive basis of an ideal and a Gröbner basis of its syzygy module.

syzygy module) is not in quasi-stable position. Then, we perform a coordinate transformation and restart.

This paper is organized as follows: First of all, we recall the main ideas of the GVW algorithm in the next section. Then, we will develop an involutive version of the GVW algorithm, first discussing the more complex Pommaret case. There, we also introduce an *index of safety* which helps us finding a suitable coordinate transformation for the restart. In the subsequent section, we gain a Janet version and prove its correctness. Afterwards, we give some remarks on implementation and derive the index of safety. There, we also discuss the benefits and issues of the usage of a POT- or TOP-lift and present some statistics from the implementation of the algorithm in Maple 2019. In the last section, we summarize our obtained results and give an outlook for future works that may be based on the presented theory.

Lastly, we want to note that large parts of this work are based on the Master's thesis [11]. For lack of space, we will omit in the sequel the proofs of a few simple, but technical lemmas. These proofs and many more details can be found in the Master's thesis[2].

2. Preliminaries

In the context of the GVW algorithm, we are trying to find a Gröbner basis of an ideal $I \coloneqq \langle F\rangle$ in the polynomial ring $R \coloneqq K[x_1, \dots, x_n]$ over a field K with $\mathrm{char}(K) = 0$, and its syzygy module $Syz(F)$ where $F \coloneqq \{f_1, \dots, f_m\}$. The main idea is to operate on the set

$$M \coloneqq \{(\boldsymbol{u}, v) \in R^m \times R \mid \boldsymbol{u}^T \boldsymbol{f} = v\},$$

where $\boldsymbol{f}$ is the vector with entries f_i.

If $v = 0$, then $\boldsymbol{u} \in Syz(F)$. Also, if we know $\boldsymbol{u}$, then we can calculate v. Thus, if v is the result of reduction steps of another element in I, $\boldsymbol{u}$ encodes the "history" of these reduction steps.

A major idea of the GVW algorithm is to combine the knowledge about the u-part and v-part to speed up the computation of the two Gröbner bases mentioned above. For this, Gao et al. follow a signature-based strategy.

We shall emphasise here that the *signature* of a pair $(\boldsymbol{u}, v) \in M$ is the leading term of the u-part, i.e. $\mathrm{lt}_{\prec_2} \boldsymbol{u}$, where $\prec_2$ is a term order on $\mathbb{T}_n^m \coloneqq \{\boldsymbol{e}_i x^\mu \mid \mu \in \mathbb{N}_0^n, 1 \le i \le m\}$ and where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ are the standard basis vectors of R^m. We also set $\mathbb{T}_n \coloneqq \mathbb{T}_n^1$.

As we are working on M, we can choose a different term order for the v-part. But we are only interested in using $\prec_1 = \prec_{degrevlex}$ as this is, to our knowledge, the only interesting term order in the context of Pommaret basis computations. Furthermore, we will demand two properties. First, the two term orders should be *compatible* in some sense, i.e.

$$\mathrm{lt}_{\prec_2}(\boldsymbol{u} \cdot v) = \mathrm{lt}_{\prec_1}(v) \cdot \mathrm{lt}_{\prec_2}(\boldsymbol{u})$$

should apply for all vector terms $\boldsymbol{u}$ and terms v. The formal definition is the following.

Definition 2.1. Let $\prec_1$ be a term order on R and $\prec_2$ one on R^m. We say $\prec_2$ is *compatible* to $\prec_1$ if for arbitrary terms x^μ, x^ν in R the equivalence

$$x^\mu \prec_1 x^\nu \Leftrightarrow x^\mu \boldsymbol{e}_i \prec_2 x^\nu \boldsymbol{e}_i \;\; \forall 1 \le i \le m$$

holds.

In the context of the GVW algorithm this is the only restriction to our term orders. In our context, however, we also require the term order $\prec_2$ to be of *type* ω, i.e. between any two (vector) terms there are only finitely many terms.

[2] http://www.mathematik.uni-kassel.de/~izgin/publications.php?lang=en.

Also, for the sake of simplicity, we shall drop the indices of the term orders and indicate the different term orders by the usage of bold letters only for the u-part (i.e. for $\prec_2$). Furthermore, we write $\operatorname{lt}(v) = 0$, if $v = 0$ and $\operatorname{lt}(\boldsymbol{u}) = 0$, if $\boldsymbol{u} = \boldsymbol{0}$.

Now, having set the term orders, the next step is to introduce reduction steps. Here, we introduce two classes of reduction steps that will play a major role for this theory.

Definition 2.2. Let $p_1 = (\boldsymbol{u}_1, v_1), p_2 = (\boldsymbol{u}_2, v_2) \in R^m \times R$. We say p_1 is *reducible* by p_2 if

(i) $v_1 \neq 0 \neq v_2$ and $\operatorname{lt}(v_2) \mid \operatorname{lt}(v_1)$,
(ii) $\operatorname{lt}(t\boldsymbol{u}_2) \preceq \operatorname{lt}(\boldsymbol{u}_1)$ with $t = \frac{\operatorname{lt}(v_1)}{\operatorname{lt}(v_2)}$.

We set $c := \frac{\operatorname{lc}(v_1)}{\operatorname{lc}(v_2)}$. Then a *reduction step* of p_1 by p_2 is given by a reduction step in the v-part performed on M, i.e.

$$p_1 - ctp_2 = (\boldsymbol{u}_1 - ct\boldsymbol{u}_2, v_1 - ctv_2) = \left(\boldsymbol{u}_1 - \frac{\operatorname{lm}(v_1)}{\operatorname{lm}(v_2)}\boldsymbol{u}_2, v_1 - \frac{\operatorname{lm}(v_1)}{\operatorname{lm}(v_2)}v_2\right). \tag{2.1}$$

If the signature of p_1 does not change in a reduction step, the reduction is called *regular*, and *super* otherwise.

Also, we call p_1 regular/super reducible by $N \subseteq R^m \times R$, if p_1 is regular/super reducible by some $p \in N$. Furthermore, we denote by Sig(N) the set of all signatures of elements in N.

Lemma 2.3. *A reduction step of p_1 by p_2 defined in (2.1) is super if and only if*

$$\operatorname{lt}(t\boldsymbol{u}_2) = \operatorname{lt}(\boldsymbol{u}_1) \quad \textit{and} \quad \frac{\operatorname{lc}(v_1)}{\operatorname{lc}(v_2)} = \frac{\operatorname{lc}(\boldsymbol{u}_1)}{\operatorname{lc}(\boldsymbol{u}_2)}.$$

Definition 2.4. Let $p_1,\, p_2 \in R^m \times R$ with $v_2 = 0$ (so $\boldsymbol{u}_2$ is a syzygy). We say p_1 is *reducible by a syzygy* $p_2 = (\boldsymbol{u}_2, 0)$ if

$$\boldsymbol{u}_1 \neq \boldsymbol{0} \neq \boldsymbol{u}_2 \quad \text{and} \quad \operatorname{lt}(\boldsymbol{u}_2) \mid \operatorname{lt}(\boldsymbol{u}_1).$$

A *reduction step* of p_1 by p_2 is given by a reduction step of $\boldsymbol{u}_1$ by $\boldsymbol{u}_2$ performed on M, i.e.

$$p_1 - \frac{\operatorname{lm}(\boldsymbol{u}_1)}{\operatorname{lm}(\boldsymbol{u}_2)}p_2 = \left(\boldsymbol{u}_1 - \frac{\operatorname{lm}(\boldsymbol{u}_1)}{\operatorname{lm}(\boldsymbol{u}_2)}\boldsymbol{u}_2, v_1\right).$$

Such a reduction step always reduces the signature of p_1, and hence, a reduction by a syzygy is always called super.

Remark 2.5. We note that for any super reduction we have $\operatorname{lt}(\boldsymbol{u}_2) \mid \operatorname{lt}(\boldsymbol{u}_1)$. Moreover, it is worth mentioning that a syzygy, by definition, is only reducible by a syzygy.

Next, we want to “lift” the notion of a Gröbner basis to M.

Definition 2.6. A finite subset $G \subseteq M$ is called a *strong Gröbner basis* of M, if every non-zero pair in M is reducible by G.

Now we present an important proposition that justifies the notion of a strong Gröbner basis. We will skip the proof, but it can be found in [5, 6].

Proposition 2.7. *Let $G = \{(\boldsymbol{u}_1, v_1), \dots, (\boldsymbol{u}_k, v_k)\}$ be a strong Gröbner basis of M. Then*

(i) $G_0 := \{\boldsymbol{u}_i \mid v_i = 0,\ 1 \leq i \leq k\}$ is a Gröbner basis of $Syz(F)$.
(ii) $G_1 := \{v_i \mid 1 \leq i \leq k\}$ is a Gröbner basis of $I = \langle F\rangle$.

With that proposition we are interested in knowing if one can calculate a strong Gröbner basis efficiently. Indeed, Gao et al. presented an algorithm to compute a strong Gröbner basis as we are going to see. In particular, in the next section we aim to lift the theorems to involutive divisions.

Definition 2.8. Let $N \subseteq R^m \times R$ and $p = (\boldsymbol{u}_1, v_1) \in M$.

- p is said to be *eventually super reducible* by N, if a regular normal form[3] of p is super reducible. As p can be regular irreducible, we call p also eventually super reducible, if it is merely super reducible and not regular reducible at all.
- p is said to be *covered* by $q = (\boldsymbol{u}_2, v_2) \in N$ if $\operatorname{lt}(\boldsymbol{u}_2) \mid \operatorname{lt}(\boldsymbol{u}_1)$ and $\frac{\operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec \operatorname{lt}(v_1)$ hold. We also may just say that p is covered by N.

If p is covered by q, this means that $\frac{\operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}_2)} q$ has the same signature as p but a smaller v-part. Hence, we have found a way to reduce p indirectly to a pair with smaller v-part and we may not have to look at p anymore. But this is a claim worth proving. Indeed, the next theorem asserts that this is a good way to look at it. But before we present it, we introduce one last notion.

Definition 2.9. Let $p_1 = (\boldsymbol{u}_1, v_1),\ p_2 = (\boldsymbol{u}_2, v_2) \in R^m \times R$ and $v_1 \neq 0 \neq v_2$.
For $i, j \in \{1, 2\}$ we set $t_i := \operatorname{lt}(v_i)$ and $t_{ij} := \operatorname{lcm}(t_i, t_j)$. Furthermore, we define

$$T := \max_{\prec_2}\{\tfrac{t_{12}}{t_1} \operatorname{lt}(\boldsymbol{u}_1), \tfrac{t_{12}}{t_2} \operatorname{lt}(\boldsymbol{u}_2)\}.$$

Without loss of generality, let $T = \frac{t_{12}}{t_1} \operatorname{lt}(\boldsymbol{u}_1)$. Moreover, let $c = \frac{\operatorname{lc}(v_1)}{\operatorname{lc}(v_2)}$. If we have

$$\operatorname{lt}(\tfrac{t_{12}}{t_1} \boldsymbol{u}_1 - c \tfrac{t_{12}}{t_2} \boldsymbol{u}_2) = T \tag{2.2}$$

then we call $\frac{t_{12}}{t_1} p_1$ the *J-pair* of p_1 and p_2. Moreover, a reduction step is given by

$$\tfrac{t_{12}}{t_1} p_1 - c \tfrac{t_{12}}{t_2} p_2 = (\tfrac{t_{12}}{t_1} \boldsymbol{u}_1 - c \tfrac{t_{12}}{t_2} \boldsymbol{u}_2,\ \tfrac{1}{\operatorname{lc}(v_1)} S(v_1, v_2)), \tag{2.3}$$

and is regular by definition (see (2.2)). Here, $S(v_1, v_2)$ is the S-polynomial of v_1 and v_2.

So instead of calling the pair in (2.3) J-pair, Gao et al. suggest to go one step back in the reduction process and to call $\frac{t_{12}}{t_1} p$ a J-pair. Doing so, we have two things worth pointing out: First, we potentially do not have to look at all S-polynomials as some of them may not have come from a regular reduction step and hence, will not satisfy (2.2). Moreover, by definition we can use the property that a J-pair is at least once regular reducible. This will be important for the proof of the next theorem. However, we will provide a proof for the involutive J-criterion in the next section. Thus, we just refer to [6] for the proof in this section. Also, it might be interesting to mention at this point that our involutive J-pairs in general will not be involutively regular reducible at least once. Hence, we will have to give a proof for the involutive case where we cannot use that involutive J-pairs are involutively regular reducible by definition.

But let us first focus on the given case. From the next theorem (see [6]), we will be able to generate an algorithm for computing a strong Gröbner basis. We will give a pseudo code for the involutive case and skip it here.

Theorem 2.10 (J-criterion). *Let $G := \{(\boldsymbol{u}_1, v_1), \ldots, (\boldsymbol{u}_k, v_k)\} \subseteq M$ be a finite subset of M such that $\langle Sig(G)\rangle = \mathbb{T}_n^m$. Then the following statements are equivalent:*

(i) G is a strong Gröbner basis of M.
(ii) Every J-pair of elements in G is eventually super reducible by G.
(iii) Every J-pair of elements in G is covered by G.

This theorem is the foundation for the proof of correctness of the GVW algorithm. Now, we want to pick up some ideas of this theory to achieve similar results for the involutive case.

[3] A regular normal form is the result of only regular reduction steps until no regular reduction is possible anymore. A regular normal form does not have to be unique as we are, in general, not reducing regular with respect to a Gröbner basis in the v-part.

3. Involutive GVW algorithm

As we are using involutive divisions from now on, it is crucial to have some knowledge about them (in particular about the Pommaret and Janet division) and involutive bases. For such an overview we refer to [18].

Throughout the rest of this article, let $G \subseteq M$ be a finite set and $B_u \times B_v \subseteq \mathbb{N}_0^n \times \mathbb{N}_0^n$, where B_u is the set of exponent vectors of signatures in G and B_v the set of leading exponents of elements in the v-part of G.

Definition 3.1. We write $p_i := (\boldsymbol{u}_i, v_i)$ for $i = 1, 2$. Let in particular $p_1 \in M$ and $p_2 \in G$. Finally, let L be an involutive division.

a) p_1 is *involutively covered by* p_2 if

$$\mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1) \quad \text{and} \quad \frac{\mathrm{lt}(\boldsymbol{u}_1)}{\mathrm{lt}(\boldsymbol{u}_2)}\,\mathrm{lt}(v_2) \prec \mathrm{lt}(v_1).$$

We say that p_1 is *involutively covered by* $G \subseteq M$ if it is involutively covered by some element in G.

b1) p_1 is said to be *involutively regular reducible by* p_2 if the following conditions hold:
   (i) $v_1 \neq 0 \neq v_2$,
   (ii) $\mathrm{lt}(v_2) \mid_{L,B_v} \mathrm{lt}(v_1)$ and
   (iii) for $(\boldsymbol{u}, v) := p_1 - \frac{\mathrm{lm}(v_1)}{\mathrm{lm}(v_2)} p_2$ we have $\mathrm{lt}(\boldsymbol{u}) = \mathrm{lt}(\boldsymbol{u}_1)$.

Moreover, we say that p_1 is *involutively super reducible by* p_2 if conditions (i), (ii), and
(iii') $\mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1)$ and $\mathrm{lt}(\boldsymbol{u}) \prec \mathrm{lt}(\boldsymbol{u}_1)$
are satisfied.

b2) If p_2 is a syzygy, i.e. $v_2 = 0$, then p_1 is called *involutively super reducible by* p_2 if

$$\boldsymbol{u}_1 \neq \boldsymbol{0} \neq \boldsymbol{u}_2 \quad \text{and} \quad \mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1).$$

For an involutively super reduction by a syzygy we perform a reduction of the u-part, namely

$$p_1 - \frac{\mathrm{lm}(\boldsymbol{u}_1)}{\mathrm{lm}(\boldsymbol{u}_2)} p_2.$$

b3) p_1 is said to be *involutively reducible by* p_2 if it is reducible in the sense of b1) or b2). Moreover, p_1 is involutively reducible by G if it is involutively reducible by some element in G.

c) A pair $p \in M$ is called *eventually involutively super reducible* by $G \subseteq M$ if there is a chain – a length of zero is allowed – of involutively regular reduction steps by G leading to an *involutively regular normal form*[4] of p which in turn is involutively super reducible by G.

d) We write $p_2 \mid_{L,B} p_1$ if $\mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1)$ and $\mathrm{lt}(v_2) \mid_{L,B_v} \mathrm{lt}(v_1)$.

Remark 3.2. Note that in b1) (iii') the condition $\mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1)$ is essential. Therefore, in contrast to [6], the conditions b1) (i)–(ii) and

$$\frac{\mathrm{lm}(v_1)}{\mathrm{lm}(v_2)}\,\mathrm{lt}(\boldsymbol{u}_2) \preceq \mathrm{lt}(\boldsymbol{u}_1)$$

do *not* imply involutive reducibility. In particular, it can happen that from these conditions we encounter a super reduction which is not involutive, i.e. we have $p_2 \mid p_1$ but not $p_2 \mid_{L,B} p_1$.

Next, we translate the definition of a strong Gröbner basis to the involutive case.

Definition 3.3. A finite set $G \subseteq M$ is called a *strong L-basis*[5] of M, if any non-zero pair $(\boldsymbol{u}, v) \in M$ is involutively reducible by G with respect to the involutive division L.

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[4]This means that no more involutively regular reduction steps are possible.
[5]This notion of a strong L-basis is not related to strong or weak involutive bases. However, it can be shown, that from a strong L-basis, two weak involutive bases will arise (see [11, Prop. 4.3.3], whose proof can easily be adapted to a general involutive division L).

Gao et al. developed a computational approach to obtain a strong Gröbner basis using S-polynomials which are associated with a criterion for computing Gröbner bases [6]. For the involutive analogue we refer to [18, Def. 4.1.1, Prop. 4.1.4]. Hence, we get the following definition for involutive J-pairs.

