Janet Bases and Resolutions in CoCoALib

Mario Albert, Matthias Fetzer, and Werner M. Seiler

Institut für Mathematik, Universität Kassel, 34132 Kassel, Germany
{albert,fetzer,seiler}@mathematik.uni-kassel.de

Abstract. Recently, the authors presented a novel approach to computing resolutions and Betti numbers using Pommaret bases. For Betti numbers, this algorithm is for most examples much faster than the classical methods (typically by orders of magnitude). As the problem of δ-regularity often makes the determination of a Pommaret basis rather expensive, we extend here our algorithm to Janet bases. Although in δ-singular coordinates, Janet bases may induce larger resolutions than the corresponding Pommaret bases, our benchmarks demonstrate that this happens rarely and has no significant effect on the computation costs.

1 Introduction

Computing resolutions represents a fundamental task in algebraic geometry and commutative algebra. For some problems like computing derived functors one needs indeed the full resolution, i. e. including the differential. For other applications, the Betti numbers measuring the size of the resolution are sufficient, as they contain already important geometric and topological information.

Determining a minimal resolution is generally rather expensive. For a module of projective dimension p, the costs correspond roughly to those of p Gröbner bases computation. Theoretically, computing only the Betti numbers should be considerably cheaper, as one does not need the differential. However, all implementations we are aware of read off the Betti numbers of the minimal resolutions and thus in practice their costs are the same as for the whole resolution.

In the recent article [2], we presented a novel approach to computing resolutions and Betti numbers based on a combination of the theory of Pommaret bases [12], a special form of involutive bases, and algebraic discrete Morse theory [15]. Within this approach, it is possible to compute Betti numbers without first determining a whole resolution. In fact, it is even possible to determine individual Betti numbers without the remaining ones.

While Pommaret bases are theoretically very nice, as they provide simple access to many invariants [12], they face from a practical point of view the problem of δ-regularity, i. e. for positive-dimensional ideals they generally exist only after a sufficiently generic coordinate transformation. There are deterministic approaches to the construction of (hopefully rather sparse) δ-regular coordinates [8], but nevertheless the computation of a Pommaret basis is usually significantly more expensive than that of a Janet basis (a more detailed analysis of this topic will appear in the forthcoming work [3]).

Here, we show that the ideas of [2] also work Janet instead of Pommaret bases and thus remove an important bottleneck in their application. As a by-product, we show that the degree of a Janet basis can never be smaller than that of a Pommaret basis. For the resolution induced by the Janet basis this implies that in general it is longer than the minimal one and can extend to higher degrees.

2 Involutive Bases and Free Resolutions

Involutive bases are Gröbner bases with additional combinatorial properties. They were introduced by Gerdt and Blinkov [5, 6] who combined Gröbner bases with ideas from the algebraic theory of partial differential equations. Surveys over their basic theory and further references can be found in [11] or [13, Chapts. 3/4].

Throughout this work, $\Bbbk$ denotes an arbitrary field and $\mathcal{P} = \Bbbk[x_1, \dots, x_n] = \Bbbk[\mathcal{X}]$ the polynomial ring in n variables over $\Bbbk$ together with the standard grading. The standard basis of the free module $\mathcal{P}^t$ is denoted by $\{\mathbf{e}_1, \dots, \mathbf{e}_t\}$. Our conventions for the Janet division require that we define term orders always "reverse" to the usual conventions, i. e. we revert the ordering of the variables. In the sequel, $0 \neq \mathcal{U} \subseteq \mathcal{P}^t$ will always be a graded submodule and all appearing elements $\mathbf{f} \in \mathcal{P}^t$ are homogeneous.

The basic idea underlying involutive bases is that each generator $\mathbf{f}$ in a basis may only be multiplied by polynomials in a restricted set of variables, its *multiplicative variables* $\mathcal{X}(\mathbf{f}) \subseteq \mathcal{X}$. The remaining variables are called the *non-multiplicative* ones $\overline{\mathcal{X}}(\mathbf{f}) = \mathcal{X} \setminus \mathcal{X}(\mathbf{f})$. Different involutive bases differ in the way the multiplicative variables are chosen. We will use here only Janet bases.

For them, the assignment of the multiplicative variables depends not only on the generator $\mathbf{f}$, but on the whole basis. Given a finite set $\mathcal{F}$ of terms and a term $x^\mu \mathbf{e}_\alpha \in \mathcal{F}$, we introduce for each $1 \leq k \leq n$ and each $1 \leq \alpha \leq t$ the subsets

$$(\mu_{k+1}, \dots, \mu_n)_\alpha = \{x^\nu \mathbf{e}_\alpha \in \mathcal{F} \mid \nu_{k+1} = \mu_{k+1}, \dots, \nu_n = \mu_n\} \subseteq \mathcal{F}$$

and put $x_k \in \mathcal{X}_{J,\mathcal{F}}(x^\mu \mathbf{e}_\alpha)$, if $\mu_k = \max\{\nu_k \mid x^\nu \mathbf{e}_\alpha \in (\mu_{k+1}, \dots, \mu_n)_\alpha\}$. For finite sets $\mathcal{F}$ of polynomial vectors, we reduce to the monomial case via a term order $\prec$ by setting $\mathcal{X}_{J,\mathcal{F},\prec}(\mathbf{f}) = \mathcal{X}_{J,\operatorname{lt}\mathcal{F}}(\operatorname{lt}\mathbf{f})$.

Definition 1. *A finite set of terms $\mathcal{H} \subset \mathcal{P}^t$ is a* Janet basis *of the monomial module $\mathcal{U} = \langle \mathcal{H} \rangle$, if as a $\Bbbk$-linear space $\mathcal{U} = \bigoplus_{\mathbf{h} \in \mathcal{H}} \Bbbk[\mathcal{X}_{J,\mathcal{H}}(\mathbf{h})] \cdot \mathbf{h}$. For every term contained in the* involutive cone $\Bbbk[\mathcal{X}_{J,\mathcal{H}}(\mathbf{h})] \cdot \mathbf{h}$, *we call* $\mathbf{h}$ *an* involutive divisor*. A finite polynomial set $\mathcal{H} \subset \mathcal{P}^t$ is a* Janet basis *of the polynomial submodule $\mathcal{U} = \langle \mathcal{H} \rangle$ for the term order $\prec$, if all elements of $\mathcal{H}$ possess distinct leading terms and these terms form a Janet basis of the leading module $\operatorname{lt}\mathcal{U}$.*

Every submodule admits a Janet basis for any term order [5, 11]. Arbitrary involutive bases can be characterised similarly to Gröbner bases [5, 11]. The S-polynomials in the theory of Gröbner bases are replaced by products of the generators with one of their *non*-multiplicative variables. A key difference is the uniqueness of involutive standard representations.

