COMPUTING QUOT SCHEMES VIA MARKED BASES OVER QUASI-STABLE MODULES

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Abstract. Let $\Bbbk$ be a field of arbitrary characteristic, A a Noetherian $\Bbbk$-algebra and consider the polynomial ring $A[\mathbf{x}] = A[x_0, \dots, x_n]$. We consider homogeneous submodules of $A[\mathbf{x}]^m$ having a special set of generators, a marked basis over a quasi-stable module. Such a marked basis shares many interesting properties with a Groebner basis, including the existence of a Noetherian reduction relation. The set of submodules of $A[\mathbf{x}]^m$ having a marked basis over a given quasi-stable module possesses an affine scheme structure which we will exhibit. Furthermore, the syzygies of a module with such a marked basis are generated by a marked basis, too (over a suitable quasi-stable module in $\oplus_{i=1}^{m'} A[\mathbf{x}](-d_i)$). We apply marked bases and related properties to the investigation of Quot functors (and schemes). More precisely, for a given Hilbert polynomial, we explicitely construct (up to the action of a general linear group) an open cover of the corresponding Quot functor, made up of open functors represented by affine schemes. This provides a new proof that the Quot functor is the functor of points of a scheme. We also exhibit a procedure to obtain the equations defining a given Quot scheme as a subscheme of a suitable Grassmannian. Thanks to the good behaviour of marked bases with respect to Castelnuovo-Mumford regularity, we can adapt our methods in order to study the locus of the Quot scheme given by an upper bound on the regularity of its points.

Introduction

Marked bases may be considered as a form of Gröbner bases which do not depend on a term order. Instead one chooses for each generator some term as head term such that the head terms generate a prescribed monomial ideal. For a long time, it was believed that it was not possible to use a marked basis which is not a Gröbner basis with respect to some term order while preserving the good features of Gröbner bases such as the termination of the standard normal form algorithm. Indeed, it was shown in [25] that the standard normal form algorithm always terminates, if and only if the head terms are chosen via a term order. However, [25] contains no results about other normal form algorithms and it was proven in [4, 10] that the involutive normal form algorithm for the Pommaret division will terminate whenever the head terms generate a strongly stable ideal over a coefficient field of characteristic zero.

The present paper is concerned with generalising and deepening the results of [1, 4, 10, 22] for the investigation of *Quot schemes*. The Quot functor was introduced by Grothendieck in [16], where he also proved that it is the functor of points of a projective scheme. A Hilbert scheme is a special case of a Quot scheme. In the present paper, we consider the Quot functor that associates to a $\Bbbk$-scheme Z the set of quotients of $\mathcal{O}^m_{\mathbb{P}^n_Z}$ with a given Hilbert polynomial and flat over Z (see Section 7). We will not use the fact that the Quot functor is the functor of points of a scheme, but actually we will give an independent proof of the existence of the Quot scheme (Corollary 10.2) only assuming that the Quot functor is a Zariski sheaf [24, Section 5.1.3].

After setting some notations and recalling some useful notions and results (Sections 1, 2), the first part of the paper is devoted to the investigation of the properties of marked sets, bases and schemes over a quasi-stable module (Sections 3 and 4) and of the syzygies of a marked basis (Section 5). Let $\Bbbk$ be a field *of arbitrary characteristic* and A a Noetherian $\Bbbk$-algebra. For variables $\mathbf{x} := \{x_0, \dots, x_n\}$ and a *weight vector* $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{Z}^m$, we consider homogeneous submodules of the graded $A[\mathbf{x}]$-module $A[\mathbf{x}]^m_{\mathbf{d}} := \oplus_{i=1}^m A[\mathbf{x}](-d_i)$. We will define marked bases over a *quasi-stable* monomial module U, i.e. over a monomial module possessing a Pommaret

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basis, for free submodules of $A[\mathbf{x}]^m_{\mathbf{d}}$ and investigate to what extent the algebraic properties of Pommaret bases shown in [28] carry over to marked bases. It will turn out that marked bases provide us with simple upper bounds for some homological invariants of the module they generate, such as Betti numbers, (Castelnuovo-Mumford) regularity and projective dimension (Corollary 5.8). Furthermore, we prove that the set of modules generated by a marked basis over a given quasi-stable module has an affine scheme structure and we show an algorithmic procedure to compute explicit equations defining this scheme (Theorem 4.1).

In the second part of the paper, we use marked bases in the context of a very specific application, namely for the derivation and the study of equations for Quot schemes and of special loci on them similarly to what is shown in [1, 7, 5] for Hilbert schemes in the characteristic zero case. In Section 6, we consider the usual action of $g \in \mathrm{PGL}(n+1)$ over a finite subset F of $A[\mathbf{x}]^m$. We show that for every finite $F \subset A[\mathbf{x}]^m$ we can algorithmically construct a transformation $g \in \mathrm{PGL}(n+1)$ such that the transformed set $\tilde{g} \centerdot F$ is marked over a quasi-stable module.

In Section 7, we recall the definition of the Quot functor and its embedding in a suitable Grassmannian functor. We first prove that, up to the action of PGL, a Grassmann functor has a cover made up of open subsets depending on quasi-stable modules (Section 8). In Section 9, we intersect this open cover with a Quot scheme and prove that, for a given Quot scheme, we have an open cover (up to the action of $\mathrm{PGL}(n+1)$) whose open subsets are suitable marked schemes over quasi-stable modules belonging to the Quot scheme (Theorem 9.5). The same holds if, instead of considering the whole marked scheme, we are interested in the points respecting an upper bound on the regularity: in this case, the open subsets are marked schemes over quasi-stable modules that respect the bound on the regularity, too.

Starting from this open cover, we obtain in Section 10 global equations defining a Quot scheme (resp. its locus defined by an upper bound on the regularity) as a closed (resp. locally closed) subscheme of a suitable projective space (Theorem 10.3). We end the paper with an explicit example (Section 11).

1. Notations and Generalities

For every $n > 0$, we consider the variables $x_0, \dots, x_n$, ordered as $x_0 < \cdots < x_{n-1} < x_n$ (see [27, 28]). This is a non-standard way to sort the variables, but it is suitable for our purposes. In some of the papers we refer to, variables are ordered in the opposite way, hence the interested reader should pay attention to this when browsing a reference. A *term* is a power product $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We denote by $\mathbb{T}$ the set of all terms in the variables $x_0, \dots, x_n$. We denote by $\max(x^\alpha)$ the largest variable that appears with non-zero exponent in x^α and, analogously, $\min(x^\alpha)$ is the smallest variable that appears with non-zero exponent in x^α. The *degree* of a term is $\deg(x^\alpha) = \sum_{i=0}^n \alpha_i = |\alpha|$.

Let $\Bbbk$ be a field and A be a Noetherian $\Bbbk$-algebra. Consider the polynomial ring $A[\mathbf{x}] := A[x_0, \dots, x_n]$ with the standard grading: for every $a \in A$ we set $\deg(a) = 0$. We write $A[\mathbf{x}]_t$ for the set of homogeneous polynomials of degree t in $A[\mathbf{x}]$. Since $A[\mathbf{x}] = \oplus_{i \geqslant 0} A[\mathbf{x}]_i$, we define $A[\mathbf{x}]_{\geqslant t} := \oplus_{i \geqslant t} A[\mathbf{x}]_i$. The ideals we consider in $A[\mathbf{x}]$ are always homogeneous. If $I \subset A[\mathbf{x}]$ is a homogeneous ideal, we write I_t for $I \cap A[\mathbf{x}]_t$ and $I_{\geqslant t}$ for $I \cap A[\mathbf{x}]_{\geqslant t}$. The ideal $I_{\geqslant t}$ is the *truncation* of I in degree t. If $F \subset A[\mathbf{x}]$ is a set of polynomials, we denote by (F) the ideal generated by F.

An ideal $J \subseteq A[\mathbf{x}]$ is *monomial* if it is generated by a set of terms. A monomial ideal J has a unique minimal set of generators made of terms and we call it *the monomial basis* of J, denoted by $\mathcal{B}_J$. We define $\mathcal{N}(J) \subseteq \mathbb{T}$ as the set of terms in $\mathbb{T}$ not belonging to J. For every polynomial $f \in A[\mathbf{x}]$, $\mathrm{Supp}(f)$ is the set of terms appearing in f with a non-zero coefficient: $f = \sum_{x^\alpha \in \mathrm{Supp}(f)} c_\alpha x^\alpha$ where $c_\alpha \in A$ is non-zero.

Hereafter, we will simply write *module* (resp. submodule) for an $A[\mathbf{x}]$-module (resp. submodule of an $A[\mathbf{x}]$-module). For modules and submodules over other rings, we will explicitly state the ring. A module M is *graded*, if it has a decomposition

$$M = \oplus_{j \in \mathbb{N}} M_j \text{ such that } A[\mathbf{x}]_i M_j \subseteq M_{i+j}.$$

If M is a graded module, the module $M_{\geqslant t} := \oplus_{i \geqslant t} M_i$ is the truncation of M in degree t. As usual, if M is a graded module, the module $M(d)$ is the graded module (isomorphic to M as a module) such that $M(d)_e = M_{d+e}$. We fix an integer $m \geqslant 1$ and $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{Z}^m$. We consider the free graded $A[\mathbf{x}]$-module $A[\mathbf{x}]^m_{\mathbf{d}} := \oplus_{i=1}^m A[\mathbf{x}](-d_i)e_i$, where $e_1, \dots, e_m$ are the standard free generators. Every submodule of $A[\mathbf{x}]^m_{\mathbf{d}}$ is finitely generated and from now on we will only consider graded submodules of $A[\mathbf{x}]^m_{\mathbf{d}}$.

If F is a set of homogeneous elements of $A[\mathbf{x}]^m_{\mathbf{d}}$, we write $\langle F\rangle$ for the graded $A[\mathbf{x}]$-module generated by F in $A[\mathbf{x}]^m_{\mathbf{d}}$. If $F = F_s$ for some positive integer s, we denote by $\langle F\rangle^A$ the A-module generated by F in $(A[\mathbf{x}]^m_{\mathbf{d}})_s$. In particular, if M is a graded submodule, every graded component M_j has the structure of A-submodule in $A[\mathbf{x}]^m_j$. Following [12, Chapter 15], a *term of* $A[\mathbf{x}]^m_{\mathbf{d}}$ is an element of the form $t = x^\alpha e_i$ for $i \in \{1, \dots, m\}$ and $x^\alpha \in \mathbb{T}$. Furthermore, we denote by $\mathbb{T}^m$ the set of terms in $A[\mathbf{x}]^m_{\mathbf{d}}$. Observe that $\mathbb{T}^m = \cup_{i=1}^m \mathbb{T}e_i$. For $x^\alpha e_i, x^\beta e_j$ in $\mathbb{T}^m$ we say that $x^\alpha e_i$ divides $x^\beta e_j$ if $i = j$ and x^α divides x^β. A submodule U of $A[\mathbf{x}]^m_{\mathbf{d}}$ is *monomial*, if it is generated by elements in $\mathbb{T}^m$. Any monomial submodule U of $A[\mathbf{x}]^m$ can be written as

$$U = \oplus_{k=1}^m J^{(k)} e_k \subset \oplus A[\mathbf{x}](-d_k)e_k = A[\mathbf{x}]^m_{\mathbf{d}}, \tag{1.1}$$

where each $J^{(k)}$ is the monomial ideal generated by the terms x^α such that $x^\alpha e_k \in U$. We define $\mathcal{N}(U) := \cup_{k=1}^m \mathcal{N}(J^{(k)})e_k$, where $\mathcal{N}(J^{(k)}) \subseteq \mathbb{T}$.

If $M \subset A[\mathbf{x}]^m_{\mathbf{d}}$ is a submodule such that for every degree s, the homogeneous component M_s is a free A-module, we define the *Hilbert function* of M as $h_M(s) = \mathrm{rk}(M_s)$, which is the number of generators contained in an A-basis of M_s. In this case, we will also say that M *admits a Hilbert function*. In this setting, this definition corresponds to the classical one (e.g. [12, Chapter 12]), considering the localisation of A in any of its maximal ideals. If we consider a monomial module U, for every s, U_s is always a free A-module and $h_U(s) = \sum_{k=1}^m h_{J^{(k)}}(s)$, with $J^{(k)}$ as in (1.1). If M admits a Hilbert function, then for $s \gg 0$, $h_M(s) = p(s)$, where $p(z)$ is a numerical polynomial (see also [7, Section 1]).

If $A = \Bbbk$, then Hilbert's Syzygy Theorem guarantees that every module $M \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ has a graded free resolution of length at most n. If A is an arbitrary $\Bbbk$-algebra, there exist generally modules in $A[\mathbf{x}]^m_{\mathbf{d}}$ whose minimal free resolution has an infinite length (see [12, Chapter 6, Section 1, Exercise 11]). Assume that the module $M \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ has the following graded minimal free resolution

$$0 \to E_\ell \to \cdots \to E_1 \to E_0 \to M \to 0, \tag{1.2}$$

where $E_i = \oplus_j A[\mathbf{x}](-j)^{b_{i,j}}$. The *Betti numbers* of the module M are the set of positive integers $\{b_{i,j}\}_{0\leqslant i\leqslant p, j\in\mathbb{Z}}$. The module M is t-regular if $t \geqslant j - i$ for every i, j such that $b_{i,j} \neq 0$. The *(Castelnuovo-Mumford) regularity* of M, denoted by $\mathrm{reg}(M)$, is the smallest t for which M is t-regular (see for instance [13]). If $M \subset A[\mathbf{x}]^m_{\mathbf{d}}$ admits a Hilbert function, we recall that $h_{A[\mathbf{x}]^m/M}(s) = p(s)$ for all degrees $s \geqslant \mathrm{reg}(M)$. The *projective dimension* of M, denoted by $\mathrm{pdim}(M)$, is defined as the length of the graded minimal free resolution (1.2), i. e. $\mathrm{pdim}(M) = \ell$.

From [21, Definition 3.5.7], consider the ideal $\mathfrak{m} := (x_0, \dots, x_n) \subset A[\mathbf{x}]$. The *saturation* of M, submodule of $A[\mathbf{x}]^m$, is

$$M^{\mathrm{sat}} := M : \mathfrak{m}^\infty = \bigcup_{i\in\mathbb{N}} M : \mathfrak{m}^i = \{f \in A[\mathbf{x}]^m \mid \mathfrak{m}^i f \subset M \text{ for some } i \in \mathbb{N}\}$$

2. Pommaret basis, Quasi-Stability and Stability

We now recall the definition and some properties of the Pommaret basis of a monomial module. Several of the following definitions and properties hold in a more general setting, namely for arbitrary *involutive divisions*. For a deeper insight into this topic, we refer to [27, 28] and the references therein. For an arbitrary term $x^\alpha \in \mathbb{T}$, we define the following sets:

- the *multiplicative variables of* x^α: $\mathcal{X}_{\mathcal{P}}(x^\alpha) := \{x_i \mid x_i \leqslant \min(x^\alpha)\}$,
- the *non-multiplicative variables of* x^α: $\overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha) := \{x_0, \dots, x_n\} \setminus \mathcal{X}_{\mathcal{P}}(x^\alpha)$.