Definition 3.4. Let $p := (\boldsymbol{u}, v) \in G$ and $v \neq 0$. Let $\bar{X}_{L,B_v}(v)$ be the set of non-multiplicative variables of $\mathrm{lt}(v)$. Then every element of the set

$$\{x_k p \mid x_k \in \bar{X}_{L,B_v}(v)\}$$

is called *involutive J-pair of p.* Furthermore, a finite product of multiplicative variables for $\mathrm{lt}(\boldsymbol{u})$ or $\mathrm{lt}(v)$ is called a *multiplicative term* for $\mathrm{lt}(\boldsymbol{u})$ or $\mathrm{lt}(v)$, respectively.

3.1. Involutive J-criterion (I) for the Pommaret Division

Next we list some rather technical, but simple lemmas whose proofs we skip here. We need them to show a first involutive variant of the covered-criterion. It will later be discussed however that this version will not be our basis for the implementation. Some of the following lemmas are just the involutive version of results in [6].

Lemma 3.5. *If a pair $(\boldsymbol{u}, v) \in M$ with $v \neq 0$ is involutively super reducible by a syzygy $(\boldsymbol{u}_1, 0) \in G$ then it is also involutively covered by it.*

Lemma 3.6. *Let $p := (\boldsymbol{u}, v) \in M$ be involutively regular reducible at least once by G. If a regular normal form $(\boldsymbol{u}', v')$ of p is an element of G, then p is involutively covered by $(\boldsymbol{u}', v')$.*

Lemma 3.7. *The relations "involutively covered by", "involutively reducible by a syzygy" and "involutively super reducible by a non-syzygy" are transitive on G.*

For the next lemma, we have to extend the spectrum of our notions a bit.

Definition 3.8. If for $(\boldsymbol{u}, v) \in M$ there is a $(\boldsymbol{u}', v') \in G$ with $\mathrm{lt}(\boldsymbol{u}') \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u})$, $\mathrm{lt}(v') \mid \mathrm{lt}(v)$, and $\frac{\mathrm{lt}(\boldsymbol{u})}{\mathrm{lt}(\boldsymbol{u}')} = \frac{\mathrm{lt}(v)}{\mathrm{lt}(v')}$, we call $(\boldsymbol{u}, v)$ *pseudo reducible* by $(\boldsymbol{u}', v')$.

The next lemma will mainly be used in later proofs for optimization of the final algorithms for the Pommaret division. Hence, we will only prove it for this division, if not noted otherwise. Now, as the Pommaret division is global, we are not bound by G.

Lemma 3.9. *Let $(\boldsymbol{u}, v) \in M$ with $v \neq 0$, $G \subseteq M$ finite and P be the Pommaret division.*

a) If $(\boldsymbol{u}, v)$ is eventually involutively super reducible by G where at least one involutive reduction is regular, then it is involutively covered by G. This remains true for arbitrary involutive divisions L.

b) If $(\boldsymbol{u}, v)$ is involutively covered by some pair $(\boldsymbol{u}', v') \in M$ which in turn is involutively super reducible by G, then $(\boldsymbol{u}, v)$ is involutively covered by G.

c) If $(\boldsymbol{u}, v)$ is pseudo reducible by $(\boldsymbol{u}', v') \in M$ and if $(\boldsymbol{u}', v')$ is involutively covered by G, then $(\boldsymbol{u}, v)$ is involutively covered by G.

Proof. ad a): We calculate an involutively regular normal form $p_1 := (\boldsymbol{u}_1, v_1)$ of $(\boldsymbol{u}, v)$. Note that $\mathrm{lt}(\boldsymbol{u}) = \mathrm{lt}(\boldsymbol{u}_1)$ and $\mathrm{lt}(v_1) \prec \mathrm{lt}(v)$ since we only performed regular reduction steps. According to our assumptions, p_1 is involutively super reducible by some $p_2 := (\boldsymbol{u}_2, v_2) \in G$. In the case of $v_2 = 0$, this implies

$$\mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1) = \mathrm{lt}(\boldsymbol{u}) \quad \text{and} \quad \frac{\mathrm{lt}(\boldsymbol{u}_1)}{\mathrm{lt}(\boldsymbol{u}_2)} \mathrm{lt}(v_2) = 0 \prec \mathrm{lt}(v).$$

Therefore p is involutively covered by p_2. For $v_2 \neq 0$ it follows that

$$\mathrm{lt}(\boldsymbol{u}_2) \mid_{L,B_u} \mathrm{lt}(\boldsymbol{u}_1) = \mathrm{lt}(\boldsymbol{u}) \quad \text{and} \quad \frac{\mathrm{lt}(\boldsymbol{u}_1)}{\mathrm{lt}(\boldsymbol{u}_2)} \mathrm{lt}(v_2) = \frac{\mathrm{lt}(v_1)}{\mathrm{lt}(v_2)} \mathrm{lt}(v_2) = \mathrm{lt}(v_1) \prec \mathrm{lt}(v).$$

Hence, p is always involutively covered by G.

ad b): If $(\boldsymbol{u}', v')$ is involutively super reducible by a syzygy $(\boldsymbol{u}_2, 0)$, then $(\boldsymbol{u}, v)$ is involutively covered by the same syzygy since $\operatorname{lt}(\boldsymbol{u}_2) \mid_P \operatorname{lt}(\boldsymbol{u}') \mid_P \operatorname{lt}(\boldsymbol{u})$. Hence, we assume as in part a) that $(\boldsymbol{u}_2, v_2)$ is not a syzyzgy. Then $\operatorname{lt}(\boldsymbol{u}_2) \mid_P \operatorname{lt}(\boldsymbol{u}')$ and $\frac{\operatorname{lt}(\boldsymbol{u}')}{\operatorname{lt}(\boldsymbol{u}_2)} = \frac{\operatorname{lt}(v')}{\operatorname{lt}(v_2)}$. Because $(\boldsymbol{u}, v)$ is involutively covered by $(\boldsymbol{u}', v')$ we have $\operatorname{lt}(\boldsymbol{u}') \mid_P \operatorname{lt}(\boldsymbol{u})$ and $\frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}')} \operatorname{lt}(v') \prec \operatorname{lt}(v)$. Hence, we have

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_P \operatorname{lt}(\boldsymbol{u}') \mid_P \operatorname{lt}(\boldsymbol{u}) \quad \text{and} \quad \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) = \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}')\frac{\operatorname{lt}(v_2)}{\operatorname{lt}(v')}} \operatorname{lt}(v_2) = \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}')} \operatorname{lt}(v') \prec \operatorname{lt}(v),$$

and thus, $(\boldsymbol{u}, v)$ is involutively covered by G.

ad c): As p' is involutively covered by G, there exists a pair $(\boldsymbol{u}_2, v_2) \in G$ such that

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_P \operatorname{lt}(\boldsymbol{u}') \quad \text{and} \quad \frac{\operatorname{lt}(\boldsymbol{u}')}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec \operatorname{lt}(v').$$

From $\frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}')} = \frac{\operatorname{lt}(v)}{\operatorname{lt}(v')}$ it follows that

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_P \operatorname{lt}(\boldsymbol{u}') \mid_P \operatorname{lt}(\boldsymbol{u}) \quad \text{and} \quad \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) = \frac{\operatorname{lt}(\boldsymbol{u}')\frac{\operatorname{lt}(v)}{\operatorname{lt}(v')}}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec \frac{\operatorname{lt}(v)}{\operatorname{lt}(v')} \operatorname{lt}(v') = \operatorname{lt}(v).$$

Therefore, we are done. □

The next lemma is very important for the Pommaret and Janet division. And of course, we aim to find a computational approach to compute strong L-bases. The following lemma is the first step towards this goal. Later, we will make a few more assumptions under which the statements indeed are equivalent. Nevertheless, without any further assumptions being made, we already obtain the following result.

Lemma 3.10. *Let L be an involutive division. Let $G \subseteq M$ be a finite set. Then the implications "$a) \Rightarrow b) \Rightarrow c)$" hold, where*

- *a) G is a strong L-basis of M.*
- *b) Every involutive J-pair of elements of G is eventually involutively super reducible by G.*
- *c) Every involutive J-pair of elements of G is involutively covered by G or involutively super reducible by G.*

Proof. We first prove "$a) \Rightarrow b)$". Suppose, G is a strong L-basis. Now let p be an involutive J-pair of an element of G. Since $p \in M$, we know that p is involutively reducible. If the reduction is super, we are done. Otherwise the reduction is regular, and we calculate an involutively regular normal form which lies again in M. Therefore, it is still involutively reducible and now it must be an involutively super reduction. Hence, b) is shown.

Now suppose b) is true. We write again $p := (\boldsymbol{u}, v) \in M$ for an arbitrary involutive J-pair. By definition of a J-pair we know $v \neq 0$.
Applying b), we can conclude that p is eventually involutively super reducible by G. If no regular reduction is possible, p is involutively super reducible and c) is true. However, if an involutively regular reduction is possible, we apply lemma 3.9 a) and we are done. □

We need one more rather technical lemma before we come to the actual result of this section.

Lemma 3.11. *Let $L = P$ be the Pommaret division. Let $G \subseteq M$ be a finite set. Suppose that every J-pair in G is involutively covered or involutively super reducible by G.*
Let $(\boldsymbol{u}, v) \in M$ be non-zero and suppose there is a pair $p_1 := (\boldsymbol{u}_1, v_1) \in G$ with $v_1 \neq 0$ such that

- *(i) $\operatorname{lt}(\boldsymbol{u}_1) \mid_P \operatorname{lt}(\boldsymbol{u})$ and*
- *(ii) $t \operatorname{lt}(v_1) := \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_1)} \operatorname{lt}(v_1)$ is minimal among all elements in G that satisfy condition (i).*

Then the following statements are true:

a) *tp_1 is not involutively covered by G.*
b) *If t contains a non-multiplicative variable for $\operatorname{lt}(v_1)$ then there exists a pair $p' := (\boldsymbol{u}', v') \in G$ such that $v' \neq 0$ and tp_1 is involutively super reducible by p'.*
c) *If G is involutively head autoreduced w.r.t. the v-part*[6] *then tp_1 is not involutively regular reducible by G.*

Proof. Since $(\boldsymbol{u}, v) \neq (\mathbf{0}, 0)$ we conclude $\operatorname{lt}(\boldsymbol{u}) \neq \mathbf{0}$. Now let us consider the first statement.

ad a): Assuming that a) is false, we will arrive at a contradiction as follows. We assume that tp_1 is involutively covered by a pair $(\boldsymbol{u}_2, v_2) \in G$. But this implies $\operatorname{lt}(\boldsymbol{u}_2) \mid_P t \operatorname{lt}(\boldsymbol{u}_1) = \operatorname{lt}(\boldsymbol{u})$ and $\frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec t \operatorname{lt}(v_1)$, violating condition (ii).

ad b): By our assumptions t is multiplicative for $\operatorname{lt}(\boldsymbol{u})$ and contains a non-multiplicative variable for $\operatorname{lt}(v_1)$. Let $l := \deg(t)$ and x_k be the variable with the largest index occurring in t. Since t contains a non-multiplicative variable, x_k must be non-multiplicative for $\operatorname{lt}(v_1)$, too. Note, that $x_k \mid_P t$[7]. Furthermore, $x_k p_1$ is an involutive J-pair . By our assumptions there are now two possibilities.

In the first case $x_k p_1$ is involutively covered by G. Then there exists $(\boldsymbol{u}_2, v_2) \in G$ such that

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_P x_k \operatorname{lt}(\boldsymbol{u}_1) \quad \text{and} \quad \frac{x_k \operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec x_k \operatorname{lt}(v_1).$$

But because $x_k \mid_P t$ is true and t is multiplicative for $\operatorname{lt}(\boldsymbol{u}_1)$, we obtain

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_P t \operatorname{lt}(\boldsymbol{u}_1) \quad \text{and} \quad \frac{t \operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec t \operatorname{lt}(v_1),$$

which contradicts a).

Thus, we now consider the second case and assume that $x_k p_1$ is involutively super reducible by a pair $p_3 := (\boldsymbol{u}_3, v_3) \in G$. If $v_3 = 0$, then $x_k p_1$ is also involutively covered by p_3 because v_1 is non-zero by our assumptions (see Lemma 3.5). But we have just shown in the first case that this leads to a contradiction. Therefore, we have $v_3 \neq 0$. Now, by definition we obtain the relations

$$\operatorname{lt}(v_3) \mid_P x_k \operatorname{lt}(v_1) \quad \text{and} \quad \operatorname{lt}(\boldsymbol{u}_3) \mid_P x_k \operatorname{lt}(\boldsymbol{u}_1) \mid_P t \operatorname{lt}(\boldsymbol{u}_1).$$

If in addition to this the relation $\operatorname{lt}(v_3) \mid_P t \operatorname{lt}(v_1)$ is true, we are done since $x_k p_1$ and tp_1 have the same leading coefficients (which would show that tp_1 is involutively super reducible by p_3). So let us suppose that this is not the case, i.e. there must be a non-multiplicative variable for $\operatorname{lt}(v_3)$ left in $\frac{t}{x_k}$. Then, we iterate our arguments, now taking the variable x_h appearing in $\operatorname{supp}(\frac{t}{x_k})$ with the largest index and looking at the J-pair $x_h(\boldsymbol{u}_3, v_3)$. Note that $x_h \mid_P \frac{t}{x_k}$, or equivalently $x_h x_k \mid_P t$ holds (remember that $h \leq k$).

Then, we end up again with a pair $p_4 := (\boldsymbol{u}_4, v_4) \in G$ with $v_4 \neq 0$, from which we know that it reduces $x_h p_3$ involutively super. In particular, we have

$$\operatorname{lt}(v_4) \mid_P x_h \operatorname{lt}(v_3) \mid_P x_h x_k \operatorname{lt}(v_1) \quad \text{and} \quad \operatorname{lt}(\boldsymbol{u}_4) \mid_P x_h \operatorname{lt}(\boldsymbol{u}_3) \mid_P x_h x_k \operatorname{lt}(\boldsymbol{u}_1) \mid_P t \operatorname{lt}(\boldsymbol{u}_1).$$

Repeating this procedure, we finish after at most $l = \deg(t)$ steps, obtaining a pair which satisfies all properties that we have claimed in b).

ad c): We prove this by contradiction. Suppose, that tp_1 is involutively regular reducible by a pair $p_2 := (\boldsymbol{u}_2, v_2) \in G$. Hence, $v_2 \neq 0$. Now,we are facing three cases.

Firstly, $t = 1$. This leads to $\operatorname{lt}(v_2) \mid_P \operatorname{lt}(v_1)$, and hence, to a contradiction because G is involutively head autoreduced w.r.t. the v-part.

[6] This means, that the set of v-parts of G does not contain two elements v_1, v_2 such that $\operatorname{lt}(v_1) \mid_{L,B_v} \operatorname{lt}(v_2)$.

[7] In this work, every variable with an index smaller or equal to the *class of a term* t', written $\operatorname{cls}(t') := \min\{r : x_r \mid t'\}$, is Pommaret multiplicative for t'.

Also, if $t \neq 1$ is a multiplicative term for $\operatorname{lt}(v_1)$ we have trivially $\operatorname{lt}(v_1) \mid_P t \operatorname{lt}(v_1)$ and still $\operatorname{lt}(v_2) \mid_P t \operatorname{lt}(v_1)$. Therefore, we must have $\operatorname{lt}(v_2) \mid_P \operatorname{lt}(v_1)$ or $\operatorname{lt}(v_1) \mid_P \operatorname{lt}(v_2)$ violating again our assumption in c).

Hence, only one case is possible: $t \neq 1$ contains a non-multiplicative variable for $\operatorname{lt}(v_1)$. But in this case, we can apply part b) and obtain a pair p' as described in b). Then, tp_1 cannot be involutively regular reducible by p'. So, we have $p' \neq p_2$. However, we have $\operatorname{lt}(v') \mid_P t \operatorname{lt}(v_1)$ and $\operatorname{lt}(v_2) \mid_P t \operatorname{lt}(v_1)$. This again implies $\operatorname{lt}(v_2) \mid_P \operatorname{lt}(v')$ or vice versa, both violating the condition that G is involutively head autoreduced w.r.t. the v-part. □

Note, that in Lemma 3.11 we only used the condition "G is involutively head autoreduced w.r.t. the v-part" for the proof of part c). This will be very important for our following work. Because later, it will turn out that we have to drop this condition as we cannot realize it for every input.

Nevertheless, we will prove as the first result of this paper an involutive version of the J-criterion.

Theorem 3.12 (Involutive J-criterion (I)). *Let P be the Pommaret division. Let $G \subseteq M$ be a finite set and involutively head autoreduced w.r.t. the v-part. Moreover, assume that $\langle Sig(G) \rangle_P = \mathbb{T}_n^m$. Then the statements of Lemma 3.10 are equivalent, i.e. the statements*

a) G is a strong P-basis of M.
b) Every involutive J-pair of elements of G is eventually involutively super reducible by G.
c) Every involutive J-pair of elements of G is involutively covered by G or involutively super reducible by G.

Proof. Due to Lemma 3.10 we only must show the implication "$c) \Rightarrow a)$". And we are doing this by reductio ad absurdum. For this purpose, suppose that G is not a strong P-basis of M and that c) holds. Then, by definition of a strong P-basis we know: There must exist a pair $(\mathbf{0}, 0) \neq (\boldsymbol{u}, v) \in M$ which is not involutively reducible by G. We take the one with smallest signature. We set $T := \operatorname{lt}(\boldsymbol{u})$ and observe that $T \neq \mathbf{0}$ as otherwise v would be 0, too. Now, as $\langle \operatorname{Sig}(G) \rangle_P = \mathbb{T}_n^m$ is true by our assumptions, we can choose a pair $(\boldsymbol{u}_1, v_1) \in G$ with the following two properties:

(i) $\operatorname{lt}(\boldsymbol{u}_1) \mid_P \operatorname{lt}(\boldsymbol{u})$ and
(ii) $t \operatorname{lt}(v_1) := \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_1)} \operatorname{lt}(v_1)$ is minimal among all elements in G that satisfy condition (i).