Proposition 2. *[11, Thm. 5.4] The finite set $\mathcal{H} \subset \mathcal{U}$ is a Janet basis of the submodule $\mathcal{U} \subseteq \mathcal{P}^t$ for the term order $\prec$, if and only if every element $0 \neq \mathbf{f} \in \mathcal{U}$ possesses a unique involutive standard representation $\mathbf{f} = \sum_{\mathbf{h}\in\mathcal{H}} P_{\mathbf{h}}\mathbf{h}$ where each non-zero coefficient satisfies $P_{\mathbf{h}} \in \Bbbk[\mathcal{X}_{J,\mathcal{H},\prec}(\mathbf{h})]$ and $\operatorname{lt}(P_h\mathbf{h}) \preceq \operatorname{lt}(\mathbf{f})$.*

Proposition 3. *[11, Cor. 7.3] Let $\mathcal{H} \subset \mathcal{P}^t$ be a finite set and $\prec$ a term order such that no leading term in $\operatorname{lt}\mathcal{H}$ is an involutive divisor of another one. The set $\mathcal{H}$ is a Janet basis of the submodule $\langle\mathcal{H}\rangle$ with respect to $\prec$, if and only if for every $\mathbf{h} \in \mathcal{H}$ and every non-multiplicative variable $x_j \in \overline{\mathcal{X}}_{J,\mathcal{H},\prec}(\mathbf{h})$ the product $x_j\mathbf{h}$ possesses an involutive standard representation with respect to $\mathcal{H}$.*

The classical Schreyer Theorem [10] describes how every Gröbner basis induces a Gröbner basis of its first syzygy module for a suitable chosen term order. If $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\} \subset \mathcal{P}^t$ is a finite set with s elements and $\prec$ an arbitrary term order on $\mathcal{P}^t$, then the *Schreyer order* $\prec_{\mathcal{H}}$ is the term order induced on the free module $\mathcal{P}^s$ by setting $x^\mu\mathbf{e}_\alpha \prec_{\mathcal{H}} x^\nu\mathbf{e}_\beta$, if $\operatorname{lt}(x^\mu\mathbf{h}_\alpha) \prec \operatorname{lt}(x^\nu\mathbf{h}_\beta)$ or if these leading terms are equal and $\beta < \alpha$.

The Schreyer order $\prec_{\mathcal{H}}$ depends on the ordering of $\mathcal{H}$. For the involutive version of the Schreyer Theorem, we assume that $\mathcal{H}$ is a Janet basis and order its elements in a suitable manner. We associate a directed graph with $\mathcal{H}$. Its vertices are given by the elements in $\mathcal{H}$. If $x_j \in \overline{\mathcal{X}}_{J,\mathcal{H},\prec}(\mathbf{h})$ for some generator $\mathbf{h} \in \mathcal{H}$, then $\mathcal{H}$ contains a unique generator $\bar{\mathbf{h}}$ such that $\operatorname{lt}\bar{\mathbf{h}}$ is an involutive divisor of $\operatorname{lt}(x_j\mathbf{h})$. In this case we include a directed edge from $\mathbf{h}$ to $\bar{\mathbf{h}}$. The graph thus defined is called the *J-graph* of the Janet basis $\mathcal{H}$. We require that the ordering of $\mathcal{H}$ satisfies the following condition: whenever the J-graph of $\mathcal{H}$ contains a path from $\mathbf{h}_\alpha$ to $\mathbf{h}_\beta$, then we must have $\alpha < \beta$. We then speak of a *J-ordering*. One can show that such orderings always exist [12]. In fact, one easily verifies that an explicit J-ordering is provided by any module order (applied to the leading terms of $\mathcal{H}$) that restricts to the lexicographic order when applied to terms living in the same component of $\mathcal{P}^t$.

Assume that $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ is a Janet basis of the polynomial submodule $\mathcal{U} \subseteq \mathcal{P}^t$. According to Proposition 3, we have for every non-multiplicative variable x_k of a generator $\mathbf{h}_\alpha$ an involutive standard representation $x_k\mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha;k)}\mathbf{h}_\beta$ and thus a syzygy $\mathbf{S}_{\alpha;k} = x_k\mathbf{e}_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha;k)}\mathbf{e}_\beta$. Let $\mathcal{H}_{\mathrm{Syz}}$ be the set of all these syzygies.

Lemma 4. *[12, Lemma 5.7] If the finite set $\mathcal{H} \subset \mathcal{P}$ is a J-ordered Janet basis, then we find for all admissible values of α and k with respect to the Schreyer order $\prec_{\mathcal{H}}$ that $\operatorname{lt}\mathbf{S}_{\alpha;k} = x_k\mathbf{e}_\alpha$.*

Theorem 5. *[12, Thm. 5.10] Let $\mathcal{H}$ be a J-ordered Janet basis of the submodule $\mathcal{U} \subseteq \mathcal{P}^t$. Then $\mathcal{H}_{\mathrm{Syz}}$ is a Janet basis of the syzygy module $\operatorname{Syz}(\mathcal{H})$ for the Schreyer order $\prec_{\mathcal{H}}$.*

Like the classical Schreyer Theorem, we can iterate Theorem 5 and obtain then a free resolution of the submodule $\mathcal{U}$. However, in contrast to the classical situation, the involutive version yields the full shape of the arising resolution without any further computations. We present here a bigraded version of this result which is obtained by a trivial extension of the arguments in [12]. It provides sharp upper bounds for the Betti numbers of $\mathcal{U}$.

Theorem 6. *[12, Thm. 6.1, Rem. 6.2] Let the finite set $\mathcal{H} \subset \mathcal{P}^t$ be a Janet basis of the polynomial submodule $\mathcal{U} \subseteq \mathcal{P}^t$. Furthermore, let $\beta_{0,j}^{(k)}$ be the number of generators $\mathbf{h} \in \mathcal{H}$ of degree j having k multiplicative variables and set $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$. Then $\mathcal{U}$ possesses a finite free graded resolution*[1]

$$0 \longrightarrow \bigoplus \mathcal{P}[-j]^{r_{n-d,j}} \longrightarrow \cdots \longrightarrow \bigoplus \mathcal{P}[-j]^{r_{1,j}} \longrightarrow \bigoplus \mathcal{P}[-j]^{r_{0,j}} \longrightarrow \mathcal{U} \longrightarrow 0 \tag{1}$$