Definition 2.1. Let $T \subset \mathbb{T}^m$ be a finite set of monomial generators for U. For every $\tau = x^\alpha e_k$ in T, we define the *Pommaret cone in* $A[\mathbf{x}]^m_{\mathbf{d}}$ of τ as

$$\mathcal{C}^m_{\mathcal{P}}(\tau) := \{x^\delta x^\alpha e_k \mid \delta_i = 0 \ \ forall x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha)\} \subset \mathbb{T}e_k.$$

Let U be a monomial submodule of $A[\mathbf{x}]^m_{\mathbf{d}}$. We say that $T \subset \mathbb{T}^m$ is a *Pommaret basis* of U if

$$U \cap \mathbb{T}^m = \bigsqcup_{\tau \in T} \mathcal{C}^m_{\mathcal{P}}(\tau).$$

If U is a monomial module, we denote its Pommaret basis (if it exists) by $\mathcal{P}(U)$. The existence of the Pommaret basis of a monomial module in $A[\mathbf{x}]$ is equivalent to the concept of *quasi-stability*. In the literature, one can find a number of alternative names for quasi-stability (e.g. [8, 2, 18]). We recall here the definition of quasi-stable and stable monomial modules. Both properties do not depend on the characteristic of the underlying field. A thorough reference on this subject is again [28].

Definition 2.2. [9, Definition 4.4] Let $U \subset A[\mathbf{x}]^m$ be a monomial module.

(i) U is *quasi-stable*, if for every term $x^\alpha e_k \in U \cap \mathbb{T}^m$ and for every non-multiplicative variable $x_j \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha)$, there is an exponent $s \geqslant 0$ such that $x_j^s x^\alpha / \min(x^\alpha) e_k \in U$.

(ii) U is *stable*, if for every term $x^\alpha e_k \in U \cap \mathbb{T}^m$ and for every non-multiplicative variable $x_j \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\alpha)$ we have $x_j x^\alpha / \min(x^\alpha) e_k \in U$.

Theorem 2.3. [28, Proposition 4.4, Proposition 4.6][23, Remark 2.10] *Let $U \subset A[\mathbf{x}]^m$ be a monomial ideal. U is quasi-stable if and only if it has a (finite) Pommaret basis, denoted by $\mathcal{P}(U)$. Furthermore, U is stable if and only if $\mathcal{P}(U)$ is its minimal monomial generating set. If $U \subset A[\mathbf{x}]$ is quasi-stable, then $U_{\geqslant s}$ is quasi-stable for every $s \geqslant 0$.*

Recalling that any monomial module U can be written as $U = \oplus_{k=1}^m J^{(k)} e_k$ with $J^{(k)}$ suitable monomial ideals in $A[\mathbf{x}]$ (see (1.1)), it is immediate that U is quasi-stable (resp. stable) if and only if $J^{(k)}$ is a quasi stable (resp. stable) ideal for every $k \in \{1, \dots, m\}$. If $U \subset A[\mathbf{x}]^m$ is a monomial module, then a term $x^\mu e_k$ is an *obstruction* to quasi-stability for U if there is $x_j > x_c = \min(x^\mu)$, such that for every $s \geqslant 0$, $(x_j^s x^\mu)/x_c^{\mu_c} e_k \notin U$. If the term $x^\mu e_k$ is an obstruction to quasi-stability for U, observe that in particular $x_j^{\mu_c} \dfrac{x^\mu}{x_c^{\mu_c}} e_k$ does not belong to U.

The following lemma collects some properties of a Pommaret basis and of the quasi-stable module it generates. In particular, certain invariants of the quasi-stable module can be directly read off from a Pommaret basis.

Lemma 2.4. *Let U be a quasi-stable module in $A[\mathbf{x}]^m$.*

(i) $U^{\mathrm{sat}} = U : (x_0)^\infty$;

(ii) The satiety of U is the maximal degree of a term in $\mathcal{P}(U)$ which is divisible by the smallest variable in the polynomial ring. If U is saturated, then the smallest variable of the ring does not divide any term in $\mathcal{P}(U)$.

(iii) The regularity of U is the maximal degree of a term in $\mathcal{P}(U)$.

(iv) The projective dimension of U is $n-D$ where D is the index of the variable $\min\{\min(x^\alpha) \mid x^\alpha \in \mathcal{P}(J)\}$.

(v) If $x^\eta e_k \notin U$ and $x_i x^\eta e_k \in U$, then either $x_i x^\eta e_k \in \mathcal{P}(U)$ or $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(x^\eta)$.

(vi) If $x^\eta e_k \notin U$ and $\left(x^\eta \cdot x^\delta\right) e_k = \left(x^{\delta'} x^\alpha\right) e_k \in U$ with $x^\alpha e_k \in \mathcal{P}(U)$ and $x^{\delta'} \in A[\mathcal{X}_{\mathcal{P}}(x^\alpha)]$, then $x^{\delta'} <_{lex} x^\delta$.

Proof. For $m = 1$, items (ii), (iii) and (iv) are proven in [28, Lemma 4.11, Theorems 9.2 and 8.11], item (v) is shown in [3, Lemma 3], item (vi) is a consequence of (v). We obtain the statement for U applying the results for ideals to $J^{(k)}$, $k \in \{1, \dots, m\}$ of (1.1). □

Proposition 2.5.

(i) Let $U \subset A[\mathbf{x}]^m$ be a quasi-stable module generated in degrees less than or equal to s. The module U is s-regular if and only if $U_{\geqslant s}$ is stable.

(ii) Let U be a quasi-stable module in $A[\mathbf{x}]^m$ and consider a degree $s \geqslant \mathrm{reg}(U)$. Then $U_{\geqslant s}$ is stable and the set of terms $U_s \cap \mathbb{T}^m$ is its Pommaret basis.

Proof. For the ideal case $m = 1$, we refer to [28, Lemma 2.2, Lemma 2.3, Theorem 9.2, Proposition 9.6]. For the module case, we repeat the argument of Lemma 2.5. □

3. Marked Modules

We extend the notions of marked polynomial, marked basis and marked family, investigated in [4, 9, 10, 22] for ideals, to finitely generated modules in $A[\mathbf{x}]^m_{\mathbf{d}}$. Let $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a monomial module so that $U = \oplus_{k=1}^m J^{(k)} e_k$ with $J^{(k)}$ monomial ideals in $A[\mathbf{x}]$. If U is a quasi-stable module, we denote by $\mathcal{P}(U)$ the Pommaret basis of U.

Definition 3.1. [25] A *marked polynomial* is a polynomial $f \in A[\mathbf{x}]$ together with a fixed term $x^\alpha \in \mathrm{Supp}(f)$ whose coefficient is equal to 1_A. This term is called *head term* of f and denoted by $\mathrm{Ht}(f)$. With a marked polynomial f, we associate the following sets:

- the *multiplicative variables of* f: $\mathcal{X}_{\mathcal{P}}(f) := \mathcal{X}_{\mathcal{P}}(\mathrm{Ht}(f))$;
- the *non-multiplicative variables of* f: $\overline{\mathcal{X}}_{\mathcal{P}}(f) := \overline{\mathcal{X}}_{\mathcal{P}}(\mathrm{Ht}(f))$.

Definition 3.2. A *marked homogeneous module element* is a homogeneous module element in $A[\mathbf{x}]^m_{\mathbf{d}}$ with a fixed term in its support whose coefficient is 1_A and which is called *head term*. More precisely, a marked homogeneous module element is of the form

$$f^k_\alpha = f_\alpha e_k - \sum_{l \neq k} g_l e_l \in A[\mathbf{x}]^m_{\mathbf{d}}$$

where f_α is a marked polynomial with $\mathrm{Ht}(f_\alpha) = x^\alpha$, and $\mathrm{Ht}(f^k_\alpha) = \mathrm{Ht}(f_\alpha) e_k = x^\alpha e_k$.

The following definition is fundamental for this work. It is modelled on a well-known characteristic property of Gröbner bases.

Definition 3.3. Let $T \subset \mathbb{T}^m$ be a finite set and U the module generated by it in $A[\mathbf{x}]^m_{\mathbf{d}}$. A T*-marked set* is a finite set $G \subset A[\mathbf{x}]^m_{\mathbf{d}}$ of marked homogeneous module elements f^k_α with $\mathrm{Ht}(f^k_\alpha) = x^\alpha e^k \in T$ and $\mathrm{Supp}(f^k_\alpha - x^\alpha e_k) \subset \langle \mathcal{N}(U) \rangle$ (obviously, $|G| = |T|$). A T-marked set G is a T*-marked basis* of the module $\langle G \rangle$, if $\mathcal{N}(U)_s$ is a basis of $\left(A[\mathbf{x}]^m_{\mathbf{d}}\right)_s / \langle G \rangle_s$ as an A-module, i. e. if $\left(A[\mathbf{x}]^m_{\mathbf{d}}\right)_s = \langle G \rangle_s \oplus \langle \mathcal{N}(U)_s \rangle^A$ for all s.

Lemma 3.4. *Let $T \subset \mathbb{T}^m$ be a finite set and U the module generated by it in $A[\mathbf{x}]^m_{\mathbf{d}}$. Let $M \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a module such that for every s the set $\mathcal{N}(U)_s$ generates the A-module $(A[\mathbf{x}]^m_{\mathbf{d}})_s / M_s$. Then for every degree s there exists an $U_s \cap \mathbb{T}^m$-marked set $F = F_s$ contained in M_s such that*

$$\left(A[\mathbf{x}]^m_{\mathbf{d}}\right)_s = \langle F \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A.$$

Proof. Let π be the usual projection morphism of $A[\mathbf{x}]^m_{\mathbf{d}}$ onto the quotient $A[\mathbf{x}]^m_{\mathbf{d}}/M$. For every $x^\alpha e_k \in U_s \cap \mathbb{T}^m$, we consider $\pi(x^\alpha e_k)$ and choose a representation $\pi(x^\alpha e_k) = \sum_{x^\eta e_l \in \mathcal{N}(U)_s} c^{\alpha k}_{\eta l} x^\eta e_l$, $c^{\alpha k}_{\eta l} \in A$, which exists as $\mathcal{N}(U)_s$ generates $(A[\mathbf{x}]^m_{\mathbf{d}})_s / M_s$ as an A-module. We consider the set of marked module elements $F = \{f^k_\alpha\}_{x^\alpha e_k \in U_s}$, where $f^k_\alpha := x^\alpha e_k - \pi(x^\alpha e_k)$ and $\mathrm{Ht}(f^k_\alpha) = x^\alpha e_k$.

We now prove that $A[\mathbf{x}]^m_s = \langle F \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$. We first prove that every term in $\mathbb{T}^m_s$ belongs to $\langle F \rangle^A + \langle \mathcal{N}(U)_s \rangle^A$. If $x^\beta e_l \in \mathcal{N}(U)_s$, there is nothing to prove. If $x^\beta e_l \in U_s$, then there is $f^l_\beta \in F$ such that $\mathrm{Ht}(f^l_\beta) = x^\beta e_l$, hence we can write $x^\beta e_l = f^l_\beta + (x^\beta e_l - f^l_\beta) = f^l_\beta + \pi(x^\beta e_l)$.

We conclude by proving that $\langle F \rangle^A \cap \langle \mathcal{N}(U)_s \rangle^A = \{0^m_A\}$. Let $g \in A[\mathbf{x}]^m_{\mathbf{d}}$ be an element belonging to $\langle F \rangle^A \cap \langle \mathcal{N}(U)_s \rangle^A$: $g = \sum_{f^k_\alpha \in F} \lambda_{\alpha k} f^k_\alpha \in \langle \mathcal{N}(U)_s \rangle$. Since the head terms of f^k_α cannot cancel each other, $\lambda_{\alpha k} = 0$ for every α and k and hence $g = 0$. □

We specialize now to the case that U is a quasi-stable module and $T = \mathcal{P}(U)$ its Pommaret basis. We study a reduction relation naturally induced by any basis marked over such a set T. In particular, we show that it is confluent and Noetherian just as the familiar reduction relation induced by a Gröbner basis.

Definition 3.5. Let $U \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set in $A[\mathbf{x}]^m_{\mathbf{d}}$. We introduce the following sets:

- $G^{(s)} := \left\{ x^\delta f^k_\alpha \mid f^k_\alpha \in G, x^\delta \in A[\mathcal{X}_{\mathcal{P}}(f^k_\alpha)], \deg x^\delta f^k_\alpha = s \right\}$;
- $\widehat{G}^{(s)} := \left\{ x^\delta f^k_\alpha \mid f^k_\alpha \in G, x^\delta \notin A[\mathcal{X}_{\mathcal{P}}(f^k_\alpha)], \deg x^\delta f^k_\alpha = s \right\} = \left\{ x^\delta f^k_\alpha \mid f^k_\alpha \in G \right\} \setminus G^{(s)}$;
- $\mathcal{N}(U, \langle G \rangle) := \langle G \rangle \cap \langle \mathcal{N}(U) \rangle$.

Lemma 3.6. *Let $U \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. For each product $x^\delta f^k_\alpha$ with $f^k_\alpha \in G$, each term in* $\mathrm{Supp}(x^\delta x^\alpha e_k - x^\delta f^k_\alpha)$ *either belongs to $\mathcal{N}(U)$ or is of the form $x^\eta x^\nu e_l \in \mathcal{C}^m_{\mathcal{P}}(x^\nu e_l)$ with $x^\nu e_l \in \mathcal{P}(U)$ and $x^\eta <_{lex} x^\delta$.*

Proof. It is sufficient to consider $x^\delta x^\beta e_l \in \mathrm{Supp}(x^\delta x^\alpha e_k - x^\delta f^k_\alpha) \cap U$. Then $x^\delta x^\beta \in J^{(l)}$ for some quasi-stable ideal $J^{(l)} \subset A[\mathbf{x}]$ appearing in (1.1). Therefore there exists $x^\gamma \in \mathcal{P}(J^{(l)})$ such that $x^\delta x^\beta \in \mathcal{C}_{\mathcal{P}}(x^\gamma)$. More precisely, if $x^\eta := x^\delta x^\beta / x^\gamma$, then $x^\eta <_{lex} x^\delta$ by Lemma 2.4 (vi). □

Note in the next definition the use of the set $G^{(s)}$, which means that we use here a generalisation of the involutive reduction relation associated with the Pommaret division and not of the standard reduction relation in the theory of Gröbner bases. This modification is the key for circumventing the restrictions imposed by the results of [25]. It also entails that if a term is reducible, then there is only one element in the marked basis which can be used for its reduction.

Definition 3.7. Let $U \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. We denote by $\xrightarrow{G^{(s)}}$ the transitive closure of the relation $h \xrightarrow{G^{(s)}} h - \lambda x^\eta f^k_\alpha$ where $x^\eta x^\alpha e_k$ is a term which appears in h with a non-zero coefficient $\lambda \in A$ and which satisfies both $\deg(x^\eta x^\alpha e_k) = s$ and $x^\eta f^k_\alpha \in G^{(s)}$. We will write $h \xrightarrow{G^{(s)}}_* g$, if $h \xrightarrow{G^{(s)}} g$ and $g \in \langle \mathcal{N}(U) \rangle$. Observe that if $h \in (A[\mathbf{x}]^m_{\mathbf{d}})_s$, then $h \xrightarrow{G^{(s)}} g \in (A[\mathbf{x}]^m_{\mathbf{d}})_s$.