Note, that $v_1 \neq 0$ as otherwise $(\boldsymbol{u}, v)$ would be involutively reducible by a syzygy $(\boldsymbol{u}_1, 0)$ due to condition (i). Hence, we are in the position to apply part c) of Lemma 3.11, telling us that $t(\boldsymbol{u}_1, v_1)$ is not involutively regular reducible by G. Next, we set $c := \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_1)}$ and

$$(\boldsymbol{u}', v') := (\boldsymbol{u}, v) - ct(\boldsymbol{u}_1, v_1).$$

First, we observe that $\operatorname{lt}(\boldsymbol{u}') \prec \operatorname{lt}(\boldsymbol{u}) = T$. For the v-part, there are several cases to consider. If $\operatorname{lt}(v) \neq t \operatorname{lt}(v_1)$, i.e. $v' \neq 0$, we argue as follows: Because $(\boldsymbol{u}', v')$ has a smaller signature than $(\boldsymbol{u}, v)$ it must be involutively reducible by G. For the moment, we reduce by syzygies if possible. Doing so, we only can reduce the signature, and hence, the remainder is still involutively reducible by G. But now, it is involutively reducible by a pair $(\boldsymbol{u}_2, v_2)$ with $v_2 \neq 0$. Also note that v' has not been changed during the reduction process so far.

Since $\operatorname{lt}(v) \neq t \operatorname{lt}(v_1)$, there are two cases.

- If $\operatorname{lt}(v) \prec t \operatorname{lt}(v_1)$ is true, then we have $\operatorname{lt}(v') = t \operatorname{lt}(v_1)$. Hence, we get the relations

$$\operatorname{lt}(v_2) \mid_P \operatorname{lt}(v') = t \operatorname{lt}(v_1) \quad \text{and} \quad \frac{t \operatorname{lt}(v_1)}{\operatorname{lt}(v_2)} \operatorname{lt}(\boldsymbol{u}_2) \preceq \operatorname{lt}(\boldsymbol{u}') \prec T = t \operatorname{lt}(\boldsymbol{u}_1),$$

which implies that $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by G leading to a contradiction to our result above obtained from Lemma 3.11 c).

- If, on the other hand, $t\,\mathrm{lt}(v_1) \prec \mathrm{lt}(v)$ is true, then we get $\mathrm{lt}(v') = \mathrm{lt}(v)$. Therefore we obtain

$$\mathrm{lt}(v_2) \mid_P \mathrm{lt}(v') = \mathrm{lt}(v) \quad \text{and} \quad \frac{\mathrm{lt}(v)}{\mathrm{lt}(v_2)}\,\mathrm{lt}(\boldsymbol{u}_2) \preceq \mathrm{lt}(\boldsymbol{u}') \prec T = \mathrm{lt}(\boldsymbol{u}),$$

which now implies that $(\boldsymbol{u}, v)$ is involutively regular reducible by G leading once again to a contradiction since $(\boldsymbol{u}, v)$ is not involutively reducible by G due to our assumptions from the beginning of this proof.

Accordingly, there is only one possibility left, i.e. we have $\mathrm{lt}(v) = t\,\mathrm{lt}(v_1)$. If $t = 1$ or if $t \neq 1$ is a multiplicative term for $\mathrm{lt}(v_1)$, then $\mathrm{lt}(v_1) \mid_P \mathrm{lt}(v)$, $\mathrm{lt}(\boldsymbol{u}_1) \mid_P \mathrm{lt}(\boldsymbol{u})$ and $\frac{\mathrm{lt}(v)}{\mathrm{lt}(v_1)} = \frac{\mathrm{lt}(\boldsymbol{u})}{\mathrm{lt}(\boldsymbol{u}_1)} = t$ and hence, $(\boldsymbol{u}, v)$ is involutively reducible by $(\boldsymbol{u}_1, v_1) \in G$. But this is not possible. So $t \neq 1$ has at least one non-multiplicative variable for $\mathrm{lt}(v_1)$. Applying part b) of Lemma 3.11 we obtain a pair $(\boldsymbol{u}_3, v_3) \in G$ such that $t(\boldsymbol{u}_1, v_1)$ is involutively super reducible by $(\boldsymbol{u}_3, v_3)$. But because of $t\,\mathrm{lt}(\boldsymbol{u}_1) = \mathrm{lt}(\boldsymbol{u})$ and $t\,\mathrm{lt}(v_1) = \mathrm{lt}(v)$, this implies that $(\boldsymbol{u}, v)$ is involutive reducible by $(\boldsymbol{u}_3, v_3)$ $\Big($*not* necessarily involutively super reducible since we might have $\frac{\mathrm{lc}(v)}{\mathrm{lc}(v_1)} \neq \frac{\mathrm{lc}(\boldsymbol{u})}{\mathrm{lc}(\boldsymbol{u}_1)}\Big)$, which is a contradiction to our choice of $(\boldsymbol{u}, v)$. □

Remark 3.13. We shall keep in mind that if we can prove Lemma 3.11 c) under some other assumptions, the proof of Theorem 3.12 carries over. Also, we shall not forget that all elements in M with smaller signature than $tl(\boldsymbol{u}_1) = \mathrm{lt}(\boldsymbol{u})$ would involutively reduce to $(\mathbf{0}, 0)$. Hence, the strong L-basis is finished up to this signature. We will later refer to those two facts.

But for now, we will discuss why we should change our preconditions in the first place, and, that we cannot just discard the condition "G is involutively head autoreduced w.r.t. the v-part". We will replace this condition by a weaker one, namely, that G is involutively regular autoreduced[8].

For a term order satisfying $\boldsymbol{e}_1 \prec \boldsymbol{e}_2$ we discuss the ideal $\langle x^2, x\rangle \trianglelefteq K[x]$. Next, we can check that the set $G := \{(\boldsymbol{e}_1, x^2), (\boldsymbol{e}_2, x)\}$ is indeed involutively regular autoreduced with $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathrm{Sig}(G)$. However, there are no involutive J-pairs to consider and the modified version of Theorem 3.12 would tell us that G is a strong P-basis. This is of course not true since there is no syzygy contained in G.

This shows that we cannot drop our assumptions in Theorem 3.12 so easily. But it shows simultaneously that there is no involutive reduction changing G into an involutively head autoreduced set w.r.t. the v-part. The only possible reduction would increase the signature and, thus, by definition is not involutive. Therefore, there are some ideals for which we cannot fulfill all needed assumptions in Theorem 3.12. Moreover, we now know that replacing the condition by "G is involutively regularly autoreduced" is not sufficient.

But fortunately, we will find a way around this problem by allowing *some* necessary, yet "forbidden" reduction steps. Of course, we want to avoid reduction steps as much as possible. Thus, we will also aim to obtain some criteria how to decide whether or not a forbidden reduction shall be done. Keep in mind that we want to do only regular reduction steps if possible, because this way we have control over the signature which is very helpful for applying our J-criteria.

We want to point out that this small example is also a counterexample for Theorem 5 in [1], where the authors left out preconditions for the v-part of G. Furthermore, they have used a weaker criterion in statement c) since there, a J-pair can only be discarded if it is covered (remember that they present the semi-involutive version), whereas we can discard involutively super reducible J-pairs, too.

Now, it is of course useful to have some criteria optimizing the test of the statement c) of Theorem 3.12. We have already found some of these criteria (e.g. Lemma 3.5), but there are more to discover. We will find some of them in the next subsection.

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[8]This means, that there is no element in G that can be involtuively regular reduced by G.

3.2. Involutive J-criterion (II) for the Pommaret Division

To prepare the next result of this work, we will introduce some rather technical lemmas in order to avoid an incomprehensible and long proof. In this section we set $L = P$ to be the Pommaret division, if not noted otherwise. Still, we shall note that some of the lemmas remain valid for arbitrary involutive divisions even though we prove them only for $L = P$.

Lemma 3.14. *If $(\boldsymbol{u}, v) \in M$ is involutively regular reducible by $(\boldsymbol{u}_1, v_1) \in G$ and involutively super reducible by $(\boldsymbol{u}_2, v_2) \in G$ with $v_2 \neq 0$, then there is a multiplicative term t for $\operatorname{lt}(\boldsymbol{u}_2)$ and $\operatorname{lt}(v_2)$ such that $\operatorname{lt}(\boldsymbol{u}) = t\operatorname{lt}(\boldsymbol{u}_2)$, $\operatorname{lt}(v) = t\operatorname{lt}(v_2)$ and $t(\boldsymbol{u}_2, v_2)$ is involutively regular reducible by $(\boldsymbol{u}_1, v_1)$.*

Proof. Since $(\boldsymbol{u}, v)$ is involutively super reducible by $(\boldsymbol{u}_2, v_2)$ with $v_2 \neq 0$, it follows that there exists a coefficient $c \in K$, a multiplicative term for $\operatorname{lt}(\boldsymbol{u}_2)$ and $\operatorname{lt}(v_2)$ such that

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_P \operatorname{lt}(\boldsymbol{u}), \quad \operatorname{lt}(v_2) \mid_P \operatorname{lt}(v) \quad \text{and} \quad \frac{\operatorname{lm}(v)}{\operatorname{lm}(v_2)} = \frac{\operatorname{lm}(\boldsymbol{u})}{\operatorname{lm}(\boldsymbol{u}_2)} = ct.$$

In particular, $t\operatorname{lt}(v_2) = \operatorname{lt}(v)$ and $t\operatorname{lt}(\boldsymbol{u}_2) = \operatorname{lt}(\boldsymbol{u})$. Now, because $(\boldsymbol{u}, v)$ is involutively regular reducible by $(\boldsymbol{u}_1, v_1)$, we get $\operatorname{lt}(v_1) \mid_P \operatorname{lt}(v) = t\operatorname{lt}(v_2)$. Thus, there are two possibilities:

- $\frac{t\operatorname{lt}(v_2)}{\operatorname{lt}(v_1)}\operatorname{lt}(\boldsymbol{u}_1) = \frac{\operatorname{lt}(v)}{\operatorname{lt}(v_1)}\operatorname{lt}(\boldsymbol{u}_1) \prec \operatorname{lt}(\boldsymbol{u}) = t\operatorname{lt}(\boldsymbol{u}_2)$, and hence, $t(\boldsymbol{u}_2, v_2)$ is involutively regular reducible by $(\boldsymbol{u}_1, v_1)$, or
- $\frac{t\operatorname{lt}(v_2)}{\operatorname{lt}(v_1)}\operatorname{lt}(\boldsymbol{u}_1) = \frac{\operatorname{lt}(v)}{\operatorname{lt}(v_1)}\operatorname{lt}(\boldsymbol{u}_1) = \operatorname{lt}(\boldsymbol{u}) = t\operatorname{lt}(\boldsymbol{u}_2)$ and

$$\frac{\operatorname{lc}(v)}{\operatorname{lc}(v_1)} \neq \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_1)}. \tag{3.1}$$

  We have to prove now, that $\frac{\operatorname{lc}(v_2)}{\operatorname{lc}(v_1)} \neq \frac{\operatorname{lc}(\boldsymbol{u}_2)}{\operatorname{lc}(\boldsymbol{u}_1)}$ in order to show the claim of the lemma. Because $(\boldsymbol{u}, v)$ is involutively super reducible by $(\boldsymbol{u}_2, v_2)$, we obtain

$$\frac{\operatorname{lc}(v)}{\operatorname{lc}(v_2)} = \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_2)}. \tag{3.2}$$

  Starting from (3.1), we know $\operatorname{lc}(v) \neq \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_1)}\operatorname{lc}(v_1)$. Plugging in (3.2), we end up with

$$\frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_2)}\operatorname{lc}(v_2) \neq \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_1)}\operatorname{lc}(v_1)$$

  and hence, with

$$\frac{\operatorname{lc}(v_2)}{\operatorname{lc}(\boldsymbol{u}_2)} \neq \frac{\operatorname{lc}(v_1)}{\operatorname{lc}(\boldsymbol{u}_1)},$$

  which is exactly what we needed to show. □

Lemma 3.15. *Let L be an arbitrary involutive division. Let $t \in \mathbb{T}_n$, $(\boldsymbol{u}, v) \in G$ and let $t(\boldsymbol{u}, v) \in M$ be involutively regular reducible by $(\boldsymbol{u}_1, v_1) \in G$. If $\operatorname{lt}(v_1) \mid_{L,B_v} \operatorname{lt}(v)$ then $(\boldsymbol{u}, v)$ is involutively regular reducible by $(\boldsymbol{u}_1, v_1)$.*

Proof. We know that $t(\boldsymbol{u}, v)$ is involutively regular reducible by $(\boldsymbol{u}_1, v_1)$. This, by definition, implies $\operatorname{lt}(v_1) \mid_{L,B_v} t\operatorname{lt}(v)$ and

$$\frac{t\operatorname{lt}(v)}{\operatorname{lt}(v_1)}\operatorname{lt}(\boldsymbol{u}_1) \prec t\operatorname{lt}(\boldsymbol{u}) \quad \text{or} \quad \left(\frac{t\operatorname{lt}(v)}{\operatorname{lt}(v_1)}\operatorname{lt}(\boldsymbol{u}_1) = t\operatorname{lt}(\boldsymbol{u}) \quad \text{and} \quad \frac{\operatorname{lc}(v)}{\operatorname{lc}(v_1)} \neq \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_1)}\right). \tag{3.3}$$

In addition we have $\operatorname{lt}(v_1) \mid_{L,B_v} \operatorname{lt}(v)$. Hence, we can write (3.3) without "t" and we are done. □

With this lemma we immediately can prove a first proposition aiming towards the next result of this work. It collects some properties about the pair $(\boldsymbol{u}_1, v_1)$ from the proof of Theorem 3.12, some of which are written down already in Lemma 3.11. Recall, that $t\,\mathrm{lt}(\boldsymbol{u}_1)$ was the smallest signature belonging to a pair in M which is not involutively reducible by G. Also, we have discussed that we may assume that G is at least involutively regular autoreduced.

Proposition 3.16. *Let $G \subseteq M$ be involutively regular autoreduced, $(\boldsymbol{u}_1, v_1) \in G$ and t multiplicative for $(\boldsymbol{u}_1, v_1)$. Let $t(\boldsymbol{u}_1, v_1)$ not be involutively covered by G. Moreover, assume that $(\boldsymbol{u}, v) \in M$ involutively reduces to $(\mathbf{0}, 0)$ for all $(\boldsymbol{u}, v)$ with $\mathrm{lt}(\boldsymbol{u}) \prec t\,\mathrm{lt}(\boldsymbol{u}_1)$.*

If $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2) \in G$ then $t\,\mathrm{lt}(v_1) = \mathrm{lt}(v_2)$ and $t \neq 1$.

Proof. Let $t(\boldsymbol{u}_1, v_1)$ be involutively regular reducible by $(\boldsymbol{u}_2, v_2)$. Because G is involutively regular autoreduced, we obtain $t \neq 1$. Since t is multiplicative for $\mathrm{lt}(v_1)$ this implies $\mathrm{lt}(v_1) \mid_P t\,\mathrm{lt}(v_1)$ and, as $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible, $\mathrm{lt}(v_2) \mid_P t\,\mathrm{lt}(v_1)$. Therefore, $\mathrm{lt}(v_2) \mid_P \mathrm{lt}(v_1)$ or vice versa. In the first case, we can apply Lemma 3.15 and find that already $(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$ contradicting the assumption that G is involutively regular autoreduced. Accordingly, $\mathrm{lt}(v_1) \mid_P \mathrm{lt}(v_2)$ and $\mathrm{lt}(v_1) \neq \mathrm{lt}(v_2)$ must hold. Then, by definition, there exists a multiplicative term $t' \neq 1$ for $\mathrm{lt}(v_1)$ such that $t'\,\mathrm{lt}(v_1) = \mathrm{lt}(v_2)$. Because of the fact that t' and t are both multiplicative terms for $\mathrm{lt}(v_1)$ and because of the relation

$$t'\,\mathrm{lt}(v_1) = \mathrm{lt}(v_2) \mid_P t\,\mathrm{lt}(v_1),$$

we know that $t' \mid_P t$. We now look at $t'(\boldsymbol{u}_1, v_1)$. Since $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$ and $\mathrm{lt}(v_2) \mid_P t'\,\mathrm{lt}(v_1)$, we can apply Lemma 3.15 once again, returning the statement that $t'(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$.

Our aim now is to show, that $t' = t$. Then we have shown everything claimed in the proposition. For that we recall that we already know $t' \mid_P t$ and thus, $t' \preceq t$. So, suppose that we have $t' \prec t$. Then $t'\,\mathrm{lt}(\boldsymbol{u}_1) \prec t\,\mathrm{lt}(\boldsymbol{u}_1)$. By our preconditions, this implies that $t'(\boldsymbol{u}_1, v_1)$ reduces involutively to $(\mathbf{0}, 0)$ by G where at least one reduction is involutively regular. Performing first all involutively regular reduction steps we see that $t'(\boldsymbol{u}_1, v_1)$ is eventually involutively super reducible by G. From Lemma 3.9 a) we know that $t'(\boldsymbol{u}_1, v_1)$ is involutively covered by a pair $(\boldsymbol{u}_3, v_3) \in G$. Thus, we have $\mathrm{lt}(\boldsymbol{u}_3) \mid_P t'\,\mathrm{lt}(\boldsymbol{u}_1)$ and

$$\frac{t'\,\mathrm{lt}(\boldsymbol{u}_1)}{\mathrm{lt}(\boldsymbol{u}_3)}\,\mathrm{lt}(v_3) \prec t'\,\mathrm{lt}(v_1). \tag{3.4}$$

Because of $t' \mid_P t$ and t is multiplicative for $\mathrm{lt}(\boldsymbol{u}_1)$ we obtain even $\mathrm{lt}(\boldsymbol{u}_3) \mid_P t\,\mathrm{lt}(\boldsymbol{u}_1)$. Finally, multiplying $\frac{t}{t'}$ to (3.4), we conclude that even $t(\boldsymbol{u}_1, v_1)$ is involutively covered by $(\boldsymbol{u}_3, v_3) \in G$, and hence, a contradiction to our precondition. □

Now, having these results, we can tackle the main theorem. For that, we have to introduce a certain subset of M.