of length $n-d$ where the ranks of the free modules are given by

$$r_{i,j} = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)} \,. \tag{2}$$

For the proof, one shows that the Janet basis $\mathcal{H}_j$ of the jth syzygy module $\mathrm{Syz}^j(\mathcal{H})$ with respect to the Schreyer order $\prec_{\mathcal{H}_{j-1}}$ consists of the syzygies $\mathbf{S}_{\alpha;\mathbf{k}}$ with an ordered integer sequence $\mathbf{k} = (k_1, \dots, k_j)$ where $1 \leq k_1 < \cdots < k_j \leq n$ and all variables x_{k_i} are non-multiplicative for the generator $\mathbf{h}_\alpha \in \mathcal{H}$. These syzygies are defined recursively. We denote for any $1 \leq i \leq j$ by $\mathbf{k}_i$ the sequence obtained by eliminating k_i from $\mathbf{k}$. Now $\mathbf{S}_{\alpha;\mathbf{k}}$ arises from the involutive standard representation of $x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j}$: $x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j} = \sum_{\beta=1}^{s}\sum_{\boldsymbol{\ell}} P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k})}\mathbf{S}_{\beta;\boldsymbol{\ell}}$. Here the second sum is over all ordered integer sequences $\boldsymbol{\ell}$ of length $j-1$ such that for all entries ℓ_i the variables x_{ℓ_i} is non-multiplicative for the generator $\mathbf{h}_\beta \in \mathcal{H}$. Lemma 4 implies that $\operatorname{lt} \mathbf{S}_{\alpha;\mathbf{k}} = x_{k_j}\mathbf{e}_{\alpha;\mathbf{k}_j}$ and that the coefficient $P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k})}$ depends only on those variables which are multiplicative for the syzygy $\mathbf{S}_{\beta;\boldsymbol{\ell}}$. If $\overline{\mathcal{X}}_{J,\mathcal{H},\prec}(h_\alpha) = \{x_{i_1}, x_{i_2}, \dots, x_{i_{n-k}}\}$, then we find for the first syzygies that $\mathcal{X}_{J,\mathcal{H}_{\mathrm{Syz}},\prec_{\mathcal{H}}}(\mathbf{S}_{\alpha;i_j}) = \mathcal{X} \setminus \{x_{i_{j+1}}, \dots, x_{i_{n-k}}\}$. Iteration yields the multiplicative variables for the higher syzygies. The simple form of the leading terms yields via a simple combinatorial computation the ranks $r_{i,j}$ of the modules in the resolution (1).

Corollary 7. *Let $\mathcal{H}$ be a Janet basis of the submodule $\mathcal{U} \subseteq \mathcal{P}^t$. If we set again $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$ and $q = \deg(\mathcal{H}) = \max\{\deg \mathbf{h} \mid \mathbf{h} \in \mathcal{H}\}$, then the projective dimension and the Castelnuovo-Mumford regularity of $\mathcal{U}$ are bounded by $\operatorname{pd}(\mathcal{U}) \leq n-d$ and $\operatorname{reg}(\mathcal{U}) \leq q$.*

Proof. The first estimate follows immediately from the resolution (1) induced by the Janet basis $\mathcal{H}$. Furthermore, the ith module of this resolution is obviously generated by elements of degree less than or equal to $q+i$. This observation implies that $\mathcal{U}$ is q-regular and thus the second estimate. □

Starting from an arbitrary Janet basis, more information cannot be obtained in a simple manner. If, however, δ-regular coordinates are used, i. e. if the Janet

[1] We use here the usual shift notation: $(\mathcal{P}[j])_d = \mathcal{P}_{d+j}$.

basis is simultaneously a Pommaret basis (which is generically the case), then stronger results hold: the resolution (1) is then always of minimal length, i. e. $\mathrm{pd}\,(\mathcal{U}) = n - d$ [12, Thm. 8.11] (which also implies by the Auslander-Buchsbaum formula that d is nothing but the depth of the submodule $\mathcal{U}$), and also the second estimate becomes an equality: $\mathrm{reg}\,(\mathcal{U}) = \deg\,(\mathcal{H})$ [12, Thm. 9.2]. By contrast, for a Janet basis the difference between the regularity and the degree of the basis can become arbitrarily big, as the next example demonstrates.

Example 8. Consider the polynomial ring $\mathcal{P} = \Bbbk[x_1, x_2, x_3]$. For a moment, we switch to the standard conventions $x_1 \succ x_2 \succ x_3$ and define $\mathcal{I}^{(d)}$ as the ideal generated by the lexsegment terminating at x_2^d for an arbitrary degree $d \in \mathbb{N}$. Thus $d = 3$ yields for example

$$\mathcal{I}^{(3)} = \langle x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^3\rangle\ .$$

The regularity of $\mathcal{I}^{(d)}$ is of course independent of the used ordering of the variables. For the ordering $x_1 \succ x_2 \succ x_3$, the above mentioned generating set is both a Janet and a Pommaret basis. Thus we find $\mathrm{reg}\,\mathcal{I}^{(d)} = d$.

For the ordering $x_1 \prec x_2 \prec x_3$, the ideal $\mathcal{I}^{(d)}$ possesses no Pommaret basis, as it contains no term of the form x_3^e for some $e \in \mathbb{N}$ which was necessary for the existence of a Pommaret basis. We can write the lexsegment as the union of the following two sets:

$$\begin{aligned}\mathcal{L}_1 &= \left\{x_1^e x_2^f \mid e \geq 0,\ f \geq 0,\ e + f = d\right\}\ ,\\ \mathcal{L}_2 &= \left\{x_1^e x_2^f x_3^g \mid e > 0,\ f \geq 0,\ 1 \leq g \leq d-1,\ e + f + g = d\right\}\ .\end{aligned}$$

Then the Janet basis of $\mathcal{I}^{(d)}$ is given by

$$\begin{aligned}\mathcal{H}^{(d)} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \left\{x_2^d x_3^g \mid 1 \leq g \leq d-1\right\} \cup\\ \left\{x_1^e x_2^{f+f'} x_3^g \mid x_1^e x_2^f x_3^g \in \mathcal{L}_2,\ f' > 0,\ f + f' < d\right\}.\end{aligned}$$

The degree of this basis is $2d - 1$ and hence $\deg\mathcal{H}^{(d)} - \mathrm{reg}\,\mathcal{I}^{(d)} = d - 1$ can become arbitrarily large.

3 Free Resolutions with Janet Bases

Theorem 6 describes only the shape of the induced resolution; it provides no information about the higher syzygies. In [2], it is shown how algebraic discrete Morse theory allows an explicit determination of all differentials in the case of a Pommaret basis. Now we will show here that this is also possible with Janet bases. First we recall some of the material presented in [2] and then point out where the crucial differences occur.