Proposition 3.8. *Let $U \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. Then the reduction relation $\xrightarrow{G^{(s)}}$ is Noetherian.*

Proof. It suffices to prove that for every term $x^\gamma e_k$ in U, there exists $g \in \langle \mathcal{N}(U) \rangle^A$ such that $x^\gamma e_k \xrightarrow{G^{(s)}}_* g$. Since $x^\gamma e_k \in U$, there exists a unique $x^\delta f^k_\alpha \in G^{(s)}$ such that $x^\delta \mathrm{Ht}(f^k_\alpha) = x^\gamma e_k$. Hence, $x^\gamma e_k \xrightarrow{G^{(s)}} x^\gamma e_k - x^\delta f^k_\alpha$. If we could proceed in the reduction without ever obtaining an element in $\langle \mathcal{N}(U) \rangle$, we would obtain by Lemma 3.6 an infinite lex-descending chain of terms in $\mathbb{T}$ which is impossible since lex is a well-ordering. Hence $\xrightarrow{G^{(s)}}$ is Noetherian. □

Corollary 3.9. *Let $U \subseteq A[\mathbf{x}]^m$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set. Every term $x^\beta e_k \in \mathbb{T}^m_s$ of degree s can be expressed in the form*

$$x^\beta e_l = \sum \lambda x^\delta f^k_\alpha + g, \tag{3.1}$$

where $\lambda \in A \setminus \{0_A\}$, $x^\delta f^k_\alpha \in G^{(s)}$, $g \in \langle \mathcal{N}(U) \rangle^A$ and the terms x^δ form a sequence which is strictly descending with respect to lex.

Proof. For terms in $\mathcal{N}(U)$, there is nothing to prove. For $x^\beta e_l \in U$, it suffices to consider $g \in \langle \mathcal{N}(U) \rangle^A$ such that $x^\beta e_l \xrightarrow{G^{(s)}}_* g$. The appearing polynomials $x^\delta f^k_\alpha \in G^{(s)}$ are exactly those used during the reduction $\xrightarrow{G^{(s)}}$. They fulfill the statement on the terms x^δ by Lemma 3.6. □

We now define an ordering for the polynomials $x^\delta f^k_\alpha \in G^{(s)}$ assuming that the polynomials in G are ordered (in some way): $x^\delta f^k_\alpha \prec x^{\delta'} f^{k'}_{\alpha'}$, if f_α is smaller than $f_{\alpha'}$ or if $f^k_\alpha = f^{k'}_{\alpha'}$ and $x^\delta <_{\mathrm{lex}} x^{\delta'}$. When we say in the sequel that a polynomial in a subset of $G^{(s)}$ is maximal, we refer to this order.

Lemma 3.10. *Let $U \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set. Consider a homogeneous element $g \in A[\mathbf{x}]^m_{\mathbf{d}}$ such that $h = \sum \lambda x^\delta f^k_\alpha$, with $\lambda \in A \setminus \{0\}$ and $x^\delta f^k_\alpha \in G^{(s)}$ with $s = \deg(h)$ and $x^\delta f^k_\alpha$ pairwise different. Then $h \neq 0^m_A$ and $h \notin \langle \mathcal{N}(U) \rangle^A$.*

Proof. Let $x^{\overline{\delta}} f^{\overline{k}}_{\overline{\alpha}}$ be the maximal polynomial of $G^{(s)}$ appearing in the summation $\sum \lambda x^\delta f^k_\alpha = 0$ with $\lambda \neq 0$. Then, by Lemma 3.6, the term $x^{\overline{\delta}} x^{\overline{\alpha}} e_{\overline{k}}$ does not appear in the support of any other polynomial $x^\delta f^k_\alpha$ involved in the summation. Hence, $h \neq 0^m_A$. Furthermore, since $x^{\overline{\delta}} x^{\overline{\alpha}} e_{\overline{k}} \in U$ belongs to the support of h, h does not belong to $\langle \mathcal{N}(U) \rangle$. □

Proposition 3.11. *Let $U \subseteq A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. The reduction relation $\xrightarrow{G^{(s)}}$ is confluent.*

Proof. Let h be a polynomial in $A[\mathbf{x}]^m$. We reduce it twice with $\xrightarrow{G^{(s)}}$ following different paths along the reduction: $h \xrightarrow{G^{(s)}}_* g_1 \in \langle \mathcal{N}(U) \rangle$ and $h \xrightarrow{G^{(s)}}_* g_2 \in \langle \mathcal{N}(U) \rangle$. By Corollary 3.9 applied to the terms in the support of h,

$$h = \sum \lambda x^\delta f^k_\alpha + g_1 = \sum \mu x^\delta f^k_\alpha + g_2. \tag{3.2}$$

Then $g_1 - g_2 = \sum (\lambda - \mu) x^\delta f^k_\alpha$. If there is a coefficient $\lambda - \mu \in A \setminus \{0\}$, then by Lemma 3.10, $g_1 - g_2 \notin \langle \mathcal{N}(U) \rangle$ contradicting the hypothesis. Thus $\lambda = \mu$ for every $x^\delta f^k_\alpha \in G^{(s)}$ in (3.2) and $g_1 = g_2$. ☐

Corollary 3.12. *Let $U \subseteq A[\mathbf{x}]^m$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set. Every term $x^\beta e_k \in \mathbb{T}^m_s$ of degree s has a unique representation of the form (3.1).*

The following Theorem and Corollaries collect some basic properties of sets marked over a Pommaret basis. They generalise analogous statements in [22, Theorems 1.7, 1.10] which considered only ideals and marked bases where the head terms generate a strongly stable ideal.

Theorem 3.13. *Let $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module with $q(s) := \mathrm{rk}(U_s)$ and G a $\mathcal{P}(U)$-marked set. Then, we have for every degree s the following decompositions of A-modules:*

(i) $\langle G \rangle_s = \langle G^{(s)} \rangle^A + \langle \widehat{G}^{(s)} \rangle^A$;
(ii) $(A[\mathbf{x}]^m_{\mathbf{d}})_s = \langle G^{(s)} \rangle^A \oplus \langle \mathcal{N}(U)_s \rangle^A$;
(iii) the A-module $\left\langle G^{(s)} \right\rangle^A$ *is free of rank equal to* $|G^{(s)}| = \mathrm{rk}(U_s)$ *and it is generated (as an A-module) by a unique* $U_s \cap \mathbb{T}^m$*-marked set* $\widetilde{G}^{(s)}$;
(iv) $\langle G \rangle_s = \left\langle G^{(s)} \right\rangle^A \oplus \mathcal{N}(U, \langle G \rangle)_s$.

Moreover, the following conditions are equivalent:

(v) G is a $\mathcal{P}(U)$-marked basis;
(vi) $\langle G \rangle_s = \left\langle G^{(s)} \right\rangle^A$ *for all degrees s;*
(vii) $\mathcal{N}(U, \langle G \rangle) = \{0^m_A\}$;
(viii) $\bigwedge^{q(s)+1} \langle G \rangle_s = 0_A$ *for all s.*

Proof. Item (i) is obvious.

Item (ii) is a consequence of Corollary 3.12.

For Item (iii) we use the arguments of [22, Theorem 1.7] for the ideal case: by (ii) we have the short exact sequence

$$0 \longrightarrow \langle G^{(s)} \rangle \hookrightarrow (A[\mathbf{x}]^m_{\mathbf{d}})_s \xrightarrow{\pi} \langle \mathcal{N}(U)_s \rangle \longrightarrow 0\,.$$

For each $x^\alpha e_k$ in U_s we compute the image $\pi(x^\alpha e_k) = \sum_{x^\beta e_l \in \mathcal{N}(U)_s} a_{\alpha\beta k l} x^\beta e_l$ and consider the set $\tilde{G}^{(s)} := \{\tilde{f}^k_\alpha := x^\alpha - \sum_{x^\beta e_l \in \mathcal{N}(U)_s} a_{\alpha\beta kl} x^\beta e_l \mid x^\alpha e_k \in U_s\} \subseteq \ker \pi = \langle G^{(s)} \rangle$. Let $U' := \langle U_s \rangle$. By construction, $\tilde{G}^{(s)}$ is a U'-marked set with $\mathrm{Ht}(\tilde{f}^k_\alpha) = x^\alpha e_k$. Applying (ii) to this U'-marked set, we have $\langle \tilde{G}^{(s)} \rangle + \langle \mathcal{N}(U')_s \rangle = (A[\mathbf{x}]^m_{\mathbf{d}})_s$.

Finally, since the A-module generated by $\tilde{G}^{(s)}$ is contained in $\langle G^{(s)} \rangle$ and $\mathcal{N}(U)_s = \mathcal{N}(U')_s$, the modules $\langle \tilde{G}^{(s)} \rangle$ and $\langle G^{(s)} \rangle$ coincide. Note that the set $\tilde{G}^{(s)}$ is marked on the monomial module U' which is generated by U_s, but it is not necessarily a $U_{\geqslant s}$-marked set, since $U_{\geqslant s}$ may have minimal generators of degree greater than s.

For Item (iv), we first note that, by (i) and (iii), we have $\langle G \rangle_s = \langle \widetilde{G}^{(s)} \rangle^A + \langle \widehat{G}^{(s)} \rangle^A$. Recalling that $\langle \widetilde{G}^{(s)} \rangle^A \cap \langle \mathcal{N}(U)_s \rangle^A = \{0^m_A\}$ by Lemma 3.4, it suffices to show that every $g \in \langle \widehat{G}^{(s)} \rangle^A$ can be written as $g = f + h$ with $f \in \langle \widetilde{G}^{(s)} \rangle^A$ and $h \in \langle \mathcal{N}(U)_s \rangle^A$. We express every term $x^\beta e_l \in U_s$ appearing in g with non-zero coefficient in the form $x^\beta e_l = \widetilde{f}^l_\beta + (x^\beta e_l - \widetilde{f}^l_\beta)$ where $\widetilde{f}^l_\beta$ is the unique polynomial in $\widetilde{G}^{(s)}$ with $\mathrm{Ht}(\widetilde{f}^l_\beta) = x^\beta e_l$. By construction, $h \in \mathcal{N}(U, \langle G \rangle)_s$. By (ii), we obtain the assertion.

Items (v), (vi), (vii) are equivalent by the previous items. In fact, these properties are a rephrasing of the definition of $\mathcal{P}(U)$-marked basis.

With respect to [22], the only new item is (viii), which is obviously equivalent to (vi) and (vii). In fact, by (iii) and (iv), we find that $\langle G\rangle_s = \left\langle G^{(s)}\right\rangle^A \oplus \mathcal{N}(U,\langle G\rangle)_s$ and $\mathrm{rk}\left\langle G^{(s)}\right\rangle^A = \mathrm{rk}(U_s) = q(s)$. □

Remark 3.14. If $G \subset A[\mathbf{x}]^m$ is a $\mathcal{P}(U)$-marked basis, then, by Theorem 3.13 (ii), (iii) and (vi), the $A[\mathbf{x}]$-module $\langle G\rangle$ admits a Hilbert function, which is the same as the Hilbert function of the monomial module U.

Corollary 3.15. *Let $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G be a $\mathcal{P}(U)$-marked set. The following conditions are equivalent:*

(i) G is a $\mathcal{P}(U)$-marked basis;
(ii) $\langle G\rangle_s = \left\langle G^{(s)}\right\rangle^A$ for every $s \leqslant \mathrm{reg}(U)+1$;
(iii) $\mathcal{N}(U,\langle G\rangle)_s = \{0^m_A\}$ for every $s \leqslant \mathrm{reg}(U)+1$;
(iv) $\bigwedge^{q(s)+1}\langle G\rangle_s = 0_A$ for every $s \leqslant \mathrm{reg}(U)+1$.

Proof. By the second part of Theorem 3.13, (i) implies (ii) and (ii), (iii) and (iv) are equivalent.

For the proof that (ii) implies (i), we follow the arguments used in [22, Theorem 1.10] and adapt them to the module case. We have to prove that $(A[\mathbf{x}]^m_{\mathbf{d}})_s = \langle G\rangle_s \oplus \langle \mathcal{N}(U)_s\rangle$ for every s. This is true for $s \leqslant m+1$ by hypothesis. By Theorem 3.13 (ii), (iii), we know that $(A[\mathbf{x}]^m_{\mathbf{d}})_s = \langle G^{(s)}\rangle \oplus \langle\mathcal{N}(U)_s\rangle$ and $\langle G^{(s)}\rangle \subseteq \langle G\rangle_s$, so that we have to prove $\langle G\rangle_s \subseteq \langle G^{(s)}\rangle$. Let us assume that this is not true and let t be the minimal degree for which $\langle G\rangle_t \nsubseteq \langle G^{(t)}\rangle$. Note that $t \geqslant m+2 > m$ and $\langle G\rangle_t = x_0\langle G\rangle_{t-1} + \cdots + x_n\langle G\rangle_{t-1}$. Since $\langle G\rangle_{t-1} = \langle G^{(t-1)}\rangle$, there must exist a variable x_i such that $x_i\langle G\rangle_{t-1} \nsubseteq \langle G^{(t)}\rangle$ or equivalently $x_i\langle G^{(t-1)}\rangle \nsubseteq \langle G^{(t)}\rangle$. Assume that the index i is minimal with this property and take a polynomial $x^\delta f^k_\alpha \in G^{(t-1)}$ with $x^\alpha e_k = \mathrm{Ht}(f^k_\alpha) \in \mathcal{P}(U)$ such that $x_i x^\delta f^k_\alpha \notin \langle G^{(t)}\rangle$. The variable x_i has to be greater than $\min(x^\alpha)$, since otherwise $x_i x^\delta f^k_\alpha \in G^{(t)}$. Morevover $|\delta| > 0$ since $t-1 > m$. Let $x_j = \max(x^\delta) \leqslant \min(x^\alpha) < x_i$ and $x^{\delta'} = \frac{x^\delta}{x_i}$. The polynomial is contained in $\langle G\rangle_{t-1}$ while $x_j(x_i x^{\delta'} f^k_\alpha) = x_i x^\delta f^k_\alpha$ is not contained in $\langle G^{(t-1)}\rangle$, contradicting the minimality of i. □

Corollary 3.16. *Let $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable modul such that $U = \oplus J^{(k)} e_k$ with $J^{(k)}$ a saturated ideal for every k and G a $\mathcal{P}(U)$-marked set. Then the following conditions are equivalent:*

(i) G is a $\mathcal{P}(U)$-marked basis;
(ii) $\langle G\rangle_{\mathrm{reg}(U)+1} = \left\langle G^{(\mathrm{reg}(U)+1)}\right\rangle^A$;
(iii) $\mathcal{N}(U,\langle G\rangle)_{\mathrm{reg}(U)+1} = \{0^m_A\}$;
(iv) $\bigwedge^{Q+1}\langle G\rangle_{\mathrm{reg}(U)+1} = 0_A$, where $Q := \mathrm{rk}(U_{\mathrm{reg}(U)+1})$.