Definition 3.17. Let $G \subset M$ be finite and involutively regular autoreduced. Let $(\boldsymbol{u}_1, v_1) \in G$ and t be a term. Then, the pair $t(\boldsymbol{u}_1, v_1)$ is called a *proxy of* $(\boldsymbol{u}_2, v_2)$ if the following conditions hold:

(i) We have $\mathrm{lt}(v_1) \mid_P \mathrm{lt}(v_2)$ and $t = \frac{\mathrm{lt}(v_2)}{\mathrm{lt}(v_1)}$,
(ii) t is a multiplicative term for $\mathrm{lt}(\boldsymbol{u}_1)$,
(iii) $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$ and
(iv) $t(\boldsymbol{u}_1, v_1)$ is not involutively covered by G.

The set of all proxy pairs is denoted by $PP(G)$. Furthermore, we call a proxy pair with smallest signature an *essential pair* for G.

It may seem to be generic that $PP(G)$ is the empty set because there are many conditions the elements in $PP(G)$ have to satisfy. However, it will turn out, that during the computations, this set most likely is not empty and will play a major role. Nevertheless, our goal will be to obtain a set G

at the end of the algorithm, where $PP(G) = \emptyset$ is true. We want also to mention why this notion of proxy pairs is chosen. But this might be more reasonable when we start to formulate the algorithm. Then, we will see, that they are indeed a "proxy" in some meaningful sense.

Now, we begin to show our next result. From our previous work we know that all we have to do is to redo part c) of Lemma 3.11 under the new conditions which we will introduce in the following proposition.

Proposition 3.18. *Let P be the Pommaret division. Let $G \subseteq M$ be finite and involutively regular autoreduced. Moreover, assume that $PP(G) = \emptyset$. Furthermore, assume that every J-pair of elements in G is involutively covered or involutively super reducible by G.*

Let $(\boldsymbol{u}, v) \in M$ be non-zero and suppose there is a pair $p_1 := (\boldsymbol{u}_1, v_1) \in G$ with $v_1 \neq 0$ such that

(i) $\operatorname{lt}(\boldsymbol{u}_1) \mid_P \operatorname{lt}(\boldsymbol{u})$ *and*
(ii) $t \operatorname{lt}(v_1) := \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_1)} \operatorname{lt}(v_1)$ *is minimal among all elements in G that satisfy condition (i).*

Moreover, assume that every $(\boldsymbol{u}', v') \in M$ with $\operatorname{lt}(\boldsymbol{u}') \prec t \operatorname{lt}(\boldsymbol{u}_1)$ involutively reduces to $(\boldsymbol{0}, 0)$ by G. Then tp_1 is not involutively regular reducible by G.

Proof. Let us suppose that tp_1 is involutively regular reducible by $(\boldsymbol{u}_2, v_2) \in G$. Then, applying Lemma 3.11 a) we know that tp_1 is not involutively covered by G.

For the moment, suppose that t is a multiplicative term for $\operatorname{lt}(v_1)$. With Proposition 3.16 we can conclude $t \operatorname{lt}(v_1) = \operatorname{lt}(v_2)$ and $t \neq 1$. Hence, we collect the following properties:

- We have $\operatorname{lt}(v_1) \mid_P \operatorname{lt}(v_2)$ and $t = \frac{\operatorname{lt}(v_2)}{\operatorname{lt}(v_1)}$.
- t is multiplicative for $\operatorname{lt}(\boldsymbol{u}_1)$.
- $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$ and
- $t(\boldsymbol{u}_1, v_1)$ is not involutively covered by G.

Therefore, $t(\boldsymbol{u}_1, v_1) \in PP(G) = \emptyset$ leads to a contradiction. Also note that indeed $t(\boldsymbol{u}_1, v_1)$ is an essential pair for G since there cannot be any other pair in $PP(G)$ with smaller signature because of the following arguments: All pairs in $PP(G)$ with smaller signature reduce involutively to zero and there is always at least one involutively regular reduction by definition of $PP(G)$. Hence, the pair is eventually involutively super reducible and at least once involutively regular reducible. Applying lemma 3.9 a) we know, that the pair would be involutively covered by G violating one condition for being an element in $PP(G)$.

Thus, t is not 1 and contains a non-multiplicative variable for $\operatorname{lt}(v_1)$. With Lemma 3.11 b) there exists a pair $p' := (\boldsymbol{u}', v') \in G$ such that $v' \neq 0$ and $t(\boldsymbol{u}_1, v_1)$ is involutively super reducible by p'. Now, applying Lemma 3.14, we know that there exists a multiplicative term t' for $\operatorname{lt}(v')$ and $\operatorname{lt}(\boldsymbol{u}')$ such that $t \operatorname{lt}(\boldsymbol{u}_1) = t' \operatorname{lt}(\boldsymbol{u}')$ and $t \operatorname{lt}(v_1) = t' \operatorname{lt}(v')$ and such that $t'(\boldsymbol{u}', v')$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$, too. t' cannot be 1, as otherwise G would not be involutively regular autoreduced.

Also, $t'(\boldsymbol{u}', v')$ is not involutively covered by G, because it has the same leading terms as $t(\boldsymbol{u}_1, v_1)$. Therefore, we are in the upper case again and we can apply Proposition 3.16 and get analogously $t'(\boldsymbol{u}', v') \in PP(G) = \emptyset$, leading to yet another contradiction. □

Now, the actual involutive J-criterion becomes a corollary.

Theorem 3.19 (Involutive J-criterion (II)). *Let P be the Pommaret division. Let $G \subseteq M$ be finite and involutively regular autoreduced. Moreover, assume that we have $\langle Sig(G) \rangle_P = \mathbb{T}_n^m$ and that $PP(G) = \emptyset$. Then the following statements are equivalent:*

a) G is a strong P-basis of M.
b) Every involutive J-pair of elements of G is eventually involutively super reducible by G.

c) *Every involutive J-pair of elements of G is involutively covered by G or involutively super reducible by G.*

Proof. According to Lemma 3.10 we only have to show "c) $\Rightarrow$ a)". Remark 3.13 tells us that for this purpose we only need to show a modified version of Lemma 3.11 c) and that Proposition 3.18 is that modified version. □

3.3. Algorithm: Strong P-Basis

Finally, we are able to formulate an algorithm that is suitable for computing a strong P-basis.

For reasons of efficiency we choose the strategy of smallest signature since by this our set G will not change already treated pairs due to involutively regular autoreductions. Indeed, it will turn out that this strategy is crucial for our algorithm to be correct.

Let $JP(G)$ be the set of all involutive J-pairs of a set G. We start with a set

$$G := \{(\boldsymbol{e}_i, f_i) \mid 1 \leq i \leq m\}$$

satisfying the precondition $\langle \mathrm{Sig}(G)\rangle_P = \mathbb{T}_n^m$. Since elements in $JP(G) \cup PP(G)$ cannot be syzygies we take the syzygies from G and store them in a set H. Next, we will deal with $JP(G)$ and $PP(G)$ simultaneously always searching for the element $p \in JP(G) \cup PP(G)$ with smallest signature. Our goal for $p \in JP(G)$ is to add, if not already existing, a pair p_1 to $G \cup H$ such that p is involutively covered by p_1 or involutively super reducible by it. For this, p_1 is set to be an involutively regular normal form of p. Then p is eventually involutively super reducible by p_1 and by applying lemma 3.10 we know, that we can discard p from $JP(G)$.

For $p \in PP(G)$ we always take an essential pair because of the strategy of the smallest signature. And this is exactly what we need to do by our proof of Proposition 3.18. Then, we calculate one of its involutively regular normal forms and add it to $G \cup H$. Hence, p is now involutively covered by $G \cup H$ (see Lemma 3.6) and not contained in $PP(G)$ anymore.

Adding an element to G may extend the set $JP(G) \cup PP(G)$ and thus we are interested in finding criteria to discard as much pairs as possible from these sets. Indeed, we have the following lemma.

Lemma 3.20. *Let $p \in JP(G) \cup PP(G)$.*

a) *If p is involutively covered by $JP(G) \cup PP(G)$, or*
b) *if $p \in JP(G)$ is involutively super reducible by $(JP(G) \cup PP(G)) \setminus \{p\}$, or*
c) *if $p \in PP(G)$ is involutively super reducible by $PP(G) \setminus \{p\}$, or*
d) *if p is pseudo reducible by $p' := (\boldsymbol{u}', v') \in PP(G) \setminus \{p\}$,*

then it can be discarded. In particular, for every G there is a unique essential pair not satisfying a) to d).

Proof. Since all these relations are transitive by Lemma 3.7, we may assume that p is the only element we have to consider.

ad a): Since we will add an element g to $G \cup H$ such that p is involutively covered or involutively super reducible by g we know by Lemma 3.9 a) and b) that p will be involutively covered by the new $G \cup H$.

ad b): Let p be involutively super reducible by p'. Then, we will provide a g for $G \cup H$ such that p' is involutively covered or – if $p' \in JP(G)$, involutively covered or involutively super reducible – by the new $G \cup H$. Applying Lemma 3.9 b), we know that the same holds for p. Hence, it can be discarded.

ad c) and d): We will provide a g for $G \cup H$ such that p' is involutively covered by the new $G \cup H$. Hence, by Lemma 3.9 we are done.

Now, we want to show, that for any signature occurring in $PP(G)$ there will be only one element left in $PP(G)$ after eliminating elements that satisfy a condition from a)–d). To see this, assume that for a given signature, there are two elements $(\boldsymbol{u}_1, v_1) \neq (\boldsymbol{u}_2, v_2) \in PP(G)$ with $\mathrm{lt}(\boldsymbol{u}_1) = \mathrm{lt}(\boldsymbol{u}_2)$. According to a), we must have $\mathrm{lt}(v_1) = \mathrm{lt}(v_2)$ as otherwise one of the two would be involutively covered by the other. Note, that since both elements are proxy pairs, the v-parts are not zero. Now, due to d), we know that the first pair considered would have been discarded as it is pseudo reducible by the other one. □

Remark 3.21. Because we now know, that the essential pair is unique when we use the criteria a) to d), its notion may become more transparent. Also, if we look back to the proof of the involutive J-criterion, we worked with the essential pair of G. Furthermore, when we look at proxy pairs in general, it turns out that we need them to perform a forbidden reduction step. Every time a regular normal form p has a v-part that is not an involutive normal form, we may insert an element into $PP(G)$ with the same leading term in the v-part as p, and with the property that it is now involutively regular reducible. Hence, for an involutively irreducible element $p \in G$ with reducible v-part we have introduced a "proxy" to a pair with the same leading term in the v-part, allowing us to continue the reduction with involutively regular reduction steps. Also note, the pair p is not important anymore for our involutive bases in the sense that neither its v-part nor its u-part will appear in it. However, its J-pairs might be necessary – and the same holds for the proxy pairs introduced with the help of p.

By the way, it is easy to see, that from Theorem 3.19, $G \cup H$ is a strong P-basis for M if $PP(G) = \emptyset$ and every element in $JP(G)$ is involutively covered or involutively super reducible by $G \cup H$. However, since we are aiming to compute two Pommaret bases the proof of termination is not trivial. Based on this issue, one may think of dropping the goal to compute a Pommaret basis for $Syz(F)$. This approach, we will follow in Subsection 3.5 and represents the semi-involutive variant of our algorithm. But for now, we first present an important proposition which is the involutive version of Proposition 2.7.

Proposition 3.22. *Let $G = \{(\boldsymbol{u}_1, v_1), \ldots, (\boldsymbol{u}_k, v_k)\}$ be a strong P-basis of M. Then*

(i) $G_0 := \{\boldsymbol{u}_i \mid v_i = 0,\ 1 \leq i \leq k\}$ *is a weak Pommaret basis of* $Syz(F)$.
(ii) $G_1 := \{v_i \mid v_i \neq 0,\ 1 \leq i \leq k\}$ *is a weak Pommaret basis of* $I = \langle F \rangle$.

Now, we want to note here that this proposition would provide us with a Gröbner basis of $Syz(F)$ if we were in the semi-involutive case. Indeed, it is enough to compute only a generating system of $Syz(F)$ as we could iterate the algorithm to compute a Pommaret basis of $Syz(F)$ starting with the obtained generating system from the first run. Moreover, if we use a semi-involutive variant of our involutive J-criterion potentially more elements will be discarded. This is one of the reasons why we will focus on this variant of the GVW algorithm.

But first, we put together the results from this section in an algorithm presented as a pseudo code for the fully involutive case. Before we go into the details, we want to mention that we now restrict to homogeneous ideals as termination is easier to prove. However, one may adopt this theory to the affine case through homogenization and dehomogenization arguments.

Before we present the involutive algorithm, we want to point out one notion. If I is homogeneous, $Syz(F)$ is too in some sense:

All we have to do is to introduce another notion of degrees of a vector term. Consider a term $x^\mu \in \mathbb{T}_n$. We set $\deg_F(x^\mu \boldsymbol{e}_i) := \deg(x^\mu) + \deg(f_i)$. In particular, this means for an $(\boldsymbol{u}, v) \in M$ that $\deg_F(\mathrm{lt}(\boldsymbol{u})) = \deg(\mathrm{lt}(v))$ if $v \neq 0$. Thus, if the v-part is homogeneous, the u-part is too.

Also, we want to note an optimization presented in [6]: If $(\boldsymbol{u}_1, v_1) \neq (\boldsymbol{u}_2, v_2) \in G$ are two different elements in G, then $v_2(\boldsymbol{u}_1, v_1) - v_1(\boldsymbol{u}_2, v_2)$ is a so-called *trivial syzygy* that we can add to the syzygies found so far. However, if we decided to only keep signatures rather than the whole u-part it is important to note, that the signatures of the two pairs may cancel, and thus, we know nothing about the u-part of the syzygy. In such a case, we therefore add nothing to H. Furthermore, when trivial syzygies are used in the algorithm, involutively super autoreductions of H are reasonable. Without

InvGVW($F, H_0, \prec_1, \prec_2, P, q$) (Pommaret Version)	
Input:	A set $F = \{f_1, \dots, f_m\} \subseteq R$ of homogeneous polynomials, $\prec_1 = \prec_{\text{degrevlex}}$ on R and a compatible term order $\prec_2$ on R^m of type ω, P Pommaret division, a degree bound q for elements in a Pommaret basis of I. An involutively fully reduced set H_0 of syzygies of F, where $H_0 = \emptyset$ is possible.
Output:	A weak Pommaret basis for $I = \langle F \rangle$, that also contains a Pommaret basis as a subset; and a weak Pommaret basis of $Syz(F)$ or an error message that I or $Syz(F)$ is not in quasi-stable position.
Variables:	G is an ordered set of pairs $(\boldsymbol{u}_i, v_i) \in M$ with $v_i \neq 0$. H is an ordered set of syzygies $(\boldsymbol{u}, 0)$ of F. $JP(G)$ is the set of involutive J-pairs of G. $PP(G)$ is the set of proxy pairs of G.
Step 1:	$G \leftarrow \{(\boldsymbol{e}_i, f_i) \mid 1 \leq i \leq m\}$, $H \leftarrow H_0$, $PP(G) \leftarrow \emptyset$
Step 2:	Perform an involutively regular autoreduction on G. Fill H with obtained syzygies, discard them from G, and fill $PP(G)$ with (new) proxy pairs. Calculate (new) trivial syzygies of G and add them to H. Involutively autoreduce H. Fill $JP(G)$ with (new) involutive J-pairs of G. Remove all elements from $JP(G) \cup PP(G) =: Q$ for which the degree of the v-part is greater than $q + 2$.
Step 3:	**while** $JP(G) \cup PP(G) \neq \emptyset$ **do**
Step 4:	Take elements $p := (\boldsymbol{u}, v) \in JP(G) \cup PP(G) =: Q$ with smallest signature and then with smallest leading term in the v-part. Do the following step for all choices of p
Step 5:	If • p is involutively covered by $G \cup H \cup Q =: S$, or • p is pseudo reducible by $PP(G) \setminus \{p\}$, or • $p \in PP(G)$ is involutively super reducible by $PP(G) \setminus \{p\}$, or • $p \in JP(G)$ is involutively super reducible by $S \setminus \{p\}$, then discard p and go back to step 3.
Step 6:	If there is more than one choice for p left, and one of them is from $PP(G)$, take it and discard the rest. If all are in $JP(G)$ perform an involutively regular reduction step from one of them by another one, replace p by the result of the reduction step and discard all other choices of p.
Step 7:	Calculate an involutively regular normal form $(\boldsymbol{u}', v')$ of p by G
Step 8:	**If** $v' = 0$ **then**
Step 9:	**If** $\min\left\{\deg_F(\text{lt}(\boldsymbol{u}')), \min_{(\boldsymbol{u},v)\in Q}\{\deg_F(\text{lt}(\boldsymbol{u}))\}\right\} > q + 1$ **then**
Step 10:	**return** "$Syz(F)$ is not in quasi-stable position"
Step 11:	**endif**
Step 12:	$H \leftarrow H \cup \{(\boldsymbol{u}', 0)\}$
Step 13:	**else**
Step 14:	**If** $\text{lt}(v')$ is not involutively reducible by the v-part of G and $\min\left\{\deg(\text{lt}(v'), \min_{(\boldsymbol{u},v)\in Q}\{\deg(\text{lt}(v))\}\right\} > q$ **then**
Step 15:	**return** "I is not in quasi-stable position"
Step 16:	**endif**
Step 17:	$G \leftarrow G \cup \{(\boldsymbol{u}', v')\}$. Go back to step 2.
Step 18:	**end if**
Step 19:	**end while**
Return:	$\{v_i \mid (\boldsymbol{u}_i, v_i) \in G\}$ and $\{\boldsymbol{u} \mid (\boldsymbol{u}, 0) \in H\}$

introducing trivial sygygies however, one can show that we can avoid involutive autoreductions as long as we do not have to perform coordinate transformations.