Definition 9. *A graded polynomial module* $\mathcal{M} \subseteq \mathcal{P}^t$ *has* initially linear syzygies*, if* $\mathcal{M}$ *possesses a finite presentation*

$$0 \longrightarrow \ker \eta \longrightarrow \mathcal{W} = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha \stackrel{\eta}{\longrightarrow} \mathcal{M} \longrightarrow 0 \tag{3}$$

such that with respect to some term order $\prec$ *on the free module* $\mathcal{W}$ *the leading module* $\operatorname{lt}\ker\eta$ *is generated by terms of the form* $x_j\mathbf{w}_\alpha$. *We say that* $\mathcal{M}$ *has* initially linear minimal syzygies*, if the presentation is minimal in the sense that* $\ker\eta \subseteq \mathfrak{m}^s$ *with* $\mathfrak{m} = \langle x_1, \dots, x_n\rangle$ *the homogeneous maximal ideal.*

The construction begins with the following two-sided Koszul complex $(\mathcal{F}, d_\mathcal{F})$ defining a free resolution of $\mathcal{M}$. Let $\mathcal{V}$ be a $\Bbbk$-linear space with basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (with n still the number of variables in $\mathcal{P}$) and set $\mathcal{F}_j = \mathcal{P} \otimes_\Bbbk \Lambda_j\mathcal{V} \otimes_\Bbbk \mathcal{M}$ which obviously yields a free $\mathcal{P}$-module. Choosing a $\Bbbk$-linear basis $\{m_a \mid a \in A\}$ of $\mathcal{M}$, a $\mathcal{P}$-linear basis of $\mathcal{F}_j$ is given by the elements $1 \otimes v_\mathbf{k} \otimes m_a$ with ordered sequences $\mathbf{k}$ of length j. The differential is now defined by

$$d_\mathcal{F}(1 \otimes \mathbf{v_k} \otimes m_a) = \sum_{i=1}^{j} (-1)^{i+1}\big(x_{k_i} \otimes \mathbf{v}_{\mathbf{k}_i} \otimes m_a - 1 \otimes \mathbf{v}_{\mathbf{k}_i} \otimes x_{k_i}m_a\big) \tag{4}$$

where $\mathbf{k}_i$ denotes the sequence $\mathbf{k}$ without the element k_i. Here it should be noted that the second term on the right hand side is not yet expressed in the chosen $\Bbbk$-linear basis of $\mathcal{M}$. For notational simplicity, we will drop in the sequel the tensor sign $\otimes$ and leading factors 1 when writing elements of $\mathcal{F}_\bullet$.

Under the assumption that the module $\mathcal{M}$ has initially linear syzygies via a presentation (3), Sköldberg [15] constructs a Morse matching leading to a smaller resolution $(\mathcal{G}, d_\mathcal{G})$. He calls the variables

$$\operatorname{crit}(\mathbf{w}_\alpha) = \{x_j \mid x_j\mathbf{w}_\alpha \in \operatorname{lt}\ker\eta\}\,; \tag{5}$$

critical for the generator $\mathbf{w}_\alpha$; the remaining *non-critical* ones are contained in the set $\operatorname{ncrit}(\mathbf{w}_\alpha)$. Then a $\Bbbk$-linear basis of $\mathcal{M}$ is given by all elements $x^\mu\mathbf{h}_\alpha$ with $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$ and $x^\mu \in \Bbbk[\operatorname{ncrit}(\mathbf{w}_\alpha)]$. We define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as the free submodule generated by those vertices $\mathbf{v_k h}_\alpha$ where the ordered sequences $\mathbf{k}$ are of length j and such that every entry k_i is critical for $\mathbf{w}_\alpha$. In particular $\mathcal{W} \cong \mathcal{G}_0$ with an isomorphism induced by $\mathbf{w}_\alpha \mapsto \mathbf{v}_\emptyset\mathbf{h}_\alpha$.

The description of the differential $d_\mathcal{G}$ is based on reduction paths in the associated Morse graph (for a detailed treatment of these notions, see [2, 9, 14]) and expresses the differential as a triple sum. If we assume that, after expanding the right hand side of (4) in the chosen $\Bbbk$-linear basis of $\mathcal{M}$, the differential of the complex $\mathcal{F}_\bullet$ can be expressed as

$$d_\mathcal{F}(\mathbf{v_k h}_\alpha) = \sum_{\mathbf{m},\mu,\gamma} Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v_m}(x^\mu\mathbf{h}_\gamma)\,, \tag{6}$$

then $d_\mathcal{G}$ is defined by

$$d_\mathcal{G}(\mathbf{v_k h}_\alpha) = \sum_{\boldsymbol{\ell},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_p \rho_p\big(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v_m}(x^\mu\mathbf{h}_\gamma)\big) \tag{7}$$

where the first sum ranges over all ordered sequences $\boldsymbol{\ell}$ which consists entirely of critical indices for $\mathbf{w}_\beta$. Moreover the second sum may be restricted to all values such that a polynomial multiple of $\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)$ effectively appears in $d_{\mathcal{F}}(\mathbf{v_k}\mathbf{h}_\alpha)$ and the third sum ranges over all reduction paths p going from $\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)$ to $\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta$. Finally ρ_p is the reduction associated with the reduction path p satisfying

$$\rho_p\big(\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)\big) = q_p \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta \tag{8}$$

for some polynomial $q_p \in \mathcal{P}$.

A key point for applying this construction in the context of involutive bases is that any Janet basis has initially linear syzygies. Thus given a Janet basis we have two resolutions available: (1) and the one obtained by Sköldberg's construction. The main result of this section will be that the two are isomorphic.

Lemma 10. *Let $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ be the Janet basis of the polynomial submodule $\mathcal{U} \subseteq \mathcal{P}^t$. Then $\mathcal{U}$ has initially linear syzygies*[2] *for the Schreyer order $\prec_{\mathcal{H}}$ and* $\operatorname{crit}(\mathbf{w}_\alpha) = \overline{\mathcal{X}}_{J,\mathcal{H},\prec}(\mathbf{h}_\alpha)$*, i. e. the critical variables of the generator $\mathbf{w}_\alpha$ are the non-multiplicative variables of $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$.*

The reduction paths can be divided into elementary ones of length two. There are essentially three types of reductions paths [2, Section 4]. The elementary reductions of *type 0* are not of interest [2, Lemma 4.5]. All other elementary reductions paths are of the form

$$\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}(\frac{x^\mu}{x_i}\mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^\nu \mathbf{h}_\beta)\,.$$

Here $\mathbf{k}\cup i$ is the ordered sequence which arises when i is inserted into $\mathbf{k}$; likewise $\mathbf{k} \setminus i$ stands for the removal of an index $i \in \mathbf{k}$.

Type 1: Here $\boldsymbol{\ell} = (\mathbf{k} \cup i)\backslash j$, $x^\nu = \frac{x^\mu}{x_i}$ and $\beta = \alpha$. Note that $i = j$ is allowed.
We define $\epsilon(i;\mathbf{k}) = (-1)^{|\{j\in\mathbf{k} | j>i\}|}$. Then the corresponding reduction is

$$\rho(\mathbf{v_k}x^\mu \mathbf{h}_\alpha) = \epsilon(i;\mathbf{k}\cup i)\epsilon(j;\mathbf{k}\cup i)x_j \mathbf{v}_{(\mathbf{k}\cup i)\backslash j}\big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\big)\,.$$