Proof. The equivalence of (ii), (iii) and (iv) is immediate by Theorem 3.13. We thus only prove that (i) and (iii) are equivalent. If G is a $\mathcal{P}(U)$-marked basis, then, by Theorem 3.13, we have $\mathcal{N}(U,\langle G\rangle)_{\mathrm{reg}(U)+1} = \{0^m_A\}$. We now assume that $\mathcal{N}(U,\langle G\rangle)_{\mathrm{reg}(U)+1} = \{0^m_A\}$ and prove that $\mathcal{N}(U,\langle G\rangle) = \{0^m_A\}$. By Corollary 3.15, it suffices to prove that $\mathcal{N}(U,\langle G\rangle)_s = \{0^m_A\}$ for every $s \leqslant \mathrm{reg}(U)$. If $f \in \mathcal{N}(U,\langle G\rangle)_s$ with $s \leqslant \mathrm{reg}(J)$, then $x_0^{\mathrm{reg}(U)+1-s} f \in \mathcal{N}(U,\langle G\rangle)_{\mathrm{reg}(U)+1}$ by Lemma 2.4 (ii) and (v) applied to U. Hence $f = 0^m_A$. □

Corollary 3.17. *Let $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and $W \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a finitely generated graded submodule such that $(A[\mathbf{x}]^m_{\mathbf{d}})_s = W_s \oplus \langle\mathcal{N}(U)_s\rangle^A$ for every s. Then W is generated by a $\mathcal{P}(U)$-marked basis.*

Proof. The statement is an easy consequence of Theorem 3.13 as soon as we have defined a $\mathcal{P}(U)$-marked set generating W. By the hypotheses, for every degree s and every monomial $x^\alpha e_k \in \mathcal{P}(U)$, there exists a unique element $h^k_\alpha \in \langle\mathcal{N}(U)_s\rangle^A$ such that $x^\alpha e_k - h^k_\alpha \in W_s$. The collection G of the elements $x^\alpha e_k - h^k_\alpha$ is obviously a $\mathcal{P}(U)$-marked set and generates a graded submodule of W. Moreover, $(A[\mathbf{x}]^m_{\mathbf{d}})_s = W_s \oplus \langle\mathcal{N}(U)_s\rangle^A = \langle G^{(s)}\rangle^A \oplus \langle\mathcal{N}(U)_s\rangle^A$. Therefore, $W_s = \langle G^{(s)}\rangle^A \subseteq G_s \subseteq W_s$ so that G generates W as a graded $A[\mathbf{x}]$-module. □

Finally, we provide an algorithmic method to check whether or not a marked set is even a marked basis using the reduction process introduced in Definition 3.7.

Theorem 3.18. *Let $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ be a quasi-stable module and G a $\mathcal{P}(U)$-marked set. The set G is a $\mathcal{P}(U)$-marked basis, if and only if*

$$\forall f^k_\alpha \in G, \forall x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f^k_\alpha) : x_i f^k_\alpha \xrightarrow{G^{(s)}} 0^m_A.$$

Proof. We adapt the arguments used in [9, Theorem 5.13] for the ideal case. Since "$\Rightarrow$" is a consequence of Theorem 3.13, we only prove "$\Leftarrow$". More precisely, we prove that $\langle G\rangle_s = \langle G^{(s)}\rangle$ showing that if $f^k_\alpha \in G$ and $\deg(x^{\alpha+\delta}) = s$, then $x^\delta f^k_\alpha$ is either an element of $G^{(s)}$ itself or a linear combination of polynomials in $G^{(s)}$.

If this were not true, we could choose an element $x^\delta f^k_\alpha \in \langle G^{(s)}\rangle$ with x^δ minimal with respect to $<_{lex}$. As $x^\delta f^k_\alpha \notin G^{(s)}$, at least one variable x_i appearing in x^δ with a non-zero exponent is non-multiplicative for x^α. Let $x^\delta = x_i x^{\delta'}$. By hypothesis, $x_i f^k_\alpha \xrightarrow{G^{(s)}}_* 0$, so that $x_i f^k_\alpha$ is a linear combination $\sum c_i x^{\eta_i} f^{k_i}_{\beta_i}$ of polynomials in $G^{(|\alpha|+1)}$. By Lemma 3.6, we have $x^{\eta_i} <_{lex} x_i$. Now $x^\delta f^k_\alpha = x^{\delta'}(x_i f^k_\alpha) = x^{\delta'}(\sum c_i x^{\eta_i} f^{k_i}_{\beta_i}) = \sum c_i x^{\eta_i+\delta'} f^{k_i}_{\beta_i}$, where $x^{\eta_i \delta'} <_{lex} x_i x^{\delta'} = x^\delta$. This yields a contradiction, since $x^{\eta_i+\delta} f^{k_i}_{\beta_i} \in \langle G^{(s)}\rangle$ by the minimality of x^δ. □

4. The marked family associated with a quasi-stable module

Given a quasi-stable module U with Pommaret basis $\mathcal{P}(U)$, we may consider the set of all modules in $A[\mathbf{x}]^m_{\mathbf{d}}$ which can be generated by a $\mathcal{P}(U)$-marked basis. We call this set the *marked family* associated to U. Our goal in this section consists of exhibiting a natural scheme structure of this set. More precisely, we will first define a functor $\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}$ mapping the Noetherian $\Bbbk$-algebra A to this set and then show that it is representable by an affine scheme.

If $\sigma : A \to B$ is a morphism of $\Bbbk$-algebras, we will also denote by σ its natural extension to a morphism $A[\mathbf{x}] \to B[\mathbf{x}]$. We now consider the functor of the $\mathcal{P}(U)$-marked bases from the category of Noetherian $\Bbbk$-algebras to the category of sets

$$\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)} : \underline{\text{Noeth } \Bbbk\text{-Alg}} \longrightarrow \underline{\text{Sets}}$$

that associates to any Noetherian $\Bbbk$-algebra A the set

$$\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}(A) := \{G \subset A[\mathbf{x}]^m_{\mathbf{d}} \mid G \text{ is a } \mathcal{P}(U)\text{-marked basis}\},$$

or, equivalently by Corollary 3.17,

$$\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}(A) := \{W \subset A[\mathbf{x}]^m_{\mathbf{d}} \mid W \text{ is generated by a } \mathcal{P}(U)\text{-marked basis}\}$$

and to any morphism $\sigma : A \to B$ the map

$$\begin{array}{rccc}\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}(\sigma): & \underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}(A) & \longrightarrow & \underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}(B) \\ & G & \longmapsto & \sigma(G)\,. \end{array}$$

Note that the image $\sigma(G)$ is indeed again a $\mathcal{P}(U)$-marked basis, as any head term in the $\mathcal{P}(U)$-marked basis G has the coefficient 1_A which is mapped by σ into 1_B and we are effectively applying the functor $-\otimes_A B$ to the decomposition $(A[\mathbf{x}]^m_{\mathbf{d}})_s = \langle G^{(s)}\rangle^A \oplus \langle \mathcal{N}(U)_s\rangle^A$ at every degree s.

The above introduced functor turns out to be representable by an affine scheme that can be explicitly constructed by the following procedure. We consider the $\Bbbk$-algebra $\Bbbk[C]$ where C denotes the finite set of variables $\{C_{\alpha\eta kl} \mid x^\alpha e_k \in \mathcal{P}(U), x^\eta e_l \in \mathcal{N}(U), \deg(x^\eta e_l) = \deg(x^\alpha e_k)\}$ and construct the $\mathcal{P}(U)$-marked set $\mathcal{G} \subset \Bbbk[C][\mathbf{x}]^m_{\mathbf{d}}$ consisting of all elements

$$F^k_\alpha = \Big(x^\alpha - \sum_{x^\eta \in \mathcal{N}(J^{(k)})_{|\alpha|}} C_{\alpha\eta kk} x^\eta\Big) e_k - \sum_{\substack{l\neq k, x^\eta e_l \in \mathcal{N}(J^{(l)})e_l \\ \deg(x^\eta e_l)=\deg(x^\alpha e_k)}} C_{\alpha\eta kl} x^\eta e_l \qquad (4.1)$$

with $x^\alpha e_k \in \mathcal{P}(U)$. Then, we compute all the complete reductions $x_i F^k_\alpha \xrightarrow{\mathcal{G}^{(s)}}_* L^k_{i\alpha}$ for every term $x^\alpha e_k \in \mathcal{P}(U)$ and every non-multiplicative variable $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(F^k_\alpha)$ and collect the coefficients of the monomials $x^\eta e_j \in \mathcal{N}(U)$ of all the reduced elements $L^k_{i\alpha}$ in a set $\mathcal{R} \subset \Bbbk[C]$.

Theorem 4.1. *The functor $\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}$ is represented by the affine scheme* $\mathrm{Spec}(\Bbbk[C]/(\mathcal{R}))$ *that we denote by $\mathbf{Mf}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}$.*

Proof. We observe that each element f^k_α of a $\mathcal{P}(U)$-marked set G in $A[\mathbf{x}]^m_\mathbf{d}$ can be written in the following form:

$$f^k_\alpha = \left(x^\alpha - \sum_{x^\eta \in \mathcal{N}(J^{(k)})_{|\alpha|}} c_{\alpha\eta kk} x^\eta \right) e_k - \sum_{\substack{l \neq k, x^\eta e_l \in \mathcal{N}(J^{(l)}) e_l \\ \deg(x^\eta e_l) = \deg(x^\alpha e_k)}} c_{\alpha\eta kl} x^\eta e_l, \quad c_{\alpha\eta kl} \in A.$$

Therefore, G can be obtained by specialising in $\mathcal{G}$ the variables $C_{\alpha\eta kl}$ to the constants $c_{\alpha\eta kl} \in A$. Moreover, G is a $\mathcal{P}(U)$-marked basis if and only $x_i f^k_\alpha \xrightarrow{G^{(s)}}_* 0$ for every $x^\alpha e_k \in \mathcal{P}(U)$ and $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f^k_\alpha)$. Equivalently, G is a $\mathcal{P}(U)$-marked basis if and only if the evaluation morphism $\varphi : \Bbbk[C] \to A$, $\varphi(C_{\alpha\eta kl}) = c_{\alpha\eta kl}$ factors through $\Bbbk[C]/(\mathcal{R})$, namely, if and only if the following diagram commutes

$$\begin{tikzcd} \Bbbk[C] \arrow[r, "\varphi"] \arrow[d] & A \\ \Bbbk[C]/(\mathcal{R}) \arrow[ur] & \end{tikzcd} .$$

□

Remark 4.2. The arguments presented in the proof of Theorem 4.1 generalise those presented in [9, 22] for ideals to our more general framework of modules. As a consequence of this result, we know that the scheme defined as $\mathrm{Spec}(\Bbbk[C]/(\mathcal{R}))$ only depends on the submodule U and not on the possibly different procedures for constructing it: any other procedure that gives a set of "minimal" conditions on the coefficients C that are necessary and sufficient to guarantee that a $\mathcal{P}(U)$-marked set G is a $\mathcal{P}(U)$-marked basis generates an ideal $\mathcal{R}'$ such that $\Bbbk[C]/(\mathcal{R}) = \Bbbk[C]/(\mathcal{R}')$.

5. $\mathcal{P}(U)$-marked Bases and Syzygies

We now study syzygies of a $\mathcal{P}(U)$-marked basis and we formulate a $\mathcal{P}(U)$-marked version of the involutive Schreyer theorem [28, Theorem 5.10]. For notational simplicity, this section is formulated for $m = 1$, that is for ideals in $A[\mathbf{x}]$, but it is straightforward to extend everything to submodules $A[\mathbf{x}]^m_\mathbf{d}$ generated by a marked basis over a quasi-stable module.

Let J be a quasi-stable monomial ideal in $A[\mathbf{x}]$ and I an ideal in $A[\mathbf{x}]$ generated by a $\mathcal{P}(J)$-marked basis G. Let m be the cardinality of $\mathcal{P}(J)$. We denote the terms in $\mathcal{P}(J)$ by $x^{\alpha(k)}$ and the polynomials in G by $f_{\alpha(k)}$, with $k \in \{1, \dots, m\}$.

Lemma 5.1. *Every polynomial $f \in I$ can be uniquely written in the form $f = \sum_{l=1}^m P_l f_{\alpha(l)}$ with $f_{\alpha(l)} \in G$ and $P_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$.*

Proof. This is a consequence of Corollary 3.9 and Theorem 3.13 (vi). □

Take an arbitrary element $f_{\alpha(k)} \in G$ and choose an arbitrary non-multiplicative variable $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(k)})$ of it. We can determine, via the reduction process $\xrightarrow{G^{(s)}}$, for each $f_{\alpha(l)} \in G$ a unique polynomial $P^{k;i}_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ such that $x_i f_{\alpha(k)} = \sum_{l=1}^m P^{k;i}_l f_{\alpha(l)}$. This relation corresponds to the *fundamental syzygy*

$$S_{k;i} = x_i e_k - \sum_{l=1}^m P^{k;i}_l e_l \ .$$

We denote the set of all fundamental syzygies by

$$G_{\mathrm{Syz}} = \{S_{k;i} \mid k \in \{1,\dots,m\},\ x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(k)})\}\,.$$

We consider the syzygies in G_{Syz} as elements of $A[\mathbf{x}]^m_{\mathbf{d}}$ with $\mathbf{d} = (\deg(x^{\alpha(1)}),\dots,\deg(x^{\alpha(m)}))$.

Lemma 5.2. *Let $S = \sum_{l=1}^m S_l e_l$ be an arbitrary syzygy of the $\mathcal{P}(J)$-marked basis G with coefficients $S_l \in A[\mathbf{x}]$. Then $S_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ for all $1 \leqslant l \leqslant m$ if and only if $S = 0^m_A$.*

Proof. If $S \in \mathrm{Syz}(G)$, then $\sum_{l=1}^m S_l f_{\alpha(l)} = 0$. According to Lemma 5.1, each $f \in I$ can be uniquely written in the form $f = \sum_{l=1}^m P_l f_{\alpha(l)}$ with $f_{\alpha(l)} \in G$ and $P_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$. In particular, this holds for $0_A \in I$. Thus $0_A = S_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ for all l and hence $S = 0^m_A$. ☐

Lemma 5.3. *Let U be the monomial module $U = \oplus_{l=1}^m(\overline{\mathcal{X}}_{\mathcal{P}}(x^{\alpha(l)}))e_l$ where $(\overline{\mathcal{X}}_{\mathcal{P}}(x^{\alpha(l)}))$ is the ideal generated by $\overline{\mathcal{X}}_{\mathcal{P}}(x^{\alpha(l)})$ in $A[\mathbf{x}]$. Then U is a quasi-stable module with Pommaret basis $\mathcal{P}(U) = \{x_i e_l \mid 1 \leqslant l \leqslant m, x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})\}$ and G_{Syz} is a $\mathcal{P}(U)$-marked set in $A[\mathbf{x}]^m_{\mathbf{d}}$.*

Proof. By [28, Lemma 5.9], we can immediately conclude that U is a quasi-stable module and that the set $\{x_i e_l \mid 1 \leqslant l \leqslant m, x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})\}$ is the Pommaret basis of U. We define $\mathrm{Ht}(S_{i;l}) = x_i e_l$ and easily see that G_{Syz} is a $\mathcal{P}(U)$-marked-set: by definition of U, every term $x^\mu e_k$ in $\mathrm{Supp}(S_{l;i} - x_i e_l)$ belongs to $\mathcal{N}(U)$, because $x^\mu \in \mathcal{X}_{\mathcal{P}}(f_{\alpha(k)})$. ☐

Observe that for every fundamental syzygy $S_{k;i} \in G_{\mathrm{Syz}}$, $\mathcal{X}_{\mathcal{P}}(S_{k;i}) = \{x_0,\dots,x_i\}$. As in Section 3, we define for every degree s the following set of polynomials in $\langle G_{\mathrm{Syz}}\rangle$:

$$G^{(s)}_{\mathrm{Syz}} = \{x^\delta S_{k;i} \mid S_{k;i} \in G_{\mathrm{Syz}}, x^\delta \in \mathcal{X}_{\mathcal{P}}(S_{k;i}), \deg(x^\delta S_{k;i}) = s\}.$$

Lemma 5.4. *The set $G^{(s)}_{\mathrm{Syz}}$ generates the A-module $\mathrm{Syz}(G)_s$ for every s.*

Proof. Let $S = \sum_{l=1}^m S_l e_l$ be an arbitrary non-vanishing syzygy in $\mathrm{Syz}(G)_s$. By Lemma 5.2, there is at least one index k such that the coefficient S_k contains a term x^μ depending on a non- multiplicative variable $x_i \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(k)})$. Among all such values of k and μ, we choose the term $x^\mu e_k$ which is lexicographically maximal. Then, $x^\mu e_k$ belongs to the quasi-stable module U, hence there is $x^\delta S_{k;j} \in G^{(s)}_{\mathrm{Syz}}$ such that $x^\delta x_j = x^\mu$. We define $S' = S - \lambda x^\delta S_{k;j}$, where $\lambda \neq 0_A$ is the coefficient of $x^\mu e_k$ in S.