3.4. Correctness of the involutive GVW algorithm

After presenting the main algorithm, we have to consider of course that we aim to calculate a Pommaret basis which does not exist in general. Hashemi et al. have shown in [10], however, that with the help of coordinate transformations, the standard algorithm [18, Algo. 4.5] will terminate for computing a Pommaret basis of an homogeneous ideal and for the degree reverse lexicographic order. But there are many different ways how to decide if a coordinate transformation is needed and if so, which would fit the best (see [18, Prop. 5.3.4],[10, Prop. 3.2]). We will tackle this question later. But for now, we want to prove, that our core algorithm is correct, if it terminates. For the termination with Pommaret bases as the only possible output we will of course have to add some steps with transformations.

Remark 3.23. We want to note one property of the involutive GVW algorithm, that follows from the strategy of smallest signatures and the definition of involutive reduction on M: Let us have added $(\boldsymbol{u}, v)$ to $G \cup H$ at some point in the algorithm. As involutive reduction steps are not allowed to increase the signature, and as we build up our strong P-basis from smallest signatures, we know for sure that the strong P-basis is finished up to elements with strictly smaller signature than $\operatorname{lt}(\boldsymbol{u})$[9]. Moreover, if we take a degree compatible term order for the u-part, we can conclude:

If $\deg(\operatorname{lt}(\boldsymbol{u})) = k+1$ for some $k \in \mathbb{N}$, we know that we have a *strong P-basis up to degree* k, i.e. a set $G \cup H$ such that every element in M with a signature of degree less than or equal to k involutively reduces to $(\boldsymbol{0}, 0)$ by[10] $G \cup H$. But as every Pommaret basis of an homogeneous ideal I has elements of degree at most[11] q, and $q+1$ for $Syz(F)$, respectively [18, Cor. 5.5.18], all we need to do is to build a strong P-basis up to the degree $q+1$. If I or $Syz(F)$ are not in quasi-stable position, we then will find also elements of degree $q+2$ that will increase the involutive span of I or $Syz(F)$ [18, Prop. 5.3.7]. Thus, a first step towards the proof of correctness is to show that the algorithm will produce elements of ascending signature.

Indeed, we will prove that after we have enlarged G by an element $(\boldsymbol{u}, v)$, all new elements that are about to be added to $JP(G) \cup PP(G)$ have a signature greater than $\operatorname{lt}(\boldsymbol{u})$. Indeed, we could show in the next lemma that there are no two elements in $G \cup H$ with the same signature, if we follow the algorithm described so far. But since we must allow coordinate transformations this may not be the case anymore after a transformation.

Lemma 3.24. *Let $p := (\boldsymbol{u}, v) \in JP(G) \cup PP(G)$. If we have inserted an involutively regular normal form $p' := (\boldsymbol{u}', v')$ of p into $G \cup H$, then after a finite number of loop iterations, we will consider a pair from $JP(G) \cup PP(G)$ with a signature that is strictly greater than $\operatorname{lt}(\boldsymbol{u}')$.*

Proof. When we insert p' into $G \cup H$, there are finitely many elements in $JP(G) \cup PP(G)$ left with the same signature. Because of step 6, all those elements in $JP(G) \cup PP(G)$ are now involutively covered by the involutively regular autoreduced G or H, and hence can be discarded. Thus, all elements left in $JP(G) \cup PP(G)$ have a greater signature than p'. Therefore, we may assume that the next element we consider has not entered $JP(G) \cup PP(G)$, yet.

Now, if p' is a syzygy, we do not enlarge $JP(G) \cup PP(G)$. If p' is not a syzygy, then we perform an involutively regular autoreduction, only changing elements in G with signature greater than or equal to the one of p'. Thus, all new J-pairs that are added to $JP(G)$ have a signature greater than the one of p'. Also, we claim that elements added to $PP(G)$ have a strictly larger signature than p' has. To see this, we consider two cases.

Firstly, assume that v' is not an involutive normal form, i.e. there exists $(\boldsymbol{u}_1, v_1) \in G$ such that $\operatorname{lt}(v_1) \mid_P \operatorname{lt}(v')$ and $\frac{\operatorname{lt}(v')}{\operatorname{lt}(v_1)} \operatorname{lt}(\boldsymbol{u}_1) \succeq \operatorname{lt}(\boldsymbol{u}')$ as otherwise G would not be involutively regular autoreduced. Suppose, that $\frac{\operatorname{lt}(v')}{\operatorname{lt}(v_1)} \operatorname{lt}(\boldsymbol{u}_1) = \operatorname{lt}(\boldsymbol{u}')$. So, if we enter $g := \frac{\operatorname{lt}(v')}{\operatorname{lt}(v_1)}(\boldsymbol{u}_1, v_1)$ to $PP(G)$, then g is involutively regular reducible by p'. Thus, we must have $\frac{\operatorname{lc}(v')}{\operatorname{lc}(v_1)} \neq \frac{\operatorname{lc}(\boldsymbol{u}')}{\operatorname{lt}(\boldsymbol{u}_1)}$. However, then we know that the reduction $(\boldsymbol{u}', v') - \frac{\operatorname{lm}(v')}{\operatorname{lm}(v_1)}(\boldsymbol{u}_1, v_1)$ is involutively regular, too, and thus, G not involutively regular autoreduced. Hence, we have $\frac{\operatorname{lt}(v')}{\operatorname{lt}(v_1)} \operatorname{lt}(\boldsymbol{u}_1) \succ \operatorname{lt}(\boldsymbol{u}')$.

Secondly, assume that there is a pair $(\boldsymbol{u}_2, v_2) \in G$ such that $\operatorname{lt}(v') \mid_P \operatorname{lt}(v_2)$ applies and $\frac{\operatorname{lt}(v_2)}{\operatorname{lt}(v')} \operatorname{lt}(\boldsymbol{u}') \succeq \operatorname{lt}(\boldsymbol{u}_2)$. Our candidate for $PP(G)$ is $h := \frac{\operatorname{lt}(v_2)}{\operatorname{lt}(v')}(\boldsymbol{u}', v')$. So, what we have to show is $\operatorname{lt}(v_2) \neq \operatorname{lt}(v')$. For this, suppose it is not true, i.e. we have $\operatorname{lt}(v_2) = \operatorname{lt}(v')$.

[9] Here, we do not want to discuss if it is even finished up to elements with signature $\operatorname{lt}(\boldsymbol{u})$.
[10] Remember that we are only interested in a strong P-basis for $\prec_1 = \prec_{\text{degrevlex}}$.
[11] Remember that q is an upper bound, or the Castelnuovo-Mumford regularity.

Therefore, we have in total $\mathrm{lt}(v_2) \mid_P \mathrm{lt}(v')$, and from $\frac{\mathrm{lt}(v_2)}{\mathrm{lt}(v')}\mathrm{lt}(\boldsymbol{u}') \succeq \mathrm{lt}(\boldsymbol{u}_2)$ we obtain the relation $\mathrm{lt}(\boldsymbol{u}') \succeq \frac{\mathrm{lt}(v')}{\mathrm{lt}(v_2)}\mathrm{lt}(\boldsymbol{u}_2)$. Now, if especially $\mathrm{lt}(\boldsymbol{u}') \succ \frac{\mathrm{lt}(v')}{\mathrm{lt}(v_2)}\mathrm{lt}(\boldsymbol{u}_2)$ holds, then $(\boldsymbol{u}', v')$ is involutively regular reducible by $(\boldsymbol{u}_2, v_2)$ leading to a contradiction. Therefore, we may assume that $\mathrm{lt}(\boldsymbol{u}') = \frac{\mathrm{lt}(v')}{\mathrm{lt}(v_2)}\mathrm{lt}(\boldsymbol{u}_2) = \mathrm{lt}(\boldsymbol{u}_2)$ holds . However, the reduction is still involutively regular, if $\frac{\mathrm{lc}(v')}{\mathrm{lc}(v_2)} \neq \frac{\mathrm{lc}(\boldsymbol{u}')}{\mathrm{lc}(\boldsymbol{u}_2)}$. So, the quotients must be equal, too. But then the reduction

$$\frac{\mathrm{lm}(v_2)}{\mathrm{lm}(v')}(\boldsymbol{u}', v') - \frac{\mathrm{lc}(v')}{\mathrm{lc}(v')}(\boldsymbol{u}_2, v_2)$$

cannot be involutively regular, which is a contradiction to the fact that h is indeed involutively regular reducible by $(\boldsymbol{u}_2, v_2)$ (as we assumed $h \in PP(G)$).

Finally, these two cases let us conclude that since every other element in the involutively regular autoreduced set G, who can lead to new elements in $PP(G)$, have at least the same signature as p', every new element in $PP(G)$ has a greater signature than the one of p'. □

Theorem 3.25. *Let P be the Pommaret division, $\prec_1 = \prec_{degrevlex}$ and $\prec_2$ a compatible term order of type ω. If the involutive GVW algorithm terminates, it is correct.*

Proof. Step 1 guarantees $\langle \mathrm{Sig}(G \cup H)\rangle_P = \mathbb{T}_n^m$ at the end as we are not removing any of the input elements from $G \cup H$, and only are able to reduce them involutively regular. Since we are performing an involutively regular autoreduction after every step we have added an element to $G \cup H$ (step 2, step 17), our $G \cup H$ is involutively regular autoreduced.

Now, if the algorithm terminates, we first dicuss the case that $JP(G) = PP(G) = \emptyset$ is true without having removed elements $(\boldsymbol{u}, v)$ with $\deg_F(\mathrm{lt}(\boldsymbol{u})) = \deg(\mathrm{lt}(v)) > q + 2$. In the context of the algorithm, this means, that every J-pair of an element in G has been studied in step 5 or 6. Lemma 3.20 tells us, that every element in $JP(G)$ removed in step 5 is now involutively covered or involutively super reducible by $G \cup H$. Also, the lemma says that every element removed from $PP(G)$ will be involutively covered by the new $G \cup H$ and hence not be a proxy pair anymore. However, we have to argue about step 6 a bit. First, assume that one of the candidates is from $PP(G)$. Then, it will be involutively regular reducible by G and an involutively regular normal form will be added to $G \cup H$. Hence, it will be involutively covered by it and so every other candidate.

Now, assume that all candidates are from $JP(G)$. As they have passed step 5, every single one of them must be involutively regular reducible by any other choice of p. We perform one of the regular reductions. Now, as the obtained pair will be involutively covered or involutively super reducible by the new $G \cup H$, all the candidates for p will be – according to Lemma 3.6 – involutively covered by the new $G \cup H$. Hence, we end up with a set $G \cup H$ for which $PP(G) = \emptyset$ and every element in $JP(G)$ is involutively covered or involutively super reducible by $G \cup H$. Applying the involutive J-criterion (II), we are done with this part.

Next, we discuss if $Q = \emptyset$ is only the case because all elements that should be contained in Q have a degree in the v-part that is greater than $q + 2$ and therefore have been removed by step 2. Because we have come so far in the algorithm without getting an error message, every element $(\boldsymbol{u}, v)$ with $\deg_F(\mathrm{lt}(\boldsymbol{u})) = \deg(\mathrm{lt}(v)) \leq q+1$ has been considered. So, if no error message stopped the algorithm (as it must be true for this case) I and $Syz(F)$ both are in quasi-stable position, as the Pommaret bases do not contain elements with a greater degree than $q + 1$ by our thoughts in Remark 3.23. Hence, the algorithm has provided us a strong P-basis up to degree $q + 1$. And according to our observations in Remark 3.23 the output is correct.

If we get the error message that $Syz(F)$ is not in quasi-stable position this means that we have added a syzygy $(\boldsymbol{u}', 0)$ to H. In particular, the syzygy is not involutive reducible by H as otherwise p would have been involutively covered by H in step 5. So, we have extended the involutive cone of $\mathrm{lt}(H)$. But every Pommaret basis of $Syz(F)$ must consist of elements with $\deg_F(\mathrm{lt}(\boldsymbol{u}')) \leq q + 1$ as we have seen above. Because of the strategy of smallest signature, $\mathrm{lt}(\boldsymbol{u}')$ will still be involutively irreduible by any H that we compute during the algorithm as $\operatorname{lt}(\boldsymbol{u}') \prec \operatorname{lt}(\boldsymbol{u})$ for all $\boldsymbol{u} \in H$ that were added after $\boldsymbol{u}'$. Because the signatures of considered pairs are ascending by Lemma 3.24 and there are only finitely many missing for a strong P-basis up to the degree $q+1$ (we have a term order of type ω), $Syz(F)$ cannot be in quasi-stable position[12] and the error message is legit.

For the last case, we assume that the algorithm finished with an error message for I. Here, we have similar arguments as for $Syz(F)$. We must have added an element of degree greater than q in the v-part which only can add elements to Q with at least degree $q+1$ in the v-part. However, all elements that are interesting for a Pommaret basis of I have degree less than $q+1$ (and elements of degree $q+1$ must involutively reduce[13] to zero by a Pommaret basis). However, as our algorithm has computed a strong P-basis up to the degree of the current signature (i.e. the one corresponding to v'), we have already a (weak) Pommaret basis of I computed as all other v-parts have degree $q+1$ or greater due to the last component of the if-statement in step 14. Still, we have added an element to G with $\deg(\operatorname{lt}(v')) > q$ which is not involutively reducible by the v-parts of G. Hence, there cannot exist a Pommaret basis of I and the corresponding error message is correct. □

Corollary 3.26. *Let $\prec_1 = \prec_{degrevlex}$ be a term order on R and $\prec_2$ a compatible term order on R^m of type ω. Then for every $p \in G$ that has been considered already, the proxy pairs are considered after finitely many steps, too.*

Proof. As by Lemma 3.24, after a finite number of steps, we look at an element with strictly larger signature, and as $\prec_2$ is of type ω, a proxy of p is considered after a finite number of steps. □

Remark 3.27. We want to mention that from this corollary one cannot conclude immediately that we are computing an involutive normal form of all considered v-parts. Because during the process of the reduction steps it may be the case that a pair does not possess a proxy because $tp_2 \notin PP(G)$. In such a case, we would stop the reductions steps. Still, this would mean that the proxy would involutively reduce to $(\mathbf{0}, 0)$ by the final $G \cup H$.

Now the next lemma[14] is the most important one to prove the termination of the semi-involutive case. Still, we will formulate the fully involutive variant. Indeed, everything we have discussed would only differ in the division on the u-part. Moreover, if we substitute "$|_{L,B_u}$" by "$|$", one can go through all the proofs and verify that they will stay the same if not become shorter. In fact, some of the proofs get easier as we do not need all of the arguments we needed for justifying why we can write "$|_{L,B_u}$". Nevertheless, as we have introduced the notions for the fully involutive case, it is more convenient to present the following lemma also with the same notions.

Lemma 3.28. *Let $p_i := (\boldsymbol{u}_i, v_i)$ be the i-th element that entered $G \cup H$ in step 12 or 17. Then, we have $p_i \nmid_P p_j$ for $i < j$. If G is involutively regular autoreduced, then $p_i \nmid_P p_j$ holds even if p_i or p_j are regular normal forms of input elements $(\boldsymbol{e}_k, f_k)$.*

Proof. First, we note that $p_i \nmid_P p_j$ for all elements in the starting set

$$G^0 := \{(\boldsymbol{e}_k, f_k) \mid k = 1, \dots, m\}$$

as $\boldsymbol{e}_i \nmid_P \boldsymbol{e}_j$ for $i < j$. This remains true after performing involutively regular autoreductions. Thus, for the rest of the proof, we can at least assume that p_j is no regular normal form of an element in G^0. Therefore, p_j has entered $G \cup H$ in step 12 or 17. Now suppose for $i < j$ we have $p_i \mid_P p_j$, i.e. $\operatorname{lt}(\boldsymbol{u}_i) \mid_P \operatorname{lt}(\boldsymbol{u}_j)$ and $\operatorname{lt}(v_i) \mid_P \operatorname{lt}(v_j)$. Therefore, there exist terms t_1, t_2 such that

$$\operatorname{lt}(v_j) = t_1 \operatorname{lt}(v_i) \quad \text{and} \quad \operatorname{lt}(\boldsymbol{u}_j) = t_2 \operatorname{lt}(\boldsymbol{u}_i).$$

[12] Otherwise the algorithm would have computed $Syz(F)$ and $\operatorname{lt}(\boldsymbol{u}')$ could not be involutively irreducible by $Syz(F)$.
[13] Here, we mean the common notion of involutive reducibility without any restrictions by a u-part.
[14] This lemma is an involutive variant of a claim embedded in Theorem 3.1 from [6].

If $t_1 \prec t_2$, then $t_1 \operatorname{lt}(\boldsymbol{u}_i) \prec t_2 \operatorname{lt}(\boldsymbol{u}_i) = \operatorname{lt}(\boldsymbol{u}_j)$. Hence, p_j is involutively regular reducible by p_i leading to a contradiction since p_j is an involutively regular normal form. Thus, $t_2 \preceq t_1$. This implies $t_2 \operatorname{lt}(v_i) \preceq t_1 \operatorname{lt}(v_i) = \operatorname{lt}(v_j)$.

Now, suppose that p_j is an involutively regular normal form of $p := (\boldsymbol{u}, v) \in JP(G) \cup PP(G)$. If p was at least once involutively regular reducible, then $\operatorname{lt}(v_j) \prec \operatorname{lt}(v)$ and hence,

$$\operatorname{lt}(\boldsymbol{u}_i) \mid_P \operatorname{lt}(\boldsymbol{u}_j) = \operatorname{lt}(\boldsymbol{u}) \quad \text{and} \quad \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_i)} \operatorname{lt}(v_i) = t_2 lt(v_i) \preceq \operatorname{lt}(v_j) \prec \operatorname{lt}(v).$$

But this means that p should have been discarded in step 5 as it is involutively covered by p_i. Thus, p is not involutively regular reducible, and in particular not an element of $PP(G)$. Therefore, $p \in JP(G)$ cannot be involutively super reducible either, because otherwise it would have been discarded and p_j would not have been calculated. So, we know that p is involutively irreducible, and hence, $p = p_j$.