Type 2: Now $\boldsymbol{\ell} = (\mathbf{k} \cup i) \setminus j$ and $x^\nu \mathbf{h}_\beta$ appears in the involutive standard representation of $\frac{x^\mu x_j}{x_i}\mathbf{h}_\alpha$ with the coefficient $\lambda_{j,i,\alpha,\mu,\nu,\beta} \in \Bbbk$. In this case, by construction of the Morse matching, we have $i \neq j$. The reduction is

$$\rho(\mathbf{v_k}x^\mu \mathbf{h}_\alpha) = -\epsilon(i;\mathbf{k}\cup i)\epsilon(j;\mathbf{k}\cup i)\lambda_{j,i,\alpha,\mu,\nu,\beta}\mathbf{v}_{(\mathbf{k}\cup i)\backslash j}(x^\nu \mathbf{h}_\beta)\,.$$

These reductions follow from the differential (4): The summands appearing there are either of the form $x_{k_i}\mathbf{v}_{\mathbf{k}_i}m_a$ or of the form $\mathbf{v}_{\mathbf{k}_i}(x_{k_i}m_a)$. For each of these summands, we have a directed edge in the Morse graph $\Gamma^A_{\mathcal{F}_\bullet}$. Thus for an elementary reduction path

$$\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\big) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^\nu \mathbf{h}_\beta)\,,$$

[2] Note that we apply here Definition 9 directly to $\mathcal{U}$ and not to $\mathcal{M} = \mathcal{P}^t/\mathcal{U}$, i.e. in (3) one must replace $\mathcal{M}$ by $\mathcal{U}$.

the second edge can originate from summands of either form. For the first form we then have an elementary reduction path of type 1 and for the second form we have type 2.

For completeness, we repeat some simple results from [2] which we need to show that the free resolution $\mathcal{G}$ is isomorphic to the resultion induced by a Janet basis $\mathcal{H}$. Some of the proofs in [2] use the class of a generator in $\mathcal{H}$, a notion arising in the context of Pommaret bases. When working with Janet bases, one has to replace it by the index of the maximal multiplicative variable.

Lemma 11. *[2, Lemma 4.3] For a non-multiplicative index*[3] $i \in \operatorname{crit}(\mathbf{h}_\alpha)$ *let* $x_i\mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha;i)}\mathbf{h}_\beta$ *be the involutive standard representation. Then we have* $d_{\mathcal{G}}(\mathbf{v}_i\mathbf{h}_\alpha) = x_i\mathbf{v}_\emptyset\mathbf{h}_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha;i)}\mathbf{v}_\emptyset\mathbf{h}_\beta$.

The next result states that if one starts at a vertex $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha)$ with certain properties and follows through all possible reduction paths in the graph, one will never get to a point where one must calculate an involutive standard representation. If there are no critical (i. e. non-multiplicative) variables present at the starting point, then this will not change throughout any reduction path. In order to generalise this lemma to higher homological degrees, one must simply replace the conditions $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$ by ordered sequences $\mathbf{k}, \boldsymbol{\ell}$ with $\mathbf{k} \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\boldsymbol{\ell} \subseteq \operatorname{ncrit}(\mathbf{h}_\beta)$.

Lemma 12. *[2, Lemma 4.4] Assume that* $i \cup \operatorname{supp}(\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$. *Then for any reduction path* $p = \mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu\mathbf{h}_\beta)$ *we have* $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$. *In particular, in this situation there is no reduction path* $p = \mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_k\mathbf{h}_\beta$ *with* $k \in \operatorname{crit}(\mathbf{h}_\beta)$.

In the sequel, we use Schreyer orders on the components of the complex $\mathcal{G}$. We define $\mathcal{H}_0$ as the Janet basis of $d_{\mathcal{G}}(\mathcal{G}_1) \subseteq \mathcal{G}_0$ with respect to the Schreyer order $\prec_{\mathcal{H}}$ induced by the term order $\prec$ on $\mathcal{P}^t$ and $\mathcal{H}_i$ as the Janet basis of $d_{\mathcal{G}}(\mathcal{G}_{i+1}) \subseteq \mathcal{G}_i$ for the Schreyer order $\prec_{\mathcal{H}_{i-1}}$.

Lemma 13. *Let $\mathcal{H}$ be a Janet basis.* $\mathbf{h}_\beta$ *is greater or equal than* $\mathbf{h}_\alpha$ *according to the J-order if* $\operatorname{lt}(\mathbf{h}_\beta)$ *is an involutive divisor of* $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$.

Proof. There is another way to compute the involutive divisor of $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$: We choose x_i, such that $\deg_i(x^\mu) \neq 0$ and compute the involutive divisor of $x_i \operatorname{lt}(\mathbf{h}_\alpha)$. Then $x^\mu \operatorname{lt}(\mathbf{h}_\alpha) = \frac{x^\mu}{x_i} x^\nu \operatorname{lt}(\mathbf{h}_\gamma)$. Then we check if $\frac{x^\mu}{x_i} x^\nu$ contains a non-multiplicative variable for $\operatorname{lt}(\mathbf{h}_\gamma)$. If not we are finished and $\operatorname{lt}(\mathbf{h}_\gamma)$ is an involutive divisor of $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$. In the other case we repeat the procedure above. This procedure must end after a finite number of steps. If this were not the case we have found a cycle in the J-graph. But this is not possible [12].

To compute the involutive divisor of $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$, we have constructed a chain $\mathbf{h}_\alpha = \mathbf{h}_{\gamma_1}, \cdots, \mathbf{h}_{\gamma_m} = \mathbf{h}_\beta$ above. Due to the procedure we see that $\mathbf{h}_{\gamma_i}$ must

[3] For notational simplicity, we will often identify sets X of variables with sets of the corresponding indices and thus simply write $i \in X$ instead of $x_i \in X$.

be smaller than $\mathbf{h}_{\gamma_{i+1}}$ according to the J-order. Hence $\mathbf{h}_\alpha$ is smaller or equal than $\mathbf{h}_\beta$ according to the J-order (in fact equality only happens when x^μ is multiplicative for $\mathbf{h}_\alpha$). □

Lemma 14. *Let $p = \mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$ be a reduction path that appears in the differential (7) (possibly as part of a longer path). If $\rho_p\big(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)\big) = x^\kappa \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$, then $\mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^{\kappa+\nu}\mathbf{v}_j\mathbf{h}_\beta) \preceq_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^\mu \mathbf{v}_i \mathbf{h}_\alpha)$.*

Proof. We prove the assertion only for an elementary reduction path p and the general case follows by induction over the path length. If p is of type 1 we can easily prove the assertion by using the same arguments as for the corresponding lemma in the Pommaret case [2, Lemma 4.6].

If p is of type 2, there exists an index $j \in \mathrm{supp}(\mu)$ (implying $j \in \mathrm{ncrit}(\mathbf{h}_\alpha)$) and thus $j \in \mathcal{X}_{J,\mathcal{H},\prec}(\mathbf{h}_\alpha)$, a multi index ν and a scalar $\lambda \in \Bbbk$ such that $\rho_p(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)) = \lambda \mathbf{v}_j(x^\nu \mathbf{h}_\gamma)$ where $x^\nu \mathbf{h}_\gamma$ appears in the involutive standard representation of $\frac{x^\mu x_i}{x_j}\mathbf{h}_\alpha$ with a non-vanishing coefficient. Lemma 12 implies now $j \in \mathrm{crit}(\mathbf{h}_\gamma)$. By construction, $\mathrm{lt}_\prec(\frac{x_i x^\mu}{x_j}\mathbf{h}_\alpha) \succeq \mathrm{lt}_\prec(x^\nu \mathbf{h}_\gamma)$.