Now we have to show that every x^ν contained in a module term $\lambda x^\nu e_l \in \mathrm{Supp}(S') \cap U$ is lexicographically smaller than x^μ. The terms of $\mathrm{Supp}(S) \cap U$ contained in $\mathrm{Supp}(S')$ are by assumption lexicographically smaller than $x^\mu e_k$. Every other term arises from $x^\delta \sum_{l=1}^m P_l^{(k;j)} e_l$. We know that $x_j f_{\alpha(k)} = \sum_{l=1}^m P_l^{(k;j)} f_{\alpha(l)}$. In particular, a term $x^{\nu'}$ in $P_l^{(k;j)}$ is lexicographically smaller than x_j, by Corollary 3.9. Therefore every term in $x^\delta \sum_{l=1}^m P_l^{(k;j)} e_\beta$ is lexicographically smaller than $x^\delta x_j = x^\mu$. If $S' \neq 0$, again by Lemma 5.2, we iterate the procedure on a lexicographical maximal term of S' containing a non-multiplicative variable. Since all new non-multiplicative terms introduced are lexicographically smaller, the reduction process must stop after a finite number of steps. As a result we get a representation $S' = \sum_{l=1}^l S'_l e_l$ such that $S'_l \in A[\mathcal{X}_{\mathcal{P}}(f_{\alpha(l)})]$ for all $1 \leqslant l \leqslant m$. But Lemma 5.2 says that this sum must be zero. ☐

Theorem 5.5 ($\mathcal{P}(U)$-marked Schreyer Theorem)**.** *Let $G = \{f_{\alpha(1)}, \dots f_{\alpha(m)}\}$ be a $\mathcal{P}(J)$-marked basis. Then G_{Syz} is a $\mathcal{P}(U)$-marked basis of $\mathrm{Syz}(G)$ with U as in Lemma 5.3.*

Proof. By Lemma 5.3, we know that G_{Syz} is a $\mathcal{P}(U)$-marked set. By Lemma 5.4, we know that $\langle G^{(s)}_{\mathrm{Syz}}\rangle^A = \langle \mathrm{Syz}(G)_s\rangle^A$ and we conclude by Theorem 3.13 (vi). ☐

Iterating this result, we arrive at a (generally non-minimal) free resolution. In contrast to the classical Schreyer Theorem for Gröbner bases, we are able to determine the ranks of all appearing free modules without any further computations.

Theorem 5.6. *Let $G = \{f_{\alpha(1)},\dots,f_{\alpha(m)}\}$, $\deg(f_{\alpha(i)}) = d_i$, be a $\mathcal{P}(J)$-marked basis and I the ideal generated by G in $A[\mathbf{x}]$. We denote by $\beta^{(k)}_{0,j}$ the number of terms $x^\alpha \in \mathcal{P}(J)$ such that*

$\deg(x^\alpha) = j$ and $\min(x^\alpha) = x_k$ and set $D = \min_{x^\alpha \in \mathcal{P}(J)}\{i \mid x_i = \min(x^\alpha)\}$. Then I possesses a finite free resolution

$$0 \longrightarrow \bigoplus A[\mathbf{x}](-j)^{r_{n-D,j}} \longrightarrow \cdots \longrightarrow \bigoplus A[\mathbf{x}](-j)^{r_{1,j}} \longrightarrow \bigoplus A[\mathbf{x}](-j)^{r_{0,j}} \longrightarrow I \longrightarrow 0 \quad (5.1)$$

of length $n - D$ where the ranks of the free modules are given by

$$r_{i,j} = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)}.$$

Proof. According to Theorem 5.5, G_{Syz} is a $\mathcal{P}(U)$-marked basis for the module $\mathrm{Syz}_1(I)$ with U as in Lemma 5.3. Applying the theorem again, we can construct a marked basis of the second syzygy module $\mathrm{Syz}_2(I)$ and so on. Recall that for every index $1 \leqslant l \leqslant m$ and for every non-multiplicative variable $x_k \in \overline{\mathcal{X}}_{\mathcal{P}}(f_{\alpha(l)})$ we have $\min(\mathrm{Ht}(S_{l;k})) = k > \min(\mathrm{Ht}(f_{\alpha(l)}))$. If D is the index of the minimal variable appearing in a head term in G, then the index of the minimal variable appearing in a head term in G_{Syz} is $D+1$. This observation yields immediately the length of the resolution (5.1). Furthermore $\deg(S_{k;i}) = \deg(f_{\alpha(i)}) + 1$, i. e. from the i-th to the $(i+1)$-th module the degree of the basis element to the corresponding syzygy grows by one.

The ranks of the modules follow from a rather straightforward combinatorial calculation. Let $\beta_{i,j}^{(k)}$ denote the number of generators of degree j of the i-th syzygy module $\mathrm{Syz}_i(G)$ with minimal variable in the head term x_k. By definition of the generators $S_{l;k}$, we find

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-1} \beta_{i-1,j-1}^{(n-t)}$$

as each generator with minimal variable smaller than k and degree $j-1$ in the marked basis of $\mathrm{Syz}_i(G)$ contributes one generator of minimal variable k and degree j to the marked basis of $\mathrm{Syz}_i(G)$. A simple but lengthy induction allows us to express $\beta_{i,j}^{(k)}$ in terms of $\beta_{0,j}^{(k)}$:

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-i} \binom{k-l-1}{i-1} \beta_{0,j-i}^{(t)}$$

Now we are able to compute the ranks of the free modules via

$$r_{i,j} = \sum_{k=1}^{n} \beta_{i,j}^{(k)} = \sum_{k=1}^{n} \sum_{t=1}^{k-i} \binom{k-t-1}{i-1} \beta_{0,j-i}^{(t)} = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)}.$$

The last equality follows from a classical identity for binomial coefficients. □

Remark 5.7. Observe that the direct summands in the resolution (5.1) depend only on the Pommaret basis $\mathcal{P}(J)$ and not on the ideal I, while the maps in (5.1) depend on I.

Corollary 5.8. *Let G be a $\mathcal{P}(J)$-marked basis and I the ideal generated by G in $A[\mathbf{x}]$. Define $r_{i,j}$ as in Theorem 5.6 and let $b_{i,j}$ be, as usual, the Betti numbers of I. Then*

- *$b_{i,j} \leqslant r_{i,j}$ for all i, j;*
- *$\mathrm{reg}(I) \leqslant \mathrm{reg}(J)$;*
- *$\mathrm{pdim}(I) \leqslant \mathrm{pdim}(J)$.*

Proof. The three inequalities follow from the free resolution (5.1) of I, recalling that $\mathrm{reg}(J) := \max_{x^\alpha \in \mathcal{P}(J)}\{\deg(x^\alpha)\}$ and $\mathrm{pdim}(J) = n - \min_{x^\alpha \in \mathcal{P}(J)}\{i \mid x_i = \min(x^\alpha)\}$. □

If G is even a Pommaret basis for the reverse lexicographic term order, i. e. if J is the leading ideal of I for this order, then we obtain the stronger results $\mathrm{reg}(I) = \mathrm{reg}(J)$ and $\mathrm{pdim}(I) = \mathrm{pdim}(J)$ (for other term orders we also get only estimates) [28, Corollaries 8.13, 9.5].

Example 5.9. Let $A[\mathbf{x}] = \Bbbk[x_0, x_1, x_2]$, J the monomial ideal with Pommaret basis $\mathcal{P}(J) = \{x_2^3, x_2^2x_1, x_2x_1, x_1x_0, x_1^2\}$ and I the polynomial ideal generated by $G = \{g_1, g_2, g_3, g_4, g_5\}$ with

$$\begin{aligned} g_1 &= x_2^3\,, & g_2 &= x_2^2x_1\,,\\ g_3 &= x_2x_1\,, & g_4 &= x_1x_0 + x_2^2\,,\\ g_5 &= x_1^2\,. \end{aligned}$$

One easily checks that G is a $\mathcal{P}(J)$-marked basis.

We explicitly compute the multiplicative representations of $x_2 \cdot g_2$, $x_2 \cdot g_3$, $x_1 \cdot g_4$, $x_2 \cdot g_4$, $x_2 \cdot g_5$ which yield the set of fundamental syzygies $G_{\mathrm{Syz}} = \{S_{2;2}, S_{3;2}, S_{4;1}, S_{4;2}, S_{5;2}\} \subset A[\mathbf{x}]^5$:

$$\begin{aligned}
x_2 \cdot g_2 &= x_1 \cdot g_1\,, & S_{2;2} &= x_2 \cdot e_2 - x_1 \cdot e_1\,,\\
x_2 \cdot g_3 &= g_2\,, & S_{3;2} &= x_2 \cdot e_3 - e_2\,,\\
x_1 \cdot g_4 &= x_0 \cdot g_5 + g_2\,, & S_{4;1} &= x_1 \cdot e_4 - x_0 \cdot e_5 - e_2\,,\\
x_2 \cdot g_4 &= x_0 \cdot g_3 + g_1\,, & S_{4;2} &= x_2 \cdot e_4 - x_0 \cdot e_3 - e_1\,,\\
x_2 \cdot g_5 &= x_1 \cdot g_3\,, & S_{5;2} &= x_2 \cdot e_5 - x_1 \cdot e_3\,.
\end{aligned}$$

The only non-multiplicative variable for G_{Syz} is $\overline{\mathcal{X}}_{\mathcal{P}}(S_{4;1}) = \{x_2\}$. Therefore we have to compute the reduction of $x_2 S_{4;1}$ which is $x_2 S_{4;1} = x_1 S_{4;2} - S_{2;2} - x_0 S_{5;2}$ and hence we get the set of fundamental syzygies of the first syzygy module $G_{\mathrm{Syz}_2} = \{x_2 e_3 - x_1 e_4 - e_1 - x_0 e_5\} \subset A[\mathbf{x}]^5$.

This leads to the following free resolution of I of length two:

$$0 \longrightarrow A[\mathbf{x}](-4) \xrightarrow{\delta_2} A[\mathbf{x}](-4) \oplus A[\mathbf{x}](-3)^4 \xrightarrow{\delta_1} \\ \xrightarrow{\delta_1} A[\mathbf{x}](-3)^2 \oplus A[\mathbf{x}](-2)^3 \xrightarrow{\delta_0} I \longrightarrow 0,$$

where

$$\delta_0 = \begin{pmatrix} x_2^3 & x_2^2 x_1 & x_2 x_1 & x_1 x_0 + x_2^2 & x_1^2 \end{pmatrix},$$
$$\delta_1 = \begin{pmatrix} -x_1 & 0 & 0 & -1 & 0 \\ x_2 & -1 & -1 & 0 & 0 \\ 0 & x_2 & 0 & -x_0 & -x_1 \\ 0 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & -x_0 & 0 & x_2 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 \\ 0 \\ x_2 \\ -x_1 \\ x_0 \end{pmatrix}.$$

This free resolution is not minimal, as these matrices contain non-vanishing constant entries. Minimising the resolution leads to the minimal free resolution of I of length one:

$$0 \longrightarrow A[\mathbf{x}](-3)^2 \xrightarrow{\delta_1'} A[\mathbf{x}](-2)^3 \xrightarrow{\delta_0'} I \longrightarrow 0.$$

Hence in the present example, we have $1 = \mathrm{pdim}(I) < \mathrm{pdim}(J) = 2$ and $2 = \mathrm{reg}(I) < \mathrm{reg}(J) = 3$.

Example 5.10. Let $A[\mathbf{x}] = \Bbbk[x_0, x_1, x_2]$, J the monomial ideal with Pommaret basis $\mathcal{P}(J) = \{x_2 x_1, x_2^2 x_1, x_2^3, x_1^3, x_2^2 x_0, x_1^2 x_0\}$ and I be the ideal generated by the $\mathcal{P}(J)$-marked basis $G = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ with

$$\begin{aligned}
g_1 &= x_2 x_1 - x_2^2 - x_1^2\,, & g_2 &= x_2^2 x_1\,,\\
g_3 &= x_2^3\,, & g_4 &= x_1^3\,,\\
g_5 &= x_2^2 x_0\,, & g_6 &= x_1^2 x_0\,,
\end{aligned}$$

where $\mathrm{Ht}(g_1) = x_2 x_1$. Observe that G is not a Gröbner basis, for any term order, due to the terms in $x_2 x_1 - g_1$.

By Theorem 5.6, we construct the following free resolution of I:

$$0 \longrightarrow A[\mathbf{x}](-5)^2 \xrightarrow{\delta_2} A[\mathbf{x}](-3) \oplus A[\mathbf{x}](-4)^6 \xrightarrow{\delta_1} \\ \xrightarrow{\delta_1} A[\mathbf{x}](-2) \oplus A[\mathbf{x}](-3)^5 \xrightarrow{\delta_0} I \longrightarrow 0. \quad (5.2)$$

It is not minimal, as the minimal free resolution of I is

$$0 \longrightarrow A[\mathbf{x}](-5)^2 \xrightarrow{\delta_2'} A[\mathbf{x}](-4)^6 \xrightarrow{\delta_1'} A[\mathbf{x}](-2) \oplus A[\mathbf{x}](-3)^4 \xrightarrow{\delta_0'} I \longrightarrow 0.$$

In this case, although the resolution (5.2) is not minimal, the bounds on projective dimension and regularity given in Corollary 5.8 are sharp.