We are in the case $t_2 \preceq t_1$. Suppose, we have $t_2 \prec t_1$. Then, $t_2 \operatorname{lt}(v_i) \prec \operatorname{lt}(v_j) = \operatorname{lt}(v)$ and $t_2 \operatorname{lt}(\boldsymbol{u}_i) = \operatorname{lt}(\boldsymbol{u}_j) = \operatorname{lt}(\boldsymbol{u})$. This would mean that p would be involutively covered by $p_i \in G$ and not be entering G in the first place. Accordingly, we have $t_2 = t_1$. But then, p is involutively reducible by p_i contradicting the observation that p is involutively irreducible by G. □

Lastly, we want to present a proof for the termination. For the fully involutive case, we can easily argue as we will find out in the next proposition.

Proposition 3.29. *Let P be the Pommaret division, $\prec_1 = \prec_{degrevlex}$ and $\prec_2$ a compatible term order of type ω. Then the fully involutive GVW algorithm terminates.*

Proof. As we can discard all elements of degree greater than $q + 2$, there are only finitely many terms left to consider since $T_{\leq q+2} := \{t \in \mathbb{T}_n \mid \deg(t) \leq q + 2\}$ is a finite dimensional subvector space of R. As the signature is strictly increasing after finitely many pairs that we have to consider (Lemma 3.24), we cannot look at the same pair infinitely many times. Thus, as we have term orders of type ω, the algorithm will terminate. □

We want to note here that once again; this proof is only that short because we are in the homogeneous case where have an a priori knowledge about the degree of an involutively regular normal form. Moreover, as a last remark, we want to point out the following result.

Remark 3.30. Because the proxy pairs are considered after finitely many steps by Corollary 3.26, we indeed compute all necessary involutive normal forms of v-parts: The proxy pairs are defined in such a way that the reduction steps in the v-part can continue, but instead of reducing v_1 by v_2 (assuming that $\operatorname{lt}(v_2) \mid_P \operatorname{lt}(v_1)$), we reduce tv_2 by v_1, where $t = \frac{\operatorname{lt}(v_1)}{\operatorname{lt}(v_2)}$. Nevertheless, the result of the reduction step is the same up to a constant factor. Thus, if a proxy survives in step 5 of the algorithm, we go one step further towards an involutive normal form of v_1. However, the algorithm will keep $(\boldsymbol{u}_1, v_1)$ in $G \cup H$, although the v-part is involutively reducible, and hence, v_1 will be not part of a strong Pommaret basis of I. Still, after the algorithm has returned his output, such elements are fairly easy to detect. We only have to check whether the v-part is involutively reducible. Then we know, that its proxy (if it exists) will be considered. Thus, we can just discard v_1 from the output and we will even obtain a strong Pommaret basis of I.

3.5. Semi-involutive GVW algorithm

At this stage, we are able to prove the termination for the semi-involutive case. One can verify all the results we have obtained so far also hold for the semi-involutive case. However, this might not trivially be the case since we do not only have weaker statements but also weaker assumptions. Still, all of the proofs can be just rewritten, only changing the involutive division in the u-part to the common one. In some contexts, one can even leave out arguments that were only necessary for the fully involutive case.

Therefore, we must drop the lines 9–11 from the pseudo code because we will no longer have a bound for elements in an involutive bases of $Syz(F)$. Thus, we cannot decide which elements we can neglect from Q. Still, the proof of correctness therefore becomes shorter as we neither have to discuss the case of an error message due to $Syz(F)$, nor the case that $Q = \emptyset$ because we have removed elements from Q by step 2. But this also means that the proof for termination will be more difficult because we cannot argue with finitely many elements that are left to be discussed. On the other hand, we can take in now "Noetherian" arguments. For the corresponding algorithm (dropping lines 9-11 and using the common division on the u-part), we are only able to proof termination for ideals I that are in quasi-stable position. In this case, we can even choose to only keep the signatures rather than the whole u-part as they are sufficient for applying the J-criterion.

In the other case (i.e. I is not in quasi-stable position), we instead will focus on computing a Pommaret basis of I. Here we have to work with the whole u-part because we cannot gain any information about the transformed syzygies when we only keep the signatures.

Now, as we have a degree bound for the v-parts, we can now neglect all elements in Q with a degree greater than $q + 1$. Note, that this implies, that we may have not found all syzygies of the Gröbner basis of $Syz(F)$, when we interrupt the algorithm at this degree bound. Nevertheless, this is not a problem for our case where we start with the Janet version of the GVW algorithm[15]. This algorithm will provide us a Gröbner basis of $Syz(F)$.

Now, our semi-involutive algorithm for the Pommaret division arises as follows from the fully involutive version:

- *Optional:* Only keep signatures rather than the whole u-part, if I is in quasi-stable position.
- Use the common division for the u-parts.
- Drop lines 9–11.

a) If I is in quasi-stable position, we do not have to neglect any elements from Q.
b) If I is not in quasi-stable position, we neglect all elements from Q with degree greater than $q + 1$ in the v-part.
c) If we cannot decide whether or not I is in quasi-stable position: If we are interested in a Gröbner basis of $Syz(F)$, we follow a), otherwise we do what is stated in b). (We implemented the idea in b).)

- Return a weak Pommaret basis of I, and a subset of a Gröbner basis of $Syz(F)$ or of $Sig(G)$.

Theorem 3.31. *Let P be the Pommaret division, $\prec_1 = \prec_{degrevlex}$ and $\prec_2$ a compatible term order of type ω. Then the semi-involutive GVW algorithm terminates.*

Proof. We prove that the algorithm terminates independently from our choice of using only signatures or neglecting elements from Q. So, we assume for this proof that we keep the whole u-part.

Suppose that the algorithm does not terminate. Then we obtain in the notation of Lemma 3.28

$$\begin{aligned} \langle \boldsymbol{u}_1 \rangle \subseteq \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle \ &\subseteq \dots \\ \langle v_1 \rangle_P \subseteq \langle v_1, v_2 \rangle_P \ &\subseteq \dots, \end{aligned}$$

where $p_i := (\boldsymbol{u}_i, v_i)$ and at least one "$\subseteq$" at same height is a "$\subsetneq$". As the upper chain must become stationary, the chain of the v-parts must be strictly ascending at some point.

Now assume, that a finite Pommaret basis exists. As the signatures are increasing by Lemma 3.24 and we have there a term order of type ω, too, we need only finitely many iterations to get beyond the degree q in the v-part. Hence, the algorithm has computed a weak Pommaret basis according to Remark 3.23. Thus, the lower chain cannot be strictly increasing, either.

On the other side, if I is not in quasi-stable position, by [18, Prop. 5.3.7] there must exist an element which has a degree greater than q and yet increase the involutive cone of the current $\mathrm{lt}(G)$.

[15] We will introduce it in Section 4.

This will still be the case after we have considered all elements in V with degree less than $q + 1$. Thus, our algorithm will terminate with an error message that I is not in quasi-stable position. □

Before we discuss the case where we need coordinate transformations we want to give a last remark.

Remark 3.32. All syzygies we find, will be encoded in some $\boldsymbol{u}_j$. But still, we would not be able to argue for the fully involutive case that the upper chain must become stationary as not all $\boldsymbol{u}_i$ refer to a syzygy. In fact, we have no good argument for the termination of the fully involutive algorithm at this point if we would not neglect all elements from Q with a degree greater than $q + 2$, because we have no better argument, yet, that both chains must become stationary besides cutting them off at the degree bound. Thus, this proof is only valid for the semi-involutive case.

3.6. Coordinate Transformations and Index of Safety

Now, we are ready to show a version of the involutive GVW algorithm with coordinate transformations. We will only discuss here the fully involutive algorithm, as it is more complex because we have to put $Syz(F)$ in quasi-stable position, too.

By a private communication from Matthias Orth, it is known that we can transform I and $Syz(F)$ step by step simultaneously into quasi-stable position. In general, we then have to restart the algorithm after each step. However, as syzygies will transform into syzygies, we can use them for our J-criteria. Hence, the strength of the involutive GVW algorithm now becomes clear. So, many of the J-pairs might be involutively covered. Also, the set of proxy pairs might not be as big as it would be without the syzygies that we obtained from the transformation of the old syzygies. Thus, we should also work on the problem, where we need to start over after a transformation. In particular, we will introduce an *index of safety* in this subsection.

Remark 3.33. Basically, we are using the involutive GVW algorithm trying to compute a Pommaret basis of I and $Syz(F)$. For this, we first calculate a Janet basis of I, obtaining a degree bound q for elements in a Pommaret basis of I (see [18, Cor. 5.5.18]). In fact, we can take the involutive GVW algorithm for the Janet division which we will present in the next section. If the Janet basis is also a Pommaret basis, we are done with I. So let's assume, that it is not a Pommaret basis. Then, because we only increase the leading term of the v-parts via $JP(G)$, we check which involutive J-pairs $x_k p \in JP(G)$ are not involutively reducible by $G \cup H$ and satisfy $\deg(\operatorname{lt}(x_k v)) > q$ or $\deg_F(\operatorname{lt}(x_k \boldsymbol{u})) > q + 1$. If so, we need to perform a coordinate transformation on the u- or v-part. Let us discuss here the v-part, the u-part works similar[16]. Let $j = cls(\operatorname{lt}(v))$. Then we transform by $x_j \mapsto x_j + x_k$. However, this forces us to start our calculations all over again. Also, we may ask ourselves, if we need to perform the transformation directly after finding such an involutive J-pair, or if we can push it back a little until we cannot do anything else but to transform the system. In fact, our algorithm does exactly this: According to steps 9 and 14, we return an error message if and only if I or $Syz(F)$ are not in quasi-stable position[17]. But in such a case, we perhaps can choose between several coordinate transformations after analyzing Q. So, we have to face the question which of the possible transformations is the best in the context of computational efficiency. Nevertheless, we have to discuss what happens after the transformation. There, we just transform F and all syzygies with a suitable coordinate transformation. Then we take it as an input for the Pommaret version of the GVW algorithm. Here of course, one is not forced to take the fully involutive variant, however, we will argue our strategy only for the fully involutive algorithms. The ideas can easily be adapted for the semi-involutive algorithms. In particular, this means for the pseudo code the following:

[16] Also, we will discuss later in this remark how to deal with an error message for $Syz(F)$ coming from the Pommaret version of the algorithm.

[17] For the argument corresponding to $Syz(F)$, one may look up the arguments in the proof of correctness of the (fully) involutive GVW algorithm.

Instead of returning an error message for I in step 15, we go through the elements in Q and check if their v-part is divisible by the v-part of a $p = (\boldsymbol{u}, v) \in G$, but involutively irreducible by G. If this is the case, we choose p with maximal $\mathrm{cls}(\mathrm{lt}(v))$. Then every J-pair of p leads to a candidate for a coordinate transformation as we have described above.

If, on the other hand, an error message for $Syz(F)$ has been returned in step 10, then we go through the signatures of Q and check if they are divisible by some $p' = (\boldsymbol{u}', v') \in G \cup H$, but not involutively reducible by the signatures of any element in $G \cup H$. Then we take under all p' satisfying this condition the one with maximal $j := \mathrm{cls}(\mathrm{lt}(\boldsymbol{u}'))$. For this signature, we have non-multiplicative variables x_k. Therefore, potential coordinate transformations are of the form $x_j \mapsto x_j + x_k$.

After gathering all possible coordinate transformations ψ_i, we do the following:

1. Transform $G \cup H$ with ψ_i into $G' \cup H'$.
2. Perform an involutively regular autoreduction of G' and insert obtained syzygies into H'.
3. Compute $Q' := JP(G') \cup PP(G')$ for the involutively regular autoreduced G' and sort it first by signature, then by the leading terms of the v-part.
4. Search the position s of the first element in the sorted Q' that cannot be discarded due to our criteria in step 5.

In 4., s is of course dependent of ψ_i, i.e. we better write $s(\psi_i)$. The largest value of $s(\psi_i)$ is called *index of safety*. It is so to speak the latest possible starting point of the algorithm after a coordinate transformation. After we have found the index of safety, we can continue the involutive GVW algorithm at step 7, taking the element in Q' at the position $\max_i \{s(\psi_i)\}$ and neglect all elements from Q' with smaller signature.

Although this strategy is straight forward to see, it might not be the best if it comes to an efficient implementation. For such one, a bigger analysis is needed. For instance, if only a few elements would not be discarded after a coordinate transformation and by accident, there is one with small signature (so the index of safety is small), one could try to analyze when the corresponding transformation is still a better choice than the one related to the index of safety. However, this might be a difficult question to answer.

Remark 3.34. It is easy to see that a POT-lift is not of type ω. However, as we have a degree bound for $Syz(F)$ in the fully involutive variant as well, we also can use a POT-lift of a term order $\prec_1$ of type ω with the following restriction: We just jump to $\boldsymbol{e}_{i+1}$ if the signature at position i exceeds degree $q+1$. Then we know, that no element at position i is of interest for our Pommaret bases, and thus can be pushed back for the moment. Thereby, we ensure that between two terms, there are only finitely many other pairs that the algorithm will consider. Whenever we say, we choose a ***POT-lift of pseudo type*** ω we mean to follow this strategy. It is worth mentioning that our algorithm follows this strategy for a POT-lift input as all elements above the degree $q+2$ are neglected and an error message only occurs if there are no elements $(\boldsymbol{u}, v)$ left with $\deg_F(lt(\boldsymbol{u})) \leq q+1$. Thus, with the POT-lift, we would not go any further in position i than to the degree $q+2$. However, elements with the degree $q+2$ that lead to an error message should be pushed back first until there are no other elements left with that degree. It is worth mentioning that this feature is not implemented in our code, yet. Therefore, a POT-lift in our implementation might generate an error for ideals that are not in quasi-stable position.

We first want to give an example that it is not guaranteed that a finite Pommaret basis of $Syz(F)$ exists only because there is one for I.

Example. $I := \langle x, y \rangle \trianglelefteq K[x, y]$, where we respect the ordering of x and y, which means that we have $G := \{(\boldsymbol{e}_1, x), (\boldsymbol{e}_2, y)\}$. With a POT-lift obeying $\boldsymbol{e}_1 \prec_{POT} \boldsymbol{e}_2$, the syzygy module is generated by $x\boldsymbol{e}_2 - y\boldsymbol{e}_1$ and thus, possesses no finite Pommaret basis.

Still, the semi-involutive algorithm here once again shows it benefits. We have to perform no coordinate transformations on the u-part, which would be necessary in the fully involutive version.

Also, the J-criteria will discard more potentially superfluous elements. But of course, we have also pointed out the disadvantages of the semi-involutive algorithm, where we might not compute the Gröbner basis of $Syz(F)$ completely before interrupting the algorithm.

4. Algorithm: Strong J-basis

The Janet division is not global and therefore we cannot be sure without further investigation that an involutive J-pair will be covered by a final G just because it was covered by a subset of it. And of course, we would build up G with the same strategy as in the Pommaret case, where we neglect involutively covered J-pairs. Fair enough, a J-pair that is involutively covered by the computed involutively regular normal form (which will be added to G) will always be involutively covered by it regardless which other elements are added to G. However, this might not be true for syzygies. In the semi-involutive variant, on the other hand, we have the ordinary division in the u-part which helps us finding superfluous J-pairs. Therefore, it is more convenient to use the semi-involutive variant of the GVW algorithm. Furthermore, we will be able to show that we do not need the concept of proxy pairs. We could go straight forward to the main theorem of this section. But before we do, we prove a lemma that points out the special relation between the Janet division and reduction steps on M.

Lemma 4.1. *Let $G \subseteq M$ be finite. If G is involutively autoreduced w.r.t. the Janet division then the v-parts of G are involutively head autoreduced.*

Proof. We follow an indirect proof. Suppose, there are two pairs $p_i := (\boldsymbol{u}_i, v_i) \in G$ for $i = 1, 2$ such that $\operatorname{lt}(v_1) \mid_{J,B_v} \operatorname{lt}(v_2)$. Because $p_i \in G$, we get $\operatorname{lt}(v_1) = \operatorname{lt}(v_2)$[18]. Now, if we have $\operatorname{lt}(\boldsymbol{u}_1) \neq \operatorname{lt}(\boldsymbol{u}_2)$, then p_1 is involutively regular reducible by p_2 or vice versa. But if the signatures are equal, then both possible reduction steps are obviously involutive in the sense of Definition 3.1. □

As we are performing only involutively regular reductions G might not be involutively autoreduced. But then we could conclude that the v-parts of G are involutively head autoreduced, which is our assumption of the first involutive J-criterion. But still, we can achieve our goal by assuming that G is involutively regular autoreduced – similar to the assumptions in the second J-criterion.

Like we did it in the previous section, we will present a proof for the fully involutive case, however we will use arguments that remain valid for the semi-involutive case.

Like in the Pommaret case we first prove a rather technical lemma corresponding to Lemma 3.11.

Lemma 4.2. *Let J be the Janet division. Let $G \subseteq M$ be a finite set. Suppose that every J-pair in G is involutively covered or involutively super reducible by G. Let $(\boldsymbol{u}, v) \in M$ be non-zero and suppose there is a pair $p_1 := (\boldsymbol{u}_1, v_1) \in G$ with $v_1 \neq 0$ such that*

(i) $\operatorname{lt}(\boldsymbol{u}_1) \mid_{J,B_u} \operatorname{lt}(\boldsymbol{u})$ *and*

(ii) $t \operatorname{lt}(v_1) := \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_1)} \operatorname{lt}(v_1)$ *is minimal among all elements in G that satisfy condition (i).*

If t contains a non-multiplicative variable for $\operatorname{lt}(v_1)$ then there exists a pair $p'' := (\boldsymbol{u}'', v'') \in G$ such that $v'' \neq 0$ and tp_1 is involutively super reducible by p''.