Here we have to distinguish between equality and strict inequality. If strict inequality holds, then also $\mathrm{lt}_\prec(x_i x^\mu) \succ \mathrm{lt}_\prec(x_j x^\nu \mathbf{h}_\gamma)$. Hence by definition of the Schreyer order we get $\mathrm{lt}_{\prec_{\mathcal{H}_1}} x^\mu \mathbf{v}_i \mathbf{h}_\alpha \succ_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^{\kappa+\nu}\mathbf{v}_j \mathbf{h}_\beta)$. In the case of equality, we note that $x^\nu\, \mathrm{lt}_\prec(\mathbf{h}_\gamma)$ must be an involutive divisor of $\frac{x_i x^\mu}{x_j}\, \mathrm{lt}_\prec(\mathbf{h}_\alpha)$. Hence Lemma 13 guarantees that $\mathbf{h}_\alpha$ is smaller than $\mathbf{h}_\gamma$ according to the J-order and hence the claim follows for this special case. □

For notational simplicity, we formulate the two decisive corollaries only for the special case of second syzygies, but they remain valid in any homological degree. They assert that there is a one-to-one correspondence between the leading terms of the syzygies contained in the free resolution (1) and of the syzygies in Sköldberg's resolution, respectively.

Corollary 15. *If $i < j$, then $\mathrm{lt}_{\prec_{\mathcal{H}_1}}\big(d_{\mathcal{G}}(\mathbf{v}_{(i,j)}\mathbf{h}_\alpha)\big) = x_j \mathbf{v}_i \mathbf{h}_\alpha$.*

Proof. We assume that the elements of the given Janet basis are numbered according to a J-order. Consider now the differential $d_{\mathcal{G}}$. We first compare the terms $x_i \mathbf{v}_j \mathbf{h}_\alpha$ and $x_j \mathbf{v}_i \mathbf{h}_\alpha$. Lemma 12 (or the minimality of these terms with respect to any order respecting the used Morse matching) entails that there are no reduction paths $[\mathbf{v}_j \mathbf{h}_\alpha \rightsquigarrow \mathbf{v}_k \mathbf{h}_\delta]$ with $k \in \mathrm{crit}\,(\mathbf{h}_\delta)$ (except trivial reduction paths of length 0). By definition of the Schreyer order, we have $x_i \mathbf{v}_j \mathbf{h}_\alpha \prec_{\mathcal{H}_1} x_j \mathbf{v}_i \mathbf{h}_\alpha$.

Now consider any other term in the sum. We will prove $x_j \mathbf{v}_i \mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa \mathbf{v}_i \mathbf{h}_\beta$, where $x^\kappa \mathbf{h}_\beta$ effectively appears in the involutive standard representation of $x_j \mathbf{h}_\alpha$. Then the claim follows from applying Lemma 14 with $x_j \mathbf{v}_i \mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa \mathbf{v}_i \mathbf{h}_\beta \succeq_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}\big(\rho_p(\mathbf{v}_i x^\kappa \mathbf{h}_\beta)\big)$.

We always have $\mathrm{lt}_\prec(x_j x_i \mathbf{h}_\alpha) \succeq \mathrm{lt}_\prec(x^\kappa x_i \mathbf{h}_\beta)$. If this is a strict inequality, then $x_j \mathbf{v}_i \mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa \mathbf{v}_i \mathbf{h}_\beta$ follows at once by definition of the Schreyer order. So now assume $\mathrm{lt}_\prec(x_j x_i \mathbf{h}_\alpha) = \mathrm{lt}_\prec(x^\kappa x_i \mathbf{h}_\beta)$. By construction, $x^\kappa \in$

$\Bbbk[\mathcal{X}_{J,\mathcal{H}}(\mathbf{h}_\beta)]$. Again by definition of the Schreyer order, the claim follows, if we can prove $\mathrm{lt}_{\prec_{\mathcal{H}_0}}(x_j x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha) \succ_{\mathcal{H}_0} \mathrm{lt}_{\prec_{\mathcal{H}_0}}(x^\kappa x_i \mathbf{v}_\emptyset \mathbf{h}_\beta)$. Since $j \in \mathrm{crit}\,(\mathbf{h}_\alpha)$ and $\mathrm{lt}\,(x_j \mathbf{h}_\alpha)$ is involutively divisible by $\mathrm{lt}\,(\mathbf{h}_\beta)$, we have $\alpha < \beta$, by definition of a J-ordering. As we have $\mathrm{lt}_\prec(x_j \mathbf{h}_\alpha) = \mathrm{lt}_\prec(x^\kappa \mathbf{h}_\beta)$, this implies $\mathrm{lt}_{\prec_{\mathcal{H}_0}}(x_j x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha) \succ_{\mathcal{H}_0} \mathrm{lt}_{\prec_{\mathcal{H}_0}}(x^\kappa x_i \mathbf{v}_\emptyset \mathbf{h}_\beta)$ and therefore $\mathrm{lt}_{\prec_{\mathcal{H}_1}}(x_j \mathbf{v}_i \mathbf{h}_\alpha) \succ_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^\kappa \mathbf{v}_i \mathbf{h}_\beta)$. □

Corollary 16. *The set* $\{d_{\mathcal{G}}(v_{\mathbf{k}} \mathbf{h}_\alpha) \mid |\mathbf{k}| = 2; \mathbf{k} \subseteq \mathrm{crit}\,(\mathbf{w}_\alpha)\}$ *is a Janet basis with respect to the term order* $\prec_{\mathcal{H}_0}$.

With Lemma 10 and these two corollaries, we are able to prove that Sköldberg's resolution is isomorphic to the resolution (1). The proof is essentially the same as for a Pommaret basis, only the mentioned lemmata and corollaries must be replaced by their Janet version.

Theorem 17. *Let* $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ *be the Janet basis of the polynomial submodule* $\mathcal{U} \subseteq \mathcal{P}^t$*, where* $\mathcal{H}$ *is a* J*-ordered Janet basis of* $\mathcal{U}$*. Then the resolution* $(\mathcal{G}, d_{\mathcal{G}})$ *is isomorphic to the resolution induced by* $\mathcal{H}$*.*

4 Benchmarks

We describe now the results from a large set of benchmarks comparing our approach with standard methods. As already discussed in [2], we focus for this comparison on the determination of Betti numbers, as currently our implementation is not yet competitive for computing minimal resolution because of the rather naive minimisation strategy used. Indeed, an important aspect for the performance of our approach is the size of the generally non-minimal resolution (1). For a given ideal $\mathcal{I}$, we call its length the *projective pseudo-dimension* $\mathrm{ppd}\,\mathcal{I}$ and the maximal degree of a generator appearing in it the *pseudo-regularity* $\mathrm{preg}\,\mathcal{I}$. It follows from the results above that for the resolution induced by a Janet basis $\mathrm{ppd}\,\mathcal{I}$ is just the maximal number of non-multiplicative variables and $\mathrm{preg}\,\mathcal{I}$ the maximal degree of a generator. As the rough measure for the size of the whole minimal solution, we define the *Betti rank* $\mathrm{brk}\,\mathcal{I}$ as the sum of all Betti numbers and similarly the *Betti pseudo-rank* $\mathrm{bprk}\,\mathcal{I}$.