In the sequel, we will apply the theory of marked bases and schemes to the study of Quot schemes. Theorem 5.6 and Corollary 5.8 seem to suggest that marked bases are particularly suitable to study loci of a Quot scheme given by bounds on the invariants of a module coming from the minimal free resolution: regularity, projective dimension, extremal Betti numbers. However, this is only partly true, due to the fact that in order to study special loci of marked schemes we need to prove inequailities like those in Corollary 5.8 on saturated ideals. Nevertheless, we will be able to study the locus of a Quot scheme given by an upper bound for the regularity, thanks to the following theorem (and corollary). The proofs are given for ideals, but they also hold for modules in $A[\mathbf{x}]^m$ generated by a marked basis over a quasi-stable module.

Theorem 5.11. *Let $J \subseteq A[\mathbf{x}]$ be a stable ideal, generated in a single degree s, and I the ideal generated by a $\mathcal{P}(J)$-marked basis G. Then J and I have the same Betti numbers.*

Proof. Since J is stable, $\mathcal{P}(J)$ is the minimal monomial generating set of J. Then we can follow the lines of the proof of [1, Theorem 4.4], thanks to Theorem 3.13, items (v), (vi). □

Corollary 5.12. *Let $J \subseteq A[\mathbf{x}]$ be a stable ideal, generated in a single degree s, and I be the ideal generated by a $\mathcal{P}(J)$-marked basis G. Then* $\operatorname{reg}(J^{\mathrm{sat}}) \geqslant \operatorname{reg}(I^{\mathrm{sat}})$.

6. Deterministic computations for stable positions

So far, from Definition 3.3 on, we considered arbitrary marked sets over a quasi-stable monomial module. We now focus on marked sets whose polynomials are generated in a single degree, which is the case of interest for our applications. We are interested in investigating how to modify a finite set of polynomials so that they become a marked set over a quasi-stable module.

Remark 6.1. Consider an arbitrary monomial module $U \subset A[\mathbf{x}]^m_{\mathbf{d}}$ generated by the terms $T = \{x^{\mu^{(1)}}e_{k_1}, \dots, x^{\mu^{(q)}}e_{k_q}\}$. Let s be the maximal degree of a term in T and assume that U is not quasi-stable, i. e. there exists a term $x^\mu e_k \in T$ and an index $j > c := \min(x^\mu)$ such that $x_j^s \dfrac{x^\mu}{x_c^{\mu_c}} e_k \notin U$. This implies that the term $x_j^{\mu_c}\dfrac{x^\mu}{x_c^{\mu_c}} e_k$ does not belong to U. If we now consider the module $\widehat{U}$ generated by $\widehat{T} = \{x_j^{\mu_c}\dfrac{x^\mu}{x_c^{\mu_c}} e_k, x^{\mu^{(1)}}e_{k_1}, \dots, x^{\mu^{(q)}}e_{k_q}\}$, then it is clear that $\widehat{U}$ is somehow nearer to quasi-stability than U. This observation is studied in much more detail for the case of ideals in [17] and [26].

With the knowledge of the remark above, we define an *elementary move* $m_{l,t,a}$ as a linear change of variables of the form $x_i \mapsto x_i$ if $i \neq l$ and $x_l \mapsto x_l + a \cdot x_t$ for suitable indices $l < t$ and a parameter $a \in \Bbbk^\times$. If we apply $m_{l,t,a}$ to a term x^μ we obtain a polynomial

$$m_{l,t,a}(x^\mu) = \sum_{i=0}^{\mu_l} \binom{\mu_l}{i} a^i x^\mu \frac{x_t^i}{x_l^i}$$

The polynomial $m_{l,t,a}(x^\mu)$ always contains at least two terms: x^μ with coefficient 1 and $\frac{x^\mu x_t^{\mu_l}}{x_l^{\mu_l}}$ with coefficient a^{μ_l}. In the case of a coefficient field of prime characteristic, any other coefficient may vanish for some values of μ_l and j. We extend the linear transformation $m_{l,t,a}$ to polynomials and sets of polynomials in the obvious way.

It is clear that any monomial module is marked on itself. If we apply a linear change of variables to a monomial module, then the transformed module is generally no longer monomial, but will have a non-monomial minimal generating set. The next proposition shows that we can construct again a marked set out of the new module.

Proposition 6.2. *For a given degree $s \geqslant 0$, let $T = \{x^{\mu(1)}e_{k_1}, \dots, x^{\mu(q)}e_{k_q}\}$ be a set of terms in $\mathbb{T}_s^m$ and let $\mathbb{K}$ be a field extension of $\Bbbk$ such that $|\mathbb{K}| > sq$. Furthermore, let $F = \{f_1, \dots, f_q\} \subset \mathbb{K}[\mathbf{x}]_s^m$ be a T-marked set. Assume that $x^\mu e_k := x^{\mu(1)}e_{k_1}$ is an obstruction to quasi-stability for $\langle T\rangle$ and set $\widehat{F} = m_{c,j,a}(F)$ for an arbitrary $a \in \mathbb{K}^\times$ and some $j > c = \min(x^\mu)$. Setting $x^{\widehat{\mu}}e_k := x_j^{\mu_c}\dfrac{x^\mu}{x_c^{\mu_c}}e_k \notin \langle T\rangle$, we denote by $\widehat{T}$ the set of terms $\{x^{\widehat{\mu}}e_k, x^{\mu(2)}e_{k_2}, \dots, x^{\mu(q)}e_{k_q}\}$ obtained from T by replacing the first generator by $x^{\widehat{\mu}}e_k$. Then there exists a set $F' \subseteq \langle \widehat{F}\rangle_s$ which is marked over $\widehat{T}$ and which can be constructed from $\widehat{F}$ via linear combinations.*

Proof. We consider the elementary move $m_{c,j,a}$ for a yet undetermined parameter $a \in \mathbb{K}^\times$. The considered term $x^\mu e_k$ transforms as follows:

$$m_{c,j,a}(x^\mu)e_k = \sum_{i=0}^{\mu_c} \binom{\mu_c}{i} a^i x^\mu \frac{x_j^i}{x_c^i} e_k \,. \tag{6.1}$$

By our choice of the index pair (c, j), the term $x^{\widehat{\mu}}e_k$ appears on the right hand side with a non-zero coefficient for the index value $i = \mu_c$.

Applying the elementary move $m_{c,j,a}$ to all polynomials $f_i \in F$ yields new generators $\widehat{f_i}$ and each $\widehat{f_i}$ still contains the term $x^{\mu(i)}e_{k_i}$ with a coefficient which is a polynomial in a with constant term 1. It may happen that the term $x^{\mu(i)}e_{k_i}$ now also appears in other generators $\widehat{f_l}$, but then its coefficient there is always a polynomial in a without a constant term. Furthermore, in $\widehat{f_1}$ the term $x^{\widehat{\mu}}e_k$ now appears. Its coefficient contains in particular the term a^{μ_c} coming from the above transformation of x^μ. If $x^{\widehat{\mu}}e_k$ also lies in the support of some other generator $\widehat{f_l}$, then its coefficient cannot contain the term a^{μ_c}, as $x^\mu e_k$ appeared only in f_1, as the set F was assumed to be marked over T.

These observations imply that, after taking suitable linear combinations of the polynomials $\widehat{f_i}$, we can arrive at a set of polynomials $F' := \{\widehat{h}_1, \ldots, \widehat{h}_q\}$ such that for $i \in \{1, \ldots, q\}$ the only term of $\widehat{T}$ appearing in $\widehat{h}_i$ is: $x^{\widehat{\mu}}e_k$ for $i = 1$, and $x^{\mu(i)}e_{k_i}$ for the other values of i.

It cannot happen that for some $i = 2, \ldots, q$ the term $x^{\mu(i)}e_{k_i}$ vanishes when we perform the linear combinations on $\widehat{h}_1, \ldots, \widehat{h}_q$, because there is exactly one term $x^{\mu(i)}e_{k_i}$ which has as coefficient a polynomial in a with constant term 1. By the same argument, it is clear that the term $x^{\widehat{\mu}}e_k$ does not vanish by performing linear combinations as its coefficient a^{μ_c} in $\widehat{h_1}$ is unique. But this implies that the set F' obtained from F by the elementary move and suitable linear combinations is marked over $\widehat{T}$. Furthermore, in each polynomial $\widehat{h}_i$ the coefficient of the head module term is a polynomial in a of degree at most s. Since we have q such coefficients, the assumption $|\mathbb{K}| > sq$ guarantees that there exists a choice for $a \in \mathbb{K}^\times$ such that none of these polynomials vanishes. □

Remark 6.3. For this proposition, it is crucial that we always consider marked sets and bases over quasi-stable modules. If we used any stronger notion of stability like stable or even strongly stable modules, then the proposition would remain true only in characteristic zero. Only in the quasi-stable case the key term $x^{\widehat{\mu}}e_k$ appears in (6.1) with coefficient 1. Obstructions to (strong) stability may appear somewhere "in the middle" of (6.1) and then it is in positive characteristic p no longer possible to guarantee that the needed term can be produced with an elementary move. One may introduce adapted "p-versions" of (strong) stability, but the corresponding monomial ideals do not necessarily exhibit all the relevant algebraic properties.

From now on, we will assume for simplicity that the field $\Bbbk$ is infinite, hence we will use coordinate transformations in $\mathrm{PGL} := \mathrm{PGL}_\Bbbk(n+1)$. For any element $g \in \mathrm{PGL}$, we denote by $\tilde{g}$ the automorphism induced by g on $A[\mathbf{x}]^m$ and by $\tilde{g}\,\boldsymbol{.}$ the corresponding action on an element. If F is a subset of $A[\mathbf{x}]^m$, $\tilde{g}\,\boldsymbol{.}\,F$ is the set obtained by applying $\tilde{g}$ to every element of F. We can now rephrase Proposition 6.2 in the following way, keeping in mind that, under the hypothesis that $\Bbbk$ is infinite, $\Bbbk$ is also Zariski dense in any field extension $\mathbb{K}$.

Corollary 6.4. *Let $F \subset A[\mathbf{x}]_s^m$ be a finite set of polynomials. Then there exists a transformation $g \in \mathrm{PGL}$ such that $\tilde{g}\,\boldsymbol{.}\,F$ is a marked set over a quasi-stable module.*

Lemma 6.5. *Consider $\ell \geqslant 0$. Let F be a saturated module in $\mathbb{K}[\mathbf{x}]^m$ for a field extension $\mathbb{K}$ of $\Bbbk$ with Hilbert polynomial $p(z)$ and $\mathrm{reg}(F) \leqslant \ell$. Then there exists a transformation $g \in \mathrm{PGL}$ and a stable module $U = \langle U_\ell \rangle \subset \mathbb{K}[\mathbf{x}]^m$ having Hilbert polynomial $p(z)$ and $\mathrm{reg}(U^{\mathrm{sat}}) \leqslant \ell$ such that $\tilde{g}\,\boldsymbol{.}\,\langle F_\ell \rangle$ belongs to $\underline{\mathbf{Mf}}^{n,m,\mathbf{d}}_{\mathcal{P}(U)}(\mathbb{K})$.*

Proof. By [28, Thm. 2.16] (or [17, Thm. 6.11, Rem. 6.13]),[1] there exists a transformation $g \in \mathrm{PGL}$ such that $\tilde{g} \centerdot \langle F_\ell \rangle$ has a Pommaret basis for the degree reverse lexicographic term order. This means that the initial module U of $\tilde{g} \centerdot \langle F_\ell \rangle$ is quasi-stable and has the same Hilbert polynomial as F. Since $\mathrm{reg}(U) = \mathrm{reg}(F_\ell) = \ell$, we have that $U = \langle U_\ell \rangle$ and U is even stable. Now it suffices to observe that $\mathrm{reg}(U^{\mathrm{sat}}) \leqslant \mathrm{reg}(U) = \ell$. □

7. Definition of Quot functor and Quot scheme with bounded regularity

Let $p(z) \in \mathbb{Q}[z]$ be the Hilbert polynomial of $\Bbbk[\mathbf{x}]^m/M$ for some homogeneous module $M \subseteq \Bbbk[\mathbf{x}]^m$. We denote by $N_m(z)$ the polynomial $m\binom{n+z}{z}$ and by $q(z)$ the polynomial $N_m(z) - p(z)$. By [11, Proposition 3.1], there exists a unique Gotzmann representation of $p(z)$:

$$p(z) = \binom{z+a_1}{a_1} + \binom{z+a_2-1}{a_2} + \cdots + \binom{z+a_r-(r-1)}{a_r}$$

where $a_1 \geqslant a_2 \geqslant \cdots a_r \geqslant 0$. We call r the Gotzmann number of $p(z)$. We recall that, by [11, Proposition 4.1], r is an upper bound for the regularity of the associated sheaf $\widetilde{M}$.

We now define the Hilbert function and the Hilbert polynomial in a more general case, following the lines of [24]. Let X be a finite type scheme over a field $\Bbbk$ together with a line bundle L. Recall that, if F is a coherent sheaf on X whose support is proper over $\Bbbk$, then the *Hilbert polynomial* $\Phi \in \mathbb{Q}[z]$ of F is defined as

$$\Phi(z) = \chi(F(z)) = \sum_{i=0}^{n} (-1)^i \dim_{\Bbbk} H^i(X, F \otimes L^{\otimes z})$$

where the dimensions of the cohomologies are finite because of the coherence and properness conditions. The fact that $\chi(F(z))$ is indeed a polynomial in z under the above assumption is a special case of what is known as Snapper's Lemma (see [20, Theorem B.7] for a proof).

Let $X \longrightarrow S$ be a finite type morphism of noetherian schemes and let L be a line bundle on X. Let $\mathcal{F}$ be any coherent sheaf on X whose schematic support is proper over S. Then for each $s \in S$ we get a polynomial $\Phi_s \in \mathbb{Q}[z]$ which is the Hilbert polynomial of the restriction $\mathcal{F}_s = F|_{X_s}$ of $\mathcal{F}$ to the fiber X_s over s calculated with respect to the line bundle $L_s = L|_{X_s}$. If $\mathcal{F}$ is flat over S, then the function $s \mapsto \Phi_s$ from the set of points of S to the polynomial ring $\mathbb{Q}[z]$ is known to be locally constant on S.

We will denote by $\mathbb{P}^n$ the n-dimensional projective space over $\Bbbk$. If Z is a $\Bbbk$-scheme, we define $\mathbb{P}^n_Z := \mathbb{P}^n \times_{\Bbbk} Z$ and if A is a finitely generated $\Bbbk$-algebra, then $\mathbb{P}^n_A$ is defined as $\mathbb{P}^n_{\mathrm{Spec}(A)}$. We are interested in the case where $X = \mathbb{P}^n_Z$, $L = \mathcal{O}_{\mathbb{P}^n_Z}(1)$ and F is a quotient of $\mathcal{O}^m_{\mathbb{P}^n_Z}$. If F is flat over Z, then the Hilbert polynomial of the fibres is locally constant. If it is constant, we call it the Hilbert polynomial of F.