Proof. If t contains a non-multiplicative variable for $\operatorname{lt}(v_1)$, this means that there is a $x_k \in \operatorname{supp}(t)$, such that $x_k(\boldsymbol{u}_1, v_1)$ is an involutive J-pair. If this J-pair is involutively covered by G, by definition, there is a $(\boldsymbol{u}_2, v_2) \in G$ such that

$$\operatorname{lt}(\boldsymbol{u}_2) \mid_{J,B_u} x_k \operatorname{lt}(\boldsymbol{u}_1) \quad \text{and} \quad \frac{x_k \operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}_2)} \operatorname{lt}(v_2) \prec x_k \operatorname{lt}(v_1). \tag{4.1}$$

Because t is a multiplicative term for $\operatorname{lt}(\boldsymbol{u}_1)$, x_k is multiplicative, too. Thus, $\operatorname{lt}(\boldsymbol{u}_1) \mid_{J,B_u} x_k \operatorname{lt}(\boldsymbol{u}_1)$. But as we have $\operatorname{le}(\boldsymbol{u}_1), \operatorname{le}(\boldsymbol{u}_2) \in B_u$, $\operatorname{lt}(\boldsymbol{u}_1) = \operatorname{lt}(\boldsymbol{u}_2)$ must hold. In particular this means that

[18] This is a property of the Janet division [18, p. 67].

$\operatorname{lt}(\boldsymbol{u}_2) \mid_{J,B_u} t \operatorname{lt}(\boldsymbol{u}_1)$ is true. We multiply both sides of the second relation in (4.1) with $\frac{t}{x_k}$ which leads to a contradiction of the choice of $(\boldsymbol{u}_1, v_1)$.
Thus, the J-pair is involutively super reducible. Then, there is a $p_3 := (\boldsymbol{u}_3, v_3) \in G$ with $v_3 \neq 0$ (as otherwise the involutive J-pair would be involutively covered by it), such that

$$\operatorname{lt}(\boldsymbol{u}_3) \mid_{J,B_u} x_k \operatorname{lt}(\boldsymbol{u}_1),$$

from which $\operatorname{lt}(\boldsymbol{u}_3) = \operatorname{lt}(\boldsymbol{u}_1) \mid_{J,B_u} t \operatorname{lt}(\boldsymbol{u}_1)$ follows. Also, we have

$$\operatorname{lt}(v_3) \mid_{J,B_v} x_k \operatorname{lt}(v_1).$$

If $\operatorname{lt}(v_3) \nmid_{J,B_v} t \operatorname{lt}(v_1)$, we can iterate these arguments, taking now a variable x_h in $\operatorname{supp}(\frac{t}{x_k})$ such that $x_h p_3$ is an involutive J-pair of p_3. Then there exists a pair $p_4 := (\boldsymbol{u}_4, v_4)$ with $v_4 \neq 0$ such that $x_h p_3$ is involutively super reducible by p_4.

This again implies $\operatorname{lt}(\boldsymbol{u}_4) = \operatorname{lt}(\boldsymbol{u}_3) = \operatorname{lt}(\boldsymbol{u}_1) \mid_{J,B_u} t \operatorname{lt}(\boldsymbol{u}_1)$ (as $\operatorname{le}(\boldsymbol{u}_4), \operatorname{le}(\boldsymbol{u}_3) \in B_u$). Furthermore, we have $\operatorname{lt}(v_4) \mid_{J,B_v} x_h \operatorname{lt}(v_3) \mid_{J,B_v} x_h x_k \operatorname{lt}(v_1)$. If even $\operatorname{lt}(v_4) \mid_{J,B_v} t \operatorname{lt}(v_1)$ holds, we are done because $x_h p_3$ and $t p_1$ have the same leading coefficients. Eventually, we construct an element $(\boldsymbol{u}'', v'') \in G$ after at most $\deg(t)$ steps such that $v' \neq 0$ and $t(\boldsymbol{u}_1, v_1)$ is involutively super reducible by $(\boldsymbol{u}'', v'')$. □

Theorem 4.3 (Involutive J-Criterion (III)). *Let J denote the Janet division. Let $G \subseteq M$ be finite and involutively regular autoreduced. Moreover, assume that $\langle Sig(G)\rangle_J = \mathbb{T}_n^m$. Then the following statements are equivalent.*

a) G is a strong J-basis of M.
b) Every involutive J-pair of elements of G is eventually involutively super reducible by G.
c) Every involutive J-pair of elements of G is involutively covered by G or involutively super reducible by G.

Proof. Because of Lemma 3.10 we only have to show "c) ⇒ a)". Basically, we follow the proof for the Pommaret case. However, this time, many things will be easier.

We give again a proof by contradiction. For this purpose, suppose that G is not a strong J-basis of M and that c) holds. Then, by definition of a strong J-basis we know: There must exist a pair $(\mathbf{0}, 0) \neq (\boldsymbol{u}, v) \in M$ which is not involutively reducible by G. We take the one with smallest signature. We set $T := \operatorname{lt}(\boldsymbol{u})$ and observe that $T \neq \mathbf{0}$ as otherwise v would be 0, too. Now, as $\langle \operatorname{Sig}(G)\rangle_J = \mathbb{T}_n^m$ is true by our assumptions, we can choose a pair $(\boldsymbol{u}_1, v_1) \in G$ with the following two properties:

(i) $\operatorname{lt}(\boldsymbol{u}_1) \mid_{J,B_u} \operatorname{lt}(\boldsymbol{u})$ and
(ii) $t \operatorname{lt}(v_1) := \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_1)} \operatorname{lt}(v_1)$ is minimal under all elements in G that satisfy condition (i).

Note, that $v_1 \neq 0$ as otherwise $(\boldsymbol{u}, v)$ would be involutively reducible by a syzygy $(\boldsymbol{u}_1, 0)$ due to condition (i).

We again claim that $t(\boldsymbol{u}_1, v_1)$ is not involutively regular reducible by G. Suppose that this is not true. Then there is a $p_2 := (\boldsymbol{u}_2, v_2) \in G$ such that $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by p_2. $t = 1$ is impossible since G is involutively regular autoreduced. Also if $t \neq 1$ contains only multiplicative variables for $\operatorname{lt}(v_1)$, we obtain $\operatorname{lt}(v_1) \mid_{J,B_v} t \operatorname{lt}(v_1)$ and $\operatorname{lt}(v_2) \mid_{J,B_v} t \operatorname{lt}(v_1)$, and hence $\operatorname{lt}(v_2) \mid_{J,B_v} \operatorname{lt}(v_1)$ or $\operatorname{lt}(v_1) \mid_{J,B_v} \operatorname{lt}(v_2)$. In either of these both cases we obtain $\operatorname{lt}(v_1) = \operatorname{lt}(v_2)$. Therefore, applying Lemma 3.15, we know that even p_1 is involutively regular reducible by p_2 which is impossible. Then, $t \neq 1$ must contain a non-multiplicative variable. Lemma 4.2 tells us, that there is a $(\boldsymbol{u}'', v'') \in G$ such that $t p_1$ is involutively super reducible by it. Thus,

$$\operatorname{lt}(\boldsymbol{u}'') \mid_{J,B_u} t \operatorname{lt}(\boldsymbol{u}_1) \quad \text{and} \quad \operatorname{lt}(v'') \mid_{J,B_v} t \operatorname{lt}(v_1) \quad \text{and} \quad \frac{t \operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}'')} = \frac{t \operatorname{lt}(v_1)}{\operatorname{lt}(v'')}$$

is true. As seen above, we can conclude $\operatorname{lt}(\boldsymbol{u}'') = \operatorname{lt}(\boldsymbol{u}_1)$. But then we get from $\frac{t\operatorname{lt}(\boldsymbol{u}_1)}{\operatorname{lt}(\boldsymbol{u}'')} = \frac{t\operatorname{lt}(v_1)}{\operatorname{lt}(v'')}$ the equality $\operatorname{lt}(v'') = \operatorname{lt}(v_1)$. Thus $\operatorname{lt}(v_1) = \operatorname{lt}(v'') \mid_{J,B_v} t\operatorname{lt}(v_1)$. But then t must be a multiplicative term for $\operatorname{lt}(v_1)$ contradicting that we have assumed for this case that t contains a non-multiplicative variable for $\operatorname{lt}(v_1)$.

This is telling us that $t(\boldsymbol{u}_1, v_1)$ is not involutively regular reducible by G. Next, we follow the strategy from the Pommaret case, setting $c := \frac{\operatorname{lc}(\boldsymbol{u})}{\operatorname{lc}(\boldsymbol{u}_1)}$ and

$$(\boldsymbol{u}', v') := (\boldsymbol{u}, v) - ct(\boldsymbol{u}_1, v_1).$$

First, we observe that $\operatorname{lt}(\boldsymbol{u}') \prec \operatorname{lt}(\boldsymbol{u}) = T$. For the v-part, there are several cases to consider.

If $\operatorname{lt}(v) \neq t\operatorname{lt}(v_1)$, i.e. $v' \neq 0$, then we argue as follows: Because $(\boldsymbol{u}', v')$ has a smaller signature than $(\boldsymbol{u}, v)$ it must be involutively reducible by G. For the moment, we reduce by syzygies if possible. Doing so, we only can reduce the signature, and hence, the remainder is still involutively reducible by G. But now, it is involutively reducible by a pair $(\boldsymbol{u}_3, v_3)$ with $v_3 \neq 0$. Also note that v' has not been changed during the reduction process so far. Since $\operatorname{lt}(v) \neq t\operatorname{lt}(v_1)$, there are two cases.

- If $\operatorname{lt}(v) \prec t\operatorname{lt}(v_1)$ is true, then we have $\operatorname{lt}(v') = t\operatorname{lt}(v_1)$. Hence, we get the relations

  $$\operatorname{lt}(v_3) \mid_{J,B_v} \operatorname{lt}(v') = t\operatorname{lt}(v_1) \quad \text{and} \quad \frac{t\operatorname{lt}(v_1)}{\operatorname{lt}(v_3)}\operatorname{lt}(\boldsymbol{u}_3) \preceq \operatorname{lt}(\boldsymbol{u}') \prec T = t\operatorname{lt}(\boldsymbol{u}_1),$$

  which implies that $t(\boldsymbol{u}_1, v_1)$ is involutively regular reducible by G leading to a contradiction to our result above.
- If, on the other hand, $t\operatorname{lt}(v_1) \prec \operatorname{lt}(v)$ is true, then we get $\operatorname{lt}(v') = \operatorname{lt}(v)$. Therefore we obtain

  $$\operatorname{lt}(v_3) \mid_{J,B_v} \operatorname{lt}(v') = \operatorname{lt}(v) \quad \text{and} \quad \frac{\operatorname{lt}(v)}{\operatorname{lt}(v_3)}\operatorname{lt}(\boldsymbol{u}_3) \preceq \operatorname{lt}(\boldsymbol{u}') \prec T = \operatorname{lt}(\boldsymbol{u}),$$

  which now implies that $(\boldsymbol{u}, v)$ is involutively regular reducible by G leading once again to a contradiction since $(\boldsymbol{u}, v)$ is not involutively reducible by G due to our assumptions from the beginning of this proof.

Accordingly, there is only one possibility left, i.e. we have $\operatorname{lt}(v) = t\operatorname{lt}(v_1)$. If $t = 1$ or if $t \neq 1$ is a multiplicative term for $\operatorname{lt}(v_1)$, then $\operatorname{lt}(v_1) \mid_{J,B_v} \operatorname{lt}(v)$, $\operatorname{lt}(\boldsymbol{u}_1) \mid_{J,B_u} \operatorname{lt}(\boldsymbol{u})$ and $\frac{\operatorname{lt}(v)}{\operatorname{lt}(v_1)} = \frac{\operatorname{lt}(\boldsymbol{u})}{\operatorname{lt}(\boldsymbol{u}_1)} = t$ and hence, $(\boldsymbol{u}, v)$ is involutively reducible by $(\boldsymbol{u}_1, v_1) \in G$. But this is not possible as $(\boldsymbol{u}, v)$ is involutively irreducible by G. So $t \neq 1$ has at least one non-multiplicative variable for $\operatorname{lt}(v_1)$. Applying Lemma 4.2 we obtain a pair $(\boldsymbol{u}_4, v_4) \in G$ such that $t(\boldsymbol{u}_1, v_1)$ is involutively super reducible by $(\boldsymbol{u}_4, v_4)$. But because of $t\operatorname{lt}(\boldsymbol{u}_1) = \operatorname{lt}(\boldsymbol{u})$ and $t\operatorname{lt}(v_1) = \operatorname{lt}(v)$, this implies that $(\boldsymbol{u}, v)$ is involutive reducible by $(\boldsymbol{u}_4, v_4)$, which is a contradiction to our choice of $(\boldsymbol{u}, v)$. □

Although we have proven the fully involutive version of this theorem, from now on we focus on the semi-involutive variant. The proof can easily be adapted. It is also a strength of the Janet version that we do not have to consider a set $PP(G)$. As we have mentioned already in the introduction, Binaei et al. have presented similar theorem in [1], that only differs in the second condition in the statement c) for the Janet division. Although we have discussed that their theorem is not correct (see last paragraph in Remark 3.13), the proof of termination remains valid for the Janet division. Therefore, termination of the following algorithm follows from [1, Thm. 6].

Thus, we end up with the following pseudo code, where we mark notions that must be treated differently in the semi-involutive case. For instance, "involutively∗ super reducible" only differs from the notion we have introduced by the division in the u-part. In the semi-involutive case we of course take the common division rather than the Janet division. The proof of correctness follows immediately from the involutive J-criterion (III) (Theorem 4.3).

SemiInvGVW$(F, H_0, \prec_1, \prec_2, J)$ (Janet Version)	
Input:	A set $F = \{f_1, \dots, f_m\} \subseteq R$ of polynomials, $\prec_1$ on R and a compatible term order $\prec_2$ on R^m, J Janet division, An involutively autoreduced set H_0 of syzygies of F, where $H_0 = \emptyset$ is possible.
Output:	A weak Janet basis for $I = \langle F \rangle$ and a Gröbner basis of $Syz(F)$.
Variables:	G is an ordered set of pairs $(\boldsymbol{u}_i, v_i) \in M$ with $v_i \neq 0$. H is an ordered set of syzygies $(\boldsymbol{u}, 0)$ of F. $JP(G)$ is the set of involutive J-pairs of G.
Step 1:	$G \leftarrow \{(\boldsymbol{e}_i, f_i) \mid 1 \leq i \leq m\}$, $H \leftarrow H_0$
Step 2:	Perform an involutively regular autoreduction on G. Fill H with obtained syzygies, discard them from G. Calculate (new) trivial syzygies of G and add them to H. Autoreduce H.Fill $JP(G)$ with (new) involutive J-pairs of G.
Step 3:	**while** $JP(G) \neq \emptyset$ **do**
Step 4:	Take an element $p := (\boldsymbol{u}, v) \in JP(G)$ with smallest signature and then with smallest leading term in the v-part.
Step 5:	If • p is covered by $G \cup H \cup JP(G) =: S$, or • p is involutively$*$ super reducible by $S \setminus \{p\}$, then discard p and go back to step 3.
Step 6:	Calculate an involutively regular normal form $(\boldsymbol{u}', v')$ of p by G
Step 7:	**If** $v' = 0$ **then**
Step 8:	$H \leftarrow H \cup \{(\boldsymbol{u}', 0)\}$
Step 9:	**else**
Step 10:	$G \leftarrow G \cup \{(\boldsymbol{u}', v')\}$. Go back to step 2.
Step 11:	**end if**
Step 12:	**end while**
Return:	$\{v_i \mid (\boldsymbol{u}_i, v_i) \in G\}$ and $\{\boldsymbol{u} \mid (\boldsymbol{u}, 0) \in H\}$

5. Remarks on Implementation

In this section, we discuss our proof of concept implementation[19] in Maple 2019. For the implementation we use the package “Groebner” to have access to some optimized functions that test, for example, which of two given terms is larger. First, we want to mention how an element $(\boldsymbol{u}, v) \in G \cup H$ is stored. Assume that $\mathrm{lm}(\boldsymbol{u}) = ct\boldsymbol{e}_i$ for some $c \in K$ and $t \in \mathbb{T}_n$. We store vectors in lists and $(\boldsymbol{u}, v)$ is represented by

$$[[c, t, i, \boldsymbol{u}], [\mathrm{lc}(v), \mathrm{lt}(v), v], \bar{X}_P(\mathrm{lt}(v))]].$$

We have implemented a sort function *FHelp* that sorts elements in G by signature and then by the leading term of the v-part. This helps us to reduce the computational time for involutively regular reduction steps. As regular reductions cannot increase the signature, we have implemented the function *FindIndex* that finds the largest element (by signature) we have to consider for (involutive) regular reductions.

We have then implemented the fully and semi-involutive GVW algorithm for the Pommaret division according to our presented pseudo code with additional coordinate transformations as described in Remark 3.33. Here we want to point out, that only a TOP-lift variant of the algorithm should be used when coordinate transformations are needed, even though the POT-lift variant works

[19] This means that in its current form the implementation is not yet optimised and will have problems with larger examples. The Maple codes of our programs are available at http://www.mathematik.uni-kassel.de/~izgin/publications.php?lang=en.

in some cases. But as described earlier it would take additional effort to implement an error free version of it.

As we wanted to check computationally that no involutively super autoreductions are needed for H, we decided not to insert trivial syzygies to H. Nevertheless, involutively super autoreductions will be necessary when coordinate transformations are needed. And in such cases, these reduction steps will be performed. We have decided to fill $JP(G)$ with all possible J-pairs in order to compare the TOP- and POT-lifts in our implementation. Nevertheless, other strategies can be implemented fairly easy by just commenting (out) some lines in the code.

Still, this means that we have computed involutive J-pairs of an element in G that may get involutively regular reduced before we have to consider its J-pairs. Thus, we would have spent time in computing the involutive J-pair and finding out that it is involutively covered by the computed involutively regular normal form.

We also have not implemented step 6 from the pseudo code as it is only a small optimization which seems not to apply very often. However, the implementation of the corresponding function requires for-loops that may take too much time, relatively speaking. The same seems to hold for finding the index of safety (with our approach) but we decided to keep the function active in order to see the additional computational effort.