We have implemented the algorithms explained above in the computer algebra system CoCoALib [1]. The implementation is very similar to the one based on Pommaret bases which we described in [2]. The main difference is that we can no longer guarantee that $\mathrm{preg}\,\mathcal{I} = \mathrm{reg}\,\mathcal{I}$ and $\mathrm{ppd}\,\mathcal{I} = \mathrm{pd}\,\mathcal{I}$. But it is straightforward to accomodate for this effect. For comparison purposes, we used as benchmark the implementations of the standard algorithms in Singular [4] and Macaulay2 [7].

For many geometrical and topological applications, it is sufficient to know only the Betti numbers; the differential of the minimal resolution is not required. To our knowledge, all current implementations read off the Betti numbers from a free resolution. By contrast, our method can determine Betti numbers without computing a complete resolution. We briefly sketch our method described in more details in [2].

Firstly, we compute only the constant part of the complex $\mathcal{G}_\bullet$. If we perform an elementary reduction of type 2 the degree of the map does not change. For an elementary reduction of type 1, the degree increases by one. Thus we obtain the constant part of $\mathcal{G}_\bullet$ by only applying reductions of type 2 on the constant part of the complex $\mathcal{F}_\bullet$. It follows from the explicit form (4) of the differential $d_\mathcal{F}$ that the left summand yields always elements of degree one and the right summand elements of degree zero. Hence, by simply skipping the left summands, we directly obtain the constant part of $\mathcal{F}_\bullet$.

Because of the above proven isomorphy between the complex $\mathcal{G}_\bullet$ and the resolution induced by the Janet basis, the bigraded ranks $r_{i,j}$ of the components of the complex $\mathcal{G}_\bullet$ can be directly determined with (2). Then, as described above, we construct (degreewise) the constant part of the matrices of $d_\mathcal{G}$. Subtracting their ranks from the corresponding $r_{i,j}$ yields the Betti numbers $b_{i,j}$. It should be noted that this approach also allows to compute directly individual Betti numbers, as the explicit expressions for the differentials in the complexes $\mathcal{F}_\bullet$ and $\mathcal{G}_\bullet$, resp., show that the submatrices relevant for the different Betti numbers are independent of each other.

Our testing environment consists of an Intel i5-4570 processor with 8GB DDR3 main memory. As operating system we used Fedora 20 and as compiler for the CoCoALib gcc 4.8.3. The running times are given in seconds and we limited the maximal time usage to two hours and the maximal memory consumption to 7.5 GB. A * marks when we run out of time and ** marks when we run out of memory. A bold line indicates that the given example is δ-singular, i. e. that no Pommaret basis exists for it in the used coordinates. As benchmarks, we took a number of standard examples given in [16]. As most of these ideals are not homogeneous, we homogenised them by adding a new smallest variable. Furthermore, we always chose $\Bbbk = \mathbb{Z}/101\mathbb{Z}$ as base field.

Singular and Macaulay2 apply the command `res` for computing a free resolution at first. In a second step both systems extract the Betti numbers from the resolution. Singular uses the classical Schreyer Theorem to compute a free resolution, which is possibly not minimal, and then determines the graded Betti numbers from it. Macaulay2 uses La Scalas method to compute a minimal free resolution and read off the graded Betti numbers. For Singular and Macaulay2 we took as input the reduced Gröbner basis of the ideal; for our algorithm the Janet basis. Because of our choice of a small coefficient field, the time needed for the determination of these input bases is neglectable (for almost all examples less than two seconds).

The benchmarks presented in Table 1 show that our approach is generally much faster than the standard methods requiring a complete resolution (often by orders of magnitude!). In particular, it scales much better when examples are getting larger. Even for δ-singular ideals, we are in general much faster than the standard methods. In fact, there is no obvious difference between δ-singular and δ-regular examples.

In Table 2 we collect some data about the examples in Table 1. The following list describes the columns:

Example	Time Macaulay2	Time Singular	Time CoCoALib
butcher8	**126.25**	**19.92**	**1.20**
camera1s	**0.09**	**6.00**	**0.13**
chandra6	0.64	8.00	0.13
cohn2	**0.03**	**1.00**	**0.03**
cohn3	**1.47**	**5.90**	**0.32**
cpdm5	14.71	5.05	0.64
cyclic6	0.99	1.26	0.37
cyclic7	1 093.66	*	37.42
cyclic8	*****	*****	**1 663.00**
des18_3	433.45	20.84	3.15
des22_24	*	**	52.19
dessin1	428.13	20.89	3.10
dessin2	*	*	32.90
f633	**591.08**	**7.70**	**49.06**
hcyclic5	**0.03**	**2.00**	**0.09**
hcyclic6	**11.00**	**47.12**	**7.41**
hcyclic7	*****	*****	**3 688.01**
hemmecke	**0.00**	**0.00**	**2.69**
hietarinta1	**443.15**	**170.29**	**4.12**
katsura6	51.41	13.90	1.22
katsura7	**	1 373.70	15.87
katsura8	*	**	412.90
kotsireas	**51.89**	**17.84**	**0.83**
mckay	**0.84**	**3.20**	**0.38**
noon5	0.13	6.00	0.27
noon6	15.14	5.07	5.25
noon7	6 979.40	821.64	122.61
rbpl	**58.81**	**22.69**	**57.91**
redcyc5	**0.02**	**2.00**	**0.01**
redcyc6	**6.79**	**1.95**	**0.13**
redcyc7	*****	*****	**8.26**
redcyc8	*****	******	**207.02**
redeco7	2.72	2.20	0.42
redeco8	355.30	11.83	5.01
redeco9	**	312.49	84.89
redeco10	**	**	2 694.05
reimer4	0.01	1.00	0.01
reimer5	1.39	5.00	0.35
reimer6	1 025.89	176.08	19.01
speer	**0.20**	**3.00**	**0.13**

Table 1. Various examples for computing Betti diagramms

Example: name of the example
#JB: number of elements in the minimal Janet basis
#GB: number of elements in the reduced Gröbner basis
$\frac{\#\mathbf{JB}}{\#\mathbf{GB}}$**:** the quotient of #JB and #GB
ppd: the projective pseudo-dimension
pd: the projective dimension
preg: the pseudo-regularity
reg: the regularity
bprk: the Betti pseudo-rank
brk: the Betti rank
$\frac{\mathbf{bprk}}{\mathbf{brk}}$**:** the quotient of bprk and brk.