In the sequel, $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}$ will denote the Quot functor $\underline{\mathrm{Sch}/\Bbbk^{\circ}} \to \underline{\mathrm{Sets}}$ that associates to any object Z of the category of schemes over $\Bbbk$ the set

$$\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(Z) = \{\mathcal{Q} \text{ quotients of } \mathcal{O}^m_{\mathbb{P}^n_Z} \text{ flat over } Z \text{ with Hilbert polynomial } p(z)\}.$$

and to any morphism of schemes $\varphi\colon Z \to Z'$ the map

$$\begin{array}{rccc} \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(\varphi)\colon & \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(Z') & \to & \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(Z) \\ & \mathcal{Q}' & \mapsto & \varphi^*\mathcal{Q}'\,. \end{array}$$

The Hilbert polynomial of $\mathcal{Q}$ is here defined via the Hilbert polynomial of each fibre of Z.

The Quot functor was introduced by Grothendieck in [16] where he also proved that this functor is the functor of points of a projective scheme. We will not use this fact, but will give instead an independent proof of the existence of the Quot scheme. Here, we only assume that

[1] These references consider only the case of ideals. However, the extension to modules along the lines of Prop. 6.2 is straightforward.

the Quot functor is a Zariski sheaf [24, Section 5.1.3]. Hence, we may consider it as a covariant functor from the category of noetherian $\Bbbk$-algebras [29, Lemma E.11]

$$\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)} \colon \underline{\Bbbk\text{-Alg}} \to \underline{\mathrm{Sets}}$$

such that for every finitely generated $\Bbbk$-algebra A

$$\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A) = \Big\{ \text{ quotients } \mathcal{Q} \text{ of } \mathcal{O}^m_{\mathbb{P}^n_A} \text{ flat over } \operatorname{Spec} A \text{ with Hilbert polynomial } p(z) \Big\}.$$

and for any $\Bbbk$-algebra morphism $f \colon A \to B$

$$\begin{array}{rccc} \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(f) \colon & \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A) & \to & \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(B) \\ & \widetilde{Q} & \mapsto & \widetilde{Q \otimes_A B} \end{array}$$

where $Q = H^0_* \mathcal{Q}$ for $\mathcal{Q} \in \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A)$. This is equivalent to consider the functor $\underline{\Bbbk\text{-Alg}} \to \underline{\mathrm{Sets}}$ that associates to every $\Bbbk$-algebra A the set

$$\begin{aligned} \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A) = \{ & \text{saturated submodules } M \text{ of } A[\mathbf{x}]^m \text{ s.t.} \\ & A[\mathbf{x}]^m/M \text{ flat with Hilbert polynomial } p(z) \}. \end{aligned}$$

and to every $\Bbbk$-algebras homomorphism $f \colon A \to B$ the function

$$\begin{array}{rccc} \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(f) \colon & \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A) & \to & \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(B) \\ & M & \mapsto & M \otimes_A B \end{array}$$

Inspired by the results in Section 5 and in [1] for Hilbert schemes, we intend to study a special subfunctor of the Quot functor defined by giving an upper bound on the Castelnuovo-Mumford regularity of the elements in $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A)$ for any $\Bbbk$-algebra A. Several proofs use the same arguments of corresponding results in [1]. If A is a local ring, we define the Castelnuovo-Mumford regularity $\mathrm{reg}(M)$ of a saturated module $M \in \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}(A)$ in the obvious way. Otherwise, we say that the Castelnuovo-Mumford regularity of M is $\min\{\mathrm{reg}(M \otimes_A A_\mathfrak{p}) \mid \mathfrak{p} \text{ prime ideal in } A\}$.

Definition 7.1. Let ℓ be an integer. The *Quot functor with bounded regularity*, denoted by $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$, is the subfunctor of $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}$ that associates to every Noetherian $\Bbbk$-algebra A the set $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(A) = \{M \in \mathcal{Q}\mathrm{uot}^{n,m}_{p(z)} \mid \mathrm{reg}(M) \leqslant \ell\}$.

It is immediate that if $\ell' \leqslant \ell$, then $\mathcal{Q}\mathrm{uot}^{n,m,[\ell']}_{p(z)}(A)$ is a subset of $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(A)$ for every $\Bbbk$-algebra A. Furthermore, if r is the Gotzmann number of $p(z)$, then $\mathcal{Q}\mathrm{uot}^{n,m,[r]}_{p(z)}$ is exactly $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}$. From now on, we fix two positive integers, ℓ and $s \geqslant \ell$. For every $\Bbbk$-algebra A and for every $M \in \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(A)$, there is a unique graded $A[\mathbf{x}]$-module generated in degree s whose saturation is M, namely $\langle M_s \rangle$. Hence, the Quot Functor with bounded regularity can be considered by [11, Lemma 5.2, Theorem 5.1] as a subfunctor of the following Grassmann functor:

$$\begin{array}{rcl} \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)} : \underline{\Bbbk\text{-Alg}} & \to & \underline{\mathrm{Sets}} \\ A & \mapsto & \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A) \end{array} \qquad \text{with } N_m(s) = m\tbinom{n+s}{s}$$

where

$$\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A) = \{A\text{-submodule } F \subseteq A[\mathbf{x}]^m_s \text{ such that } A[\mathbf{x}]^m_s/F \text{ is locally free of rank } p(s)\}.$$

Therefore, the Quot Functor with bounded regularity can be seen as a subfunctor of the Grassmann functor $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$ in the following way:

$$\begin{aligned} \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(A) = \{ & F \in \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A) \text{ with } A[\mathbf{x}]^m/\langle F \rangle \text{ flat} \\ & \text{with Hilbert polynomial } p(z) \text{ and } \mathrm{reg}(F^{\mathrm{sat}}) \leqslant \ell \}. \end{aligned}$$

and for every $\Bbbk$-algebra homomorphism $f\colon A \to B$

$$\begin{array}{rccl} \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(f)\colon & \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(A) & \to & \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}(B) \\ & F & \mapsto & F \otimes_A B\,. \end{array}$$

Furthermore, we define the natural transformation of functors

$$\mathcal{H}^{[s]} : \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)} \to \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}\,.$$

We denote by π_M the canonical projection $A[\mathbf{x}]^m_s \to A[\mathbf{x}]^m_s/M_s$. The functor $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$ is representable and the representing scheme $\mathrm{Gr}^{N_m(s)}_{p(s)}$ is called the Grassmannian. By the Plücker embedding, it can be seen as a closed subscheme of $\mathbb{P}^{\binom{N_m(s)}{p(s)}-1}$.

We will now introduce some useful subfunctors of $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$ and $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$. We set a basis $\{b_1,\dots,b_{p(s)}\}$ for $A^{p(s)}$. Consider the complete list $\mathbb{T}^m_s = \{\tau_\ell\}_{\ell=1,\dots,N_m(s)}$ of terms $\tau = x^\alpha e_i$, $|\alpha| = s$, of $\Bbbk[\mathbf{x}]^m_s$. $\mathbb{T}^m_s$ is the basis we consider for the A-module $A[\mathbf{x}]^m_s$. For every element $g \in \mathrm{PGL} := \mathrm{PGL}_{\mathbb{Q}}(n+1)$, we denote by $\tilde{g}$ also the automorphism induced by g on the Grassmann and Quot functors and $g_\bullet$ denotes the corresponding action on an element.

Consider for $\mathcal{I} = \{a_1,\dots,a_{p(s)}\} \subset \{1,\dots,N_m(s)\}$ with $|\mathcal{I}| = p(s)$ the injective morphism $\Gamma_{\mathcal{I}} : A^{p(s)} \to A[\mathbf{x}]^m_s$, $\quad b_i \mapsto \tau_{a_i}$ and for $g \in \mathrm{PGL}$ the subfunctor $\mathcal{G}^{[s]}_{\mathcal{I},g}$ that associates to every noetherian $\Bbbk$-algebra A the set

$$\mathcal{G}^{[s]}_{\mathcal{I},g}(A) = \{F \in \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A) \mid \pi_F \circ \tilde{g} \circ \Gamma_{\mathcal{I}} \text{ is surjective}\}\,.$$

The open subfunctors $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}$ provide an open cover of $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$ when $\mathcal{I}$ varies among the subsets of $\{1,\dots N_m(s)\}$ containing $p(s)$ elements [15, Lemma 8.13]. We refer to these open subfunctors as *standard open cover* of $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$. Finally, for every $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ with $|\mathcal{I}| = p(s)$ and for every $g \in \mathrm{PGL}$, we define the following open subfunctors of $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$:

$$\mathcal{Q}^{[\ell,s]}_{\mathcal{I},g}(A) := \left(\mathcal{H}^{[s]}\right)^{-1}\left(\mathcal{G}^{[s]}_{\mathcal{I},g}(A)\right) \cap \mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}. \tag{7.1}$$

Obviously, for $g = \mathrm{Id}$ and $\mathcal{I}$ varying, the subfunctors in (7.1) cover $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$.

8. Quasi-stable open cover of the Grassmannian

We associate to any set $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ the set of monomials $\mathcal{U}_{\mathcal{I}} := \{\tau_i\}_{i\in\mathcal{I}} \subset \mathbb{T}^m_s$ and its complement $\mathcal{U}^c_{\mathcal{I}}$ in $\mathbb{T}^m_s$. If $|\mathcal{I}| = p(s)$, then $|\mathcal{U}^c_{\mathcal{I}}| = N_m(s) - p(s)$. Later, we will prefer to work with a different open cover of the Quot scheme defined by considering only some special sets $\mathcal{U}_{\mathcal{I}}$.

Lemma 8.1. *Consider $\mathcal{I} = \{a_1,\dots,a_{p(s)}\} \subset \{1,\dots,N_m(s)\}$, and assume that the monomial module $U := \langle \mathcal{U}^c_{\mathcal{I}} \rangle \subset A[\mathbf{x}]^m$ is quasi-stable.*

What is F here?

(i) $F \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$ if and only if it is generated as an A-module by a $\mathcal{U}^c_{\mathcal{I}}$-marked set.

(ii) If F belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$, then for every $s' \geqslant s$ the A-module $\langle F \rangle_{s'}$ contains a free submodule of rank $\geqslant q(s')$ generated by a $\langle \mathcal{U}^c_{\mathcal{I}} \rangle \cap \mathbb{T}^m_{s'}$-marked set.

Proof.

(i) If F belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$, then $\mathcal{U}_{\mathcal{I}}$ is a generating set for the module $A[\mathbf{x}]^m_s/F$, since $\pi_F \circ \mathrm{Id} \circ \Gamma_{\mathcal{I}}$ is surjective. For every $\tau \in \mathcal{U}^c_{\mathcal{I}}$, we consider the polynomial $f_\tau = \tau - \pi_F(\Gamma_{\mathcal{I}}(\tau))$. The module element f_τ is a homogeneous marked element of $A[\mathbf{x}]^m$ with $\mathrm{Ht}(f_\tau) = \tau$ and $\tau - f_\tau \in \langle \mathcal{U}_{\mathcal{I}} \rangle^A = \langle \mathcal{N}(U)_s \rangle^A$. Hence $\{f_\tau\}_{\tau\in\mathcal{U}^c_{\mathcal{I}}}$ is a $\mathcal{U}^c_{\mathcal{I}}$-marked set contained in F. Observe that $\langle f_\tau \rangle^A \subset F$ and $\mathrm{rk}(F_s) = \mathrm{rk}\langle f_\tau \rangle$, hence $F = \langle f_\tau \rangle^F$.

Vice versa, let $G = \{f_\tau\}_{\tau\in\mathcal{U}^c_\mathcal{I}}$ be the $\mathcal{U}^c_\mathcal{I}$-marked set generating F. Then, by Corollary 3.9, for every $\tau \in A[\mathbf{x}]^m$, there is $g \in F_{|\tau|}$ such that $\tau - f = \sum_{\tau'\in\mathcal{N}(U)_{|\tau|}} a'\tau'$ with $a' \in A$. Hence the A-module F generated by $\{f_\tau\}_{\tau\in\mathcal{U}^c}$ belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$.

(ii) We denote by $G^{(s')}$ the set $\{x^\delta f_\tau \; f_\tau \in G, \deg(x^\delta f_\tau) = s', \min(\tau) \geqslant \max(x^\delta)\}$. Due to the fact that $\langle G^{(s')}\rangle^A \subset \langle F\rangle_{s'}$, the statement follows from Theorem 3.13 (iii). $\square$

The following example shows that $A[\mathbf{x}]^m/\langle\mathcal{U}^c_\mathcal{I}\rangle$ and $A[\mathbf{x}]^m/\langle F\rangle$, with $F \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$, in general do not have the same Hilbert polynomial or function.

Example 8.2. In $A[\mathbf{x}] = k[x_2, x_1, x_0]$, we consider $\mathcal{U}^c_\mathcal{I} = \{x_1x_2, x_0^2\}$ and $U := \langle\mathcal{U}^c_\mathcal{I}\rangle \subset A[\mathbf{x}]$. Let M be the submodule of $A[\mathbf{x}]$ generated by $f_1 = x_1x_2 + x_0x_1$, $f_2 = x_0^2 + x_0x_2$, which form a $\mathcal{U}^c_\mathcal{I}$-marked set. The Hilbert polynomial of $A[\mathbf{x}]/U$ is constant, while the Hilbert polynomial of $A[\mathbf{x}]/\langle f_1, f_2\rangle$ has degree 1. Hence they also do not have the same Hilbert function.

Lemma 8.3. *Let $(A, \mathfrak{m}, \mathbb{K})$ be a local ring and $F \in \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A)$. Then $F \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$, if and only if $F \otimes_A \mathbb{K} \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(\mathbb{K})$.*

Proof. By extension of the scalars, it is clear that $F \otimes_A \mathbb{K} \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(\mathbb{K})$, if $F \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$. Therefore we only prove the other direction.

Assume that $F \otimes_A \mathbb{K} \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(\mathbb{K})$ and let $\{\overline{f}_\tau\}_{\tau\in\mathcal{U}^c_\mathcal{I}}$ be the $\mathcal{U}^c_\mathcal{I}$-marked set generating $F \otimes_A \mathbb{K}$. Consider a set of polynomials $\{f_\tau\}_{\tau\in\mathcal{U}^c_\mathcal{I}} \subset F$ such that the image of each f_τ in $\mathbb{K}[\mathbf{x}]^m_s$ is $\overline{f}_\tau$. We construct for F a $q(s) \times N_m(s)$ matrix M_F. We order (in any way) the terms of $\mathbb{T}^m_s$: $x^{\alpha_1}e_{k_1}, \dots, x^{\alpha_{N_m(s)}}e_{k_{N_m(s)}}$ and the elements f_τ. The j-th column of M corresponds to the term $x^{\alpha_j}e_{k_j}$. The i-th row of M_F corresponds to the coefficients in the i-th element in $\{f_\tau\}_{\tau\in\mathcal{U}^c_\mathcal{I}}$.

Considering the images of the entries in $\mathbb{K}$, we obtain the analogous matrix M' for $\{\overline{f}_\tau\}_{\tau\in\mathcal{U}^c_\mathcal{I}}$. By hypothesis, the minor corresponding to $\mathcal{U}^c$ of this last matrix is invertible. Then the corresponding minor in M is also invertible, because A is local. In general, $\{f_\tau\}_{\tau\in\mathcal{U}^c}$ is not a $\mathcal{U}^c_\mathcal{I}$-marked set. But we can obtain a $\mathcal{U}^c_\mathcal{I}$-marked set by performing a row reduction of M such that the minor from above gets the identity matrix. $\square$

Definition 8.4. For any admissible Hilbert polynomial $p(z)$ in $A[\mathbf{x}]^m$, given an integer ℓ, we consider the integers $s \geqslant \ell$, $p(s)$, $N_m(s)$. We define the following sets:

- $\mathbb{QS}$ is the set of those quasi-stable modules in $\Bbbk[\mathbf{x}]^m$ whose minimal monomial set of generators consists of $N_m(s) - p(s)$ terms of degree s.
- $\mathbb{QS}_{p(z)}$ is the subset of $\mathbb{QS}$ containing monomial modules having Hilbert polynomial $p(z)$.
- $\mathbb{QS}^\ell_{p(z)}$ is the subset of $\mathbb{QS}_{p(z)}$ containing submodules U with $\mathrm{reg}(U^{\mathrm{sat}}) \leqslant \ell$.
- $L^{[\ell,s]}_{p(z)}$ is the closed subset of $\mathrm{Gr}^{N_m(s)}_{p(s)}$ defined by the ideal

$$\Big(g \centerdot \Delta_\mathcal{I} \mid \forall\ g \in \mathrm{PGL}(n+1),\ \forall\ \langle\mathcal{U}^c_\mathcal{I}\rangle \in \mathbb{QS}^\ell_{p(z)}\Big).$$

Proposition 8.5. *The collection of subfunctors*

$$\Big\{\mathcal{G}^{[s]}_{\mathcal{I},g} \mid g \in \mathrm{PGL}, \mathcal{I} \subset \{1,\dots,N_m(s)\}\ s.t\ \langle\mathcal{U}^c_\mathcal{I}\rangle \in \mathbb{QS}\Big\}$$

covers the Grassmann functor $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$.

Proof. We have to prove that, for every $\Bbbk$-Algebra A and every $F \in \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A)$, there exist $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ with $\langle\mathcal{U}^c_\mathcal{I}\rangle \in \mathbb{QS}$ and $g \in \mathrm{PGL}$ such that $F \in \mathcal{G}^{[s]}_{\mathcal{I},g}(A)$ or equivalently such that $g^{-1} \centerdot F \in \mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$. As the question is local, it suffices to consider the case that the ring A is local. By Lemma 8.3, we may assume that A is in fact a field.

Let $F \in \mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}(A)$ for a field A. Let $\mathcal{J}$ be the set of subsets of $\mathbb{T}^m_s$ of cardinality $N_m(s)-p(s)$. As in the proof of Lemma 8.3, we associate the $q(s)\times N_m(s)$ matrix M_F to F, considering a set of generators for the module F. For every $\mathcal{V} \in \mathcal{J}$, let $\Delta_{\mathcal{V}}(M_F)$ be the minor of M_F corresponding to $\mathcal{V} \in \mathcal{J}$. It is obvious that there is $\mathcal{V} \in \mathcal{J}$ such that $\Delta_{\mathcal{V}}(M_F) \neq 0$.

If $\langle \mathcal{V} \rangle \in \mathbb{QS}$, we are already done: if $\mathcal{I} \subset \{1, \dots, N_m(s)\}$ is such that $\mathcal{U}^c_{\mathcal{I}} = \mathcal{V}$, then F belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$. Assume that this is not the case. Then there exists an obstruction to quasi-stability: $x^\mu e_k \in \mathcal{V}$ and $j > c := \min(x^\mu)$, such that $x_j \frac{x^\mu}{x_c} e_k \notin \langle \mathcal{V} \rangle$. We denote by $\widehat{\mathcal{V}} \in \mathcal{J}$ the set obtained by replacing in $\mathcal{V}$ the obstruction to quasi-stability $x^\mu e_k$ with $x^{\widehat{\mu}} e_k := x_j \frac{x^\mu}{x_c} e_k$.

Up to an autoreduction of F, we can assume without loss of generality, that F is generated by a $\mathcal{V}$-marked set due to the fact that $\Delta_{\mathcal{V}}(M_F)$ is non-zero. Proposition 6.2 guarantees that there is a linear coordinate transformation $g \in \mathrm{PGL}$ with respect to the elementary move $m_{c,j,a}$ for an $a \in A$ such that $\widehat{F} = g^{-1} \centerdot F$ and $\widehat{F}$ is generated by a $\widehat{\mathcal{V}}$-marked set. This implies that $\Delta_{\widehat{\mathcal{V}}}(M_{\widehat{F}}) \neq 0$. If $\langle \widehat{\mathcal{V}} \rangle \in \mathbb{QS}$, we are done: if $\mathcal{I} \subset \{1, \dots, N_m(s)\}$ is such that $\mathcal{U}^c_{\mathcal{I}} = \widehat{\mathcal{V}}$, then $g^{-1} \centerdot F$ belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$. If $\langle \widehat{\mathcal{V}} \rangle \notin \mathbb{QS}$, we can repeat this construction starting from an obstruction to stability for the module $\langle \widehat{\mathcal{V}} \rangle$.

The claim of the proposition follows from a simple termination argument, which shows that we finally get a $\widehat{\mathcal{V}} \in \mathcal{J}$ and $g \in \mathrm{PGL}$ such that there is $\mathcal{I} \subset \{1, \dots, N_m(s)\}$ such that $\mathcal{U}^c_{\mathcal{I}} = \widehat{\mathcal{V}}$, $\langle \widehat{\mathcal{V}} \rangle \in \mathbb{QS}$ and $g^{-1} \centerdot F$ belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$. We introduce an ordering on $\mathcal{J}$. Given two sets $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{J}$, we first sort them according to $\prec_{\mathrm{TOP}_{\mathrm{degrevlex}}}$ (greatest term first) and then compare the two sets entry by entry again with respect to $\prec_{\mathrm{TOP}_{\mathrm{degrevlex}}}$. Then $\mathcal{V}_1 < \mathcal{V}_2$, if there is i such that for every $j < i$, the j-th entry of $\mathcal{V}_1$ is the same as the j-th entry of $\mathcal{V}_2$, while the i-th entry of $\mathcal{V}_1$ is smaller than (or equal to) the i-th entry of $\mathcal{V}_2$ with respect to $\prec_{\mathrm{TOP}_{\mathrm{degrevlex}}}$.

Our construction gives at each recursion a set $\widehat{\mathcal{V}}$ such that $\widehat{\mathcal{V}} > \mathcal{V}$ with respect to the ordering we defined. In this way, we construct a strictly ascending chain of sets in $\mathcal{J}$. Since $\mathcal{J}$ is a finite set, the chain must be finite, too. Hence, our construction only stops when there are no obstructions to quasi-stability, that is when it reaches a set $\widehat{\mathcal{V}}$ such that $\langle \widehat{\mathcal{V}} \rangle \in \mathbb{QS}$. □

Definition 8.6. A *quasi-stable subfunctor* of $\mathcal{G}\mathrm{r}^{N_m(s)}_{p(s)}$ is any element of the collection of subfunctors of Proposition 8.5.

Remark 8.7. The same statement as Proposition 8.5 is proved in [1, Proposition 5.4], concerning the Grassmannian of linear spaces of $A[\mathbf{x}]_s$. We emphasise that here we consider the action of PGL on $A[\mathbf{x}]^m$, hence [1, Proposition 5.4] does not apply. Furthermore, the proof of Proposition 8.5 gives an algorithmic strategy to explicitly construct a g such that $g^{-1} \centerdot F$ belongs to $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$ with $\langle \mathcal{U}^c_{\mathcal{I}} \rangle \in \mathbb{QS}$.

Corollary 8.8. *For every $g \in \mathrm{PGL}$ and for every $\mathcal{I} \subset \{1, \dots, N_m(s)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}} \rangle \in \mathbb{QS}$, the functor $\mathcal{G}^{[s]}_{\mathcal{I},g}$ is the functor of points of an affine scheme which we denote by $\mathrm{G}^{[s]}_{\mathcal{I},g}$ and which is naturally isomorphic to $\mathbb{A}^{(N_m(s)-p(s))\cdot p(s)}_{\Bbbk}$.*

Proof. This is analogous to [1, Proposition 5.4]. □

Proposition 8.9. *The collection of open subschemes*

$$\{\mathcal{G}^{[s]}_{\mathcal{I},g} \mid g \in \mathrm{PGL}, \mathcal{I} \subset \{1, \dots, N_m(s)\} \ s.t \ \langle \mathcal{U}^c_{\mathcal{I}} \rangle \in \mathbb{QS}^\ell_{p(z)}\}$$

covers $\mathrm{Gr}^{N_m(s)}_{p(s)} \setminus L^{\ell,s}_{p(z)}$.

Proof. It suffices to recall the definitions of $\mathbb{QS}^\ell_{p(z)}$ and $L^{[\ell,s]}_{p(z)}$ given in Definition 8.4. □

9. Stable cover and representability of Quot functors

We will now prove that for covering the Quot Functor with bounded regularity $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$, it suffices to consider those sets $\mathcal{I}$ for which the module $\langle \mathcal{U}^c_{\mathcal{I}} \rangle$ is stable, possesses the same Hilbert polynomial $p(z)$ and has a saturation where the regularity is bounded by ℓ. We will divide the proof in two steps. In Proposition 9.1, we will show that in order to cover $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ it suffices to consider those $\mathcal{I}$ for which $\langle \mathcal{U}^c_{\mathcal{I}}\rangle$ is in $\mathbb{QS}_{p(z)}$. In Theorem 9.5, we will prove that only those $\mathcal{I}$ with $\operatorname{reg}(\langle \mathcal{U}^c_{\mathcal{I}}\rangle^{\mathrm{sat}}) \leqslant \ell$ are necessary and that such a cover actually does not depend on the chosen degree $s \geqslant \ell$ for the embedding in the Grassmannian.

Proposition 9.1. *Consider $s \geqslant \ell$. The collection of subfunctors*

$$\{\mathcal{Q}^{[\ell,s]}_{\mathcal{I},g} \mid g \in \mathrm{PGL},\ \mathcal{I} \subset \{1,\dots,N_m(s)\} \ s.t \ \langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}_{p(z)}\}$$

covers the Quot functor $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$.

Proof. We consider $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ embedded by $\mathcal{H}^{[s]}$ in $\mathrm{Gr}^{N_m(s)}_{p(s)}$. By Proposition 8.5, we can immediately deduce that the Quot functor is covered by

$$\left\{\mathcal{Q}^{[\ell,s]}_{\mathcal{I},g} \mid g \in \mathrm{PGL},\ \mathcal{I} \subset \{1,\dots,N_m(s)\} \text{ s.t } \langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}\right\}.$$

We obtain the statement by proving that $\mathcal{Q}^{[\ell,s]}_{\mathcal{I},\mathrm{Id}}(A) \neq \emptyset$ for any $\mathcal{I} \subset \{1,\dots,N(s)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}$, if and only if actually $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}_{p(z)}$. In fact, this implies that for every $g \in \mathrm{PGL}$ we have $\mathcal{Q}^{[\ell,s]}_{\mathcal{I},g}(A) \neq \emptyset$, if and only if $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}_{p(z)}$. As this is a local and set-theoretical fact, we may assume that A is a field.

Consider now a module $F \in \mathcal{Q}^{[\ell,s]}_{\mathcal{I},\mathrm{Id}}(A)$, for $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}$. Due to Lemma 8.1, we know that $\langle F\rangle_s$ is generated by a $\mathcal{U}^c_{\mathcal{I}}$-marked set. By Theorem 3.13, we know that $\langle F\rangle_{s'}$ contains an A-vector space of the same dimension as $\langle \mathcal{U}^c_{\mathcal{I}}\rangle_{s'}$ for every $s' \geqslant s$. This implies, by Theorem 3.13, that $N_m(s') - p(s') = \dim(\langle F\rangle_{s'}) \geqslant \dim(\langle \mathcal{U}^c_{\mathcal{I}}\rangle_{s'})$. But the growth theorem of Macaulay [19, Lem. 23] implies that $\dim(\langle \mathcal{U}^c_{\mathcal{I}}\rangle_{s'}) \geqslant N_m(s') - p(s')$, hence we have equality and the Hilbert polynomial of $\langle \mathcal{U}^c_{\mathcal{I}}\rangle$ must be $p(z)$. ☐

In order to obtain an open cover for $\mathcal{Q}\mathrm{uot}^{n,m,[l]}_{p(z)}$ made up of less open subsets than the one given in Proposition 9.1, we need some preliminary results.

Proposition 9.2. *Consider $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}_{p(z)}$ and $\operatorname{reg}(\langle \mathcal{U}^c_{\mathcal{I}}\rangle^{\mathrm{sat}}) \leqslant s$. Let F be an element of $\mathcal{G}^{[s]}_{\mathcal{I},\mathrm{Id}}(A)$. Then $F \in \mathcal{Q}^{[r,s]}_{\mathcal{I},\mathrm{Id}}(A)$, if and only if for any $s' \geqslant s$ the A-module $\langle F\rangle_{s'}$ is free of rank $q(s')$ and generated by a $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \cap \mathbb{T}^m_{s'}$-marked basis.*

Proof. Observe that under the made hypotheses, $\langle \mathcal{U}^c_{\mathcal{I}}\rangle$ is stable. As the question is again local, we may again assume that A is a local ring. We first consider the special case that A is even a field. Let $G = \{f_\tau\}_{\tau \in \mathcal{U}^c_{\mathcal{I}}}$ be the $\mathcal{U}^c_{\mathcal{I}}$-marked set generating F. For any $s' \geqslant s$, we denote by $G^{(s')}$ the set $\{x^\delta f_\tau \mid f_\tau \in G, \deg(x^\delta f_\tau) = s', \min(\tau) \geqslant \max(x^\delta)\}$. It is immediate that $\langle G^{(s)}\rangle \subseteq F_s$. Using the same argument as in the proof of Proposition 9.1, the dimension of both vector spaces is $q(s')$. Hence they must be equal for every degree s' and this implies via Theorem 3.13 that G is a $\mathcal{U}^c_{\mathcal{I}}$-marked basis of F.

We generalize this result to the case that $(A, \mathfrak{m}, \Bbbk)$ is a local ring by the Nakayama lemma, since for any $s' \geqslant s$ the A-module $F_{s'}$ contains the free submodule $\langle G^{(s')}\rangle$ of rank $N_m(s') - p(s')$ (by Theorem 3.13) and the two $A/\mathfrak{m}$-vector spaces $F_{s'} \otimes_A A/\mathfrak{m}$ and $\langle G^{(s')}\rangle \otimes_A A/\mathfrak{m}$ coincide, as they have the same dimensions. ☐

We now easily see that the functor $\mathcal{Q}^{[\ell,\ell]}_{\mathcal{I},\mathrm{Id}}$ with $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}^{\ell}_{p(z)}$ is isomorphic to $\underline{\mathbf{Mf}}^m_{\mathcal{U}^c_{\mathcal{I}}}$.

Proposition 9.3. *Let $\mathcal{I} \subset \{1,\dots,N_m(\ell)\}$ be such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}^{\ell}_{p(z)}$.*

(i) The subfunctor $\mathcal{Q}^{[\ell,\ell]}_{\mathcal{I},\mathrm{Id}}$ is isomorphic to the marked functor $\underline{\mathbf{Mf}}^m_{\mathcal{