Nevertheless, we have done some optimizations to our code by using the functions *FindIndex* and *FHelp*. Also, it might be useful to only keep the signatures once we have exceeded the degree limit $q + 1$ since all following elements are not of interest to us. This might save time for the calculations in the degree $q + 2$.

The strength of the implementation is the following. One can choose between a TOP-lift (encoded in $ord = 2$) and a POT-lift ($ord = 1$), the full ($Syzbool = true$) and semi-involutive variant ($Syzbool = false$) and between keeping the whole u-part ($SigOnly = false$) or only the signatures ($SigOnly = true$).

Then, one can call the algorithm by the function $StrPBas(F, H_0, Syzbool, q, false)$, where H_0 is an (involutively) autoreduced set of syzygies and q a degree bound for elements in a Pommaret basis of I. The last entry is set to be *false* initially. It will be set to be *true* after we have found out, that a coordinate transformation is required. Then we call *StrPBas* recursively with the last parameter, called *RestartBool*, being *true*.

After the algorithm has returned a result G and H it should be tested. If $Syzbool = true$ and $SigOnly = false$, i.e. we are in the fully involutive case where we keep the whole u-part, the function *TestBasis* computes all non-multiplicative prolongations of any syzygy from H and performs involutive reductions. If and only if all these reductions end with a zero vector the message *"True for syzygy module"* is printed. This pays respect to the fact that the function does not test whether we have found a generating system of $Syz(F)$. If this error message is not printed, H is not a weak Pommaret basis of $Syz(F)$, and thus, this is what will be printed instead. Moreover, *TestBasis* does the same check for G regardless of what choice of parameters we have set (this is meaningful as we always aim to compute a Pommaret basis of I). Here, we print *"Output contains a Pommaret basis for the ideal"*, if and only if all reduction steps return the remainder 0 which takes two things into account: First, we know that we have a generating system as we have started with $(\boldsymbol{e}_i, f_i)$, $1 \leq i \leq m$. Secondly, it contains the result of Remark 3.30. Indeed, the detection of the negligible elements mentioned in that remark is already implemented at the end of *StrPBas*. Hence, the function returns a (strong) Pommaret basis of I. If G is not a weak Pommaret basis, the algorithm will detect it and print a corresponding message. It is worth mentioning that the non-involutive GVW algorithm may not have a reduced Gröbner basis contained in its output as there is no set $PP(G)$ that ensures that proxy pairs will be reduced.

6. Benchmarks

Although we have presented the proof of termination only for homogeneous inputs, we will present benchmark calculations with affine inputs. Here, we are not comparing with algorithms that only aim to compute a Pommaret basis or a Janet basis of I as our algorithm does more than this. Also, a comparison would be not fair since our implementation is far away from being completely optimized. Instead, we make some statistics about the different variants that are provided by our implementation. Finally, we can compare it with the Janet version of the semi-involutive variant that has been implemented by Binaei. As only signatures are saved in this implementation, we shall compare the computational time with our algorithm where we choose $SigOnly = true$ and $Syzbool = false$ (i.e. we store only signatures and use the semi-involutive variant of the algorithm). And because the algorithm does only aim to compute a Janet basis of I, we only compare zero dimensional ideals, which implies that no coordinate transformation will be required and both algorithms will compute a Pommaret basis.

However, we surely want to investigate with at least a few examples of how our algorithm works when a coordinate transformation is needed. We return the coordinate transformations (from top to bottom), the maximal value for the index of safety[20] (referred to as "max. i.o.s."), the number of syzygies, the number of elements in the Pommaret basis and the used value for q which encodes the Castelnuovo-Mumford regularity $Reg(I)$ in the related examples.

In order to underline the advantage of the algorithm when syzygies are known, we will also restart the computations with $H_0 = H$ as input, where H is a generating system or signatures of the generating system of $Syz(F)$. Thus, we have the following structure of cells:

runtime [s]: $H_0 = \emptyset$	discarded elements	regular normal forms	$\|H\|$
runtime [s]: $H_0 = H$	discarded elements	regular normal forms	

TABLE 1. Structure of cells for zero dimensional benchmark problems.

Therefore, we obtain the Tables 2 to 5 for zero dimensional benchmark problems.

POT-lift	Katsura5 (q=6)				Katsura6 (q=7)				Katsura7 (q=8)			
SigOnly=true	8.16	140	93	43	84.63	386	188	83	1241.4	1003	372	156
Syzbool=false	3.53	175	50		36.53	450	105		434.6	1113	216	
SigOnly=false	15.31	144	94	44	200.58	399	191	86				
Syzbool=true	5.03	179	50		53.5	462	105					
SigOnly=false	15.28	140	93	43	203.13	386	188	83				
Syzbool=false	5.13	175	50		53.2	450	105					

TABLE 2. Katsura benchmark runs for the POT-lift with structure as described in Table 1

One can observe that for the Chandra benchmark runs (Tables 4 and 5), the POT-lift variant is slightly faster. However, for the Katsura runs (Tables 2 and 3) the opposite is true. For Katsura7 for instance the difference is about 260s.

Also, we can say that the bigger the example is the bigger is the difference between discarded elements and thus, the amount of saved regular normal form calculations. However, if we compare the number of discarded elements from a POT-lift run with a TOP-lift run we can conclude that with a POT-lift more elements will be discarded. But as the usage of the POT-lift led to a larger runtime, it seems to be the case that too many negligible pairs were calculated. This might come from the incremental character of the GVW algorithm when a POT-lift is used.

[20]Remember that we have an index of safety for every coordinate transformation that we perform.

TOP-lift	Katsura5 (q=6)				Katsura6 (q=7)				Katsura7 (q=8)			
SigOnly=true	7.47	112	83	42	80.06	307	164	80	981.95	778	310	143
Syzbool=false	3.64	153	41		33.97	384	84		370.75	918	167	
SigOnly=false	23.03	111	84	43	364.48	304	167	83				
Syzbool=true	5.66	153	41		55.06	384	84					
SigOnly=false	22.34	112	83	42	338.84	307	164	80				
Syzbool=false	5.47	153	41		53.17	384	84					

TABLE 3. Katsura benchmark runs for the TOP-lift with structure as described in Table 1

POT-lift	Chandra4 (q=4)				Chandra5 (q=5)				Chandra6 (q=6)			
SigOnly=true	0.13	24	26	11	1.05	78	60	26	8.73	224	130	57
Syzbool=false	0.09	28	15		0.84	89	34		6.80	250	73	
SigOnly=false	0.22	24	26	11	1.52	78	60	26	12.80	224	130	57
Syzbool=true	0.16	28	15		1.02	89	34		8.58	250	73	
SigOnly=false	0.20	24	26	11	1.42	78	60	26	12.45	224	130	57
Syzbool=false	0.14	28	15		0.97	89	34		8.48	250	73	

TABLE 4. Chandra benchmark runs for the POT-lift with structure as described in Table 1

TOP-lift	Chandra4 (q=4)				Chandra5 (q=5)				Chandra6 (q=6)			
SigOnly=true	0.14	24	26	11	1.31	78	60	26	11.23	224	130	57
Syzbool=false	0.14	28	15		1.23	89	34		9.31	250	73	
SigOnly=false	0.25	24	26	11	2.02	78	60	26	19.52	224	130	57
Syzbool=true	0.16	28	15		1.41	89	34		11.98	250	73	
SigOnly=false	0.23	24	26	11	2.16	78	60	26	18.50	224	130	57
Syzbool=false	0.19	28	15		1.30	89	34		12.16	250	73	

TABLE 5. Chandra benchmark runs for the TOP-lift with structure as described in Table 1

It is also worth mentioning that the semi- and the fully involutive variants do not differ very much according to their runtimes or number of discarded elements. However, only keeping the signatures has a major impact on the runtime, at least when it comes to the Katsura benchmark runs.

Now let us compare the runtimes of our implementation with Binaei's. Remember, that we take the times from the first row as this row corresponds to the structure of Binaei's implementation.

Also, keep in mind, that instead of a POT- or TOP-lift the author chose the Schreyer ordering.

	Chandra4	Chandra5	Chandra6	Katsura5	Katsura6	Katsura7
Pommaret	0.13-0.14	1.05-1.31	8.73-11.23	7.47-8.16	80.06-84.63	981.95-1241.4
Janet	0.63	5.34	52.5	43.69	636.09	10155.31

TABLE 6. Runtime comparison for Chandra and Katsura benchmark runs with the TOP- and POT-lift between our implementation of the Pommaret version of the GVW algorithm and the Janet version with the Schreyer ordering presented in [1]

For the examples in table 6, we can conclude that our implementation is about four to five times as fast as the implementation of Binaei for the smaller examples and five to eight times as fast if it comes to Katsura runs. This is surprising in the sense that for the Pommaret case we have to consider proxy pairs. Still, some of the proxy pairs potentially will be involutive J-pairs in the Janet version of the algorithm.

Also, the answer may lie in the fact that our J-criterion potentially discards more elements than the J-criterion presented in [1].

Now, one may also think that this comes from the fact that we did not take the degree of the Janet basis as our value of q but instead searched heuristically for the Castelnuovo-Mumford regularity (which was often smaller by 1 for these examples). However, this is not true. For instance, the run for Katsura6 was repeated with $q = 8$ and finished after 91.25s with a TOP-lift. Furthermore, our algorithm only discards elements of degree that is greater than $q + 1$ (in the v-part and the semi-involutive variant). So, this cannot be the reason.

Maybe our implementation is that much faster because our term orders on R^m work better for the GVW algorithm, or because we do not compute the whole Gröbner basis of $Syz(F)$ in general. However, this is an open question at this point.

For investigating the performance of our implementation for inputs where coordinate transformations may be required, we look at the four examples in table 7. Here, we also choose *Syzbool=true* and *SigOnly=false* so that transformations of $Syz(F)$ would be detected, too. This, we mark with a "no" in the column "focus" as we do not focus on only computing a Pommaret basis of I with this choice of parameters. When we write "yes" in that column, we always set $Syzbool = false$ and, if no transformations are required, $SigOnly = true$. Otherwise $SigOnly = false$ must be chosen because we need the whole syzygy to be transformed.

As we have a TOP- and a POT-lift, we perform the computations for both term orders. In the first column we write for instance "Caprasse (T/P)" or "Caprasse (T/-)". Here, (T/P) means, that we first give the data for the TOP-lift, and separate it from the data for the POT-lift calculations by the slash-symbol. With (T/-) we mean that the computation for the POT-lift was interrupted after four hours, and that only a TOP-lift computation was possible in a reasonable amount of time with our proof of concept implementation[21]. Analogously, we use (P/-) where "P" indicates the usage of the POT-lift.

In the column "transformations" we indicate with "(I)" that the reason for the transformation was the ideal. If the syzygy module was the reason we use "(S)".

In table 7, one can see that the ideals and syzygy modules could be transformed into a quasi-stable position. Also, the index of safety is always significantly bigger for the POT-lift. It might be noteworthy that in some cases the transformations for the TOP- and POT-lift do not coincide. Moreover, for Chemequs, there was no transformation required for a POT-lift, whereas the TOP-lift run needed two. This is also reflected in the runtime, of course. In general, if we focus only on finding a Pommaret basis of I, the runtime is smaller. If only the syzygy module is not in quasi-stable position this effect becomes even more clear (see for instance at the data about Caprasse). For the Noonburg-89 runs the focus on the ideal saved not only computation time but also a coordinate transformation for the TOP-lift.

It is also worth mentioning that the example Cyclic_4 could be handled fairly easily with the POT-lift whereas the TOP-lift was interrupted after four hours of calculations. There, the algorithm stated several times that both, the ideal and the syzygy module are not in quasi-stable position.

7. Benefits and Issues of the Usage of POT- or TOP-lifts

We have focused especially on TOP- and POT-lifts. In this rather small section, we want to collect properties of the POT- and TOP-lifts in the context of the (semi-)involutive GVW algorithm.

[21] The Maple code is available at http://www.mathematik.uni-kassel.de/~izgin/publications.php?lang=en.

	focus	runtime	$\lvert H\rvert$	$\lvert G\rvert$	$Reg(I)$	transformations	max. i.o.s.
Caprasse (T/-)	no	3400	131	30	7	$t \mapsto t+x$ $z \mapsto z+y$ (S)	45
Caprasse (T/P)	yes	595/1418	22/21	38/38	7		
Chemequs (T/P)	no	1289/14	51/57	19/20	4	$y_4 \mapsto y_4+y_3$ $y_4 \mapsto y_4+y_1$ (S) / -	12 / -
Chemequs (T/P)	yes	4/4	21/25	20/20	4		
Noonburg-89 (T/P)	no	10/28	14/26	15/16	6	$c \mapsto c+z$ $c \mapsto c+y$ $z \mapsto z+y$ $c \mapsto c+x$ (I) / $c \mapsto c+z$ $c \mapsto c+y$ $c \mapsto c+x$ (I)	32 / 113
Noonburg-89 (T/P)	yes	6/24	4/4	15/16	6	$c \mapsto c+z$ $c \mapsto c+y$ $c \mapsto c+x$ (I) / $c \mapsto c+z$ $c \mapsto c+y$ $c \mapsto c+x$ (I)	32 / 113
Cyclic_4 (P/-)	no	0.39	8	8	6	$z \mapsto z+y$ (I)	2

TABLE 7. Benchmark runs for the TOP- and POT-lift and the Pommaret version of the GVW algorithm.

Now, it is well known, that with a POT-lift, the algorithm becomes incremental [6]. Furthermore, as we are performing coordinate transformations on the u-part, too, we have to find the signatures of the transformed pairs. For the u-part, this obviously is easier with a POT-lift than with a TOP-lift as we have to search for the leading term only in one position of the vector in the u-part. However, when using the POT-lift, we may be stuck easier at a signature belonging to a position i, because the POT-lift is not of type ω. Hence, we may go to the degree $q+1$ more often until we jump to elements with a signature at position $i+1$. This means, that we are calculating too many unnecessary pairs, blowing up the set G and hence increasing the costs for any operation on G, especially the calculation of new elements for $JP(G) \cup PP(G)$ for the Pommaret case. A POT-lift, in general, reacts sensitively to these signature-based algorithms as the order of the elements in F have a major impact on the efficiency as we will recall in a moment.

Whereas a TOP-lift may be more convenient for calculations, it is rather expensive when we have to perform a lot of coordinate transformations. However, it is an open question at this point which of the two lifts performs better according to the index of safety. Lastly, we want to recall one more thing:

It is not guaranteed that a Pommaret basis exists for $Syz(F)$ just because it exists for $\langle F\rangle$ (e.g. for Caprasse in table 7 for the TOP-lift). For a POT-lift, a simple example was given in the example after remark 3.34. In fact, we want to point out that in this example we already started with a Pommaret basis of $\langle F\rangle$, yet our algorithm would return an error message for $Syz(F)$ and the POT-lift. However, one can see that everything would work perfectly fine if we took a POT-lift and just changed the order of our elements in F, so that $G = \{(\boldsymbol{e}_1, y), (\boldsymbol{e}_2, x)\}$ holds. Then $\boldsymbol{e}_1 \prec_{POT} \boldsymbol{e}_2$, however, now the leading term of the syzygy $x\boldsymbol{e}_1 - y\boldsymbol{e}_2$ is $y\boldsymbol{e}_2$. This, on the other hand, points out how sensitive the involutive GVW algorithm reacts to the POT-lift.

8. Summary and Outlook

In this work, we have introduced the main ideas of the original GVW algorithm and then presented a corresponding theory for involutive divisions where we discussed very detailed the algorithm for the Pommaret division. We have developed the theory also presenting the process of finding computational achievable assumptions under which an involutive J-criterion holds. We also gave examples of why none of the new made assumptions can be dropped. Moreover, we gave a counterexample for the involutive J-criterion from [1]. We presented for our version some criteria in order to make an implementation more efficient and proved the termination of the fully and semi-involutive GVW algorithm. We also pointed out the benefits of both variants and the issues that go along with them. Then we have introduced coordinate transformations and the index of safety that arose naturally from our strategy for the algorithm. After completing the discussion of the Pommaret case, we were able to present a Janet version of the GVW algorithm. Indeed, we have found out that no proxy pairs are required. But even though we have proven a fully involutive version, we argued that without further investigations only a semi-involutive implementation of the algorithm is meaningful. In the last section, we gave remarks on our implementation and presented some benchmark computations along with some examples that tested the functionality of coordinate transformations.

With the versions of the involutive GVW algorithm for the Pommaret and Janet version, we gave an algorithm to compute Pommaret bases for homogeneous ideals and the degree reverse lexicographic order together with a compatible term order of type ω or a POT-lift of pseudo type ω. In the last section we collected some properties of these term orders. However, we left out the discussion of the Schreyer ordering which was used in [1] for the implementation.

Still, the given implementation is only a proof of concept and shall not be used for bigger examples. For further investigation of this algorithm and its properties with a POT- or TOP-lift, one may consider the following question: Does the POT-lift lead to a bigger maximal value of the index of safety compared to the one obtained with a TOP-lift?

Lastly, it should be investigated in detail how the Pommaret variant can be adapted for affine inputs. And it might be also interesting to discover why our implementation seems to be much faster than the implementation for the Janet division.

Furthermore, it might be useful to add some of the ideas presented in the introduction to the involutive GVW algorithm. In particular, one could aim to create a Hilbert-driven algorithm that uses the substituting method from [15] and the concept of mutant pairs from [20].

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Amir Hashemi
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran
e-mail: `Amir.Hashemi@iut.ac.ir`

Thomas Izgin
Institut für Mathematik, Universität Kassel, Heinrich-Plett-Str. 40, 34132 Kassel, Germany
e-mail: `izgin@mathematik.uni-kassel.de`

Daniel Robertz
School of Engineering, Computing and Mathematics, University of Plymouth, 2-5 Kirkby Place, Drake Circus, Plymouth PL4 8AA, United Kingdom
e-mail: `daniel.robertz@plymouth.ac.uk`

Werner M. Seiler
Institut für Mathematik, Universität Kassel, Heinrich-Plett-Str. 40, 34132 Kassel, Germany
e-mail: `seiler@mathematik.uni-kassel.de`