In our test set there is only a very small subset of examples where the standard algorithms perform better than our new algorithm. For example, for *hemmecke* our method needs 2.69 seconds to compute the Betti diagram, whereas Singular and Macaulay2 do not even need a measurable amount of time. If we take a look at the size of the minimal Janet basis and the reduced Gröbner basis in Table 2, we see immediately why this example is bad for our algorithm. The reduced Gröbner basis consists of 9 elements, but the minimal Janet basis contains 983 elements.[4] Therefore the Betti pseudo-rank is with 6 242 much larger than the real Betti rank of 38. As a consequence, we must spend first much time to compute the constant part of a large resolution and then even more time for reducing it. In comparatively small examples like *cyclic6* one notices in our approach overhead effects because of the need to set up complex data structures. In general, one observes that the larger the example (in particular, the larger its projective dimension) the better our algorithm fares in comparison to the standard methods.

It seems that the quotient $\frac{\#\text{JB}}{\#\text{GB}}$ provides a good indication whether or not our algorithm is fast relative to the standard methods. One could think that the quotient $\frac{\text{bprk}}{\text{brk}}$ is also a good indicator for efficiency. But in our test set we cannot identify such a correlation. In fact, even if the factor is greater than 100, somewhat surprisingly our algorithm can be faster (see *redeco10*).

There are two aspects which may explain this observation for *redeco10*. The first one is that we only perform matrix operations over the base field, which are not only much more efficient than polynomial computations but also consume much less memory. The second one could be the relatively large projective dimension 10 of *redeco10*. Classical methods have to compute roughly $\operatorname{pd}\mathcal{I}$ Gröbner bases to determine the Betti numbers. Our approach requires always only one Janet basis and some normal form computations.

Another interesting observation in Table 2 concerns the difference of the projective pseudo-dimension and the pseudo-regularity to the true values $\operatorname{pd}\mathcal{I}$ and $\operatorname{reg}\mathcal{I}$ for δ-singular ideals. In our test set only for two examples (*hcylic5* and *mckay*) the values differ and in only one of them (*mckay*) a significant difference occurs. Thus it seems that in typical benchmark examples no big differences in

[4] Although this example is not a toric ideal, it shares certain characteristic features of toric ideals. It is well-known that for such ideals special techniques must be employed.

Example	#JB	#GB	$\frac{\#\text{JB}}{\#\text{GB}}$	ppd	pd	preg	reg	bprk	brk	$\frac{\text{bprk}}{\text{brk}}$
butcher8	**64**	**54**	**1.19**	**8**	**8**	**3**	**3**	**3 732**	**2 631**	**1.42**
camera1s	**59**	**29**	**2.03**	**6**	**6**	**4**	**4**	**863**	**337**	**2.56**
chandra6	32	32	1.00	6	6	5	5	684	64	10.69
cohn2	**33**	**23**	**1.43**	**4**	**4**	**7**	**7**	**179**	**67**	**2.67**
cohn3	**106**	**92**	**1.15**	**4**	**4**	**7**	**7**	**696**	**370**	**1.88**
cpdm5	83	77	1.08	5	5	9	9	1 020	100	10.20
cyclic6	46	45	1.02	6	6	9	9	1 060	320	3.31
cyclic7	210	209	1.00	7	7	11	11	10 356	1 688	6.14
cyclic8	**384**	**372**	**1.03**	**8**	**8**	**12**	**12**	**34 136**	**6 400**	**5.33**
des18_3	104	39	2.67	8	8	4	4	8 132	2 048	3.97
des22_24	129	45	2.87	10	10	4	4	32 632	6 192	5.27
dessin1	104	39	2.67	8	8	4	4	8 132	2 048	3.97
dessin2	122	46	2.65	10	10	4	4	22 760	6 192	3.68
f633	**153**	**47**	**3.26**	**10**	**10**	**3**	**3**	**17 390**	**4 987**	**3.49**
hcyclic5	**52**	**38**	**1.37**	**6**	**5**	**11**	**10**	**932**	**32**	**29.13**
hcyclic6	**221**	**99**	**2.23**	**7**	**7**	**14**	**14**	**9 834**	**146**	**67.36**
hcyclic7	**1 182**	**443**	**2.67**	**8**	**8**	**17**	**17**	**105 957**	**1 271**	**83.37**
hemmecke	**983**	**9**	**109.22**	**4**	**4**	**61**	**61**	**6 242**	**38**	**164.26**
hietarinta1	**52**	**51**	**1.02**	**10**	**10**	**2**	**2**	**6 402**	**3 615**	**1.77**
katsura6	43	41	1.05	7	7	6	6	1 812	128	14.16
katsura7	79	74	1.07	8	8	7	7	6 900	256	26.95
katsura8	151	143	1.06	9	9	8	8	27 252	512	53.23
kotsireas	**78**	**70**	**1.11**	**6**	**6**	**5**	**5**	**1 810**	**1 022**	**1.77**
mckay	**126**	**51**	**2.47**	**4**	**4**	**15**	**9**	**840**	**248**	**3.39**
noon5	137	72	1.90	5	5	8	8	1 618	130	12.45
noon6	399	187	2.13	6	6	10	10	9 558	322	29.68
noon7	1 157	495	2.34	7	7	12	12	56 666	770	73.59
rbpl	**309**	**126**	**2.45**	**7**	**7**	**14**	**14**	**13 834**	**1 341**	**10.32**
redcyc5	**23**	**10**	**2.30**	**5**	**5**	**7**	**7**	**276**	**88**	**3.14**
redcyc6	**46**	**21**	**2.19**	**6**	**6**	**9**	**9**	**1 060**	**320**	**3.31**
redcyc7	**210**	**78**	**2.69**	**7**	**7**	**11**	**11**	**10 356**	**1 688**	**6.14**
redcyc8	**371**	**193**	**1.92**	**8**	**8**	**12**	**12**	**32 459**	**6 973**	**4.65**
redeco7	48	33	1.45	7	7	5	5	1 708	128	13.34
redeco8	96	65	1.48	8	8	6	6	6 828	256	26.67
redeco9	192	129	1.49	9	9	7	7	27 308	512	53.34
redeco10	384	257	1.49	10	10	8	8	109 228	1 024	106.67
reimer4	19	17	1.12	4	4	6	6	118	16	7.38
reimer5	55	38	1.45	5	5	9	9	694	32	21.69
reimer6	199	95	2.09	6	6	12	12	5 302	64	82.84
speer	**49**	**44**	**1.11**	**5**	**5**	**7**	**7**	**359**	**133**	**2.70**

Table 2. Statistics for examples from Table 1

the sizes of the induced resolutions for Janet and Pommaret bases, respectively, occur, although we showed in Example 8 that theoretically the difference may become arbitrarily large.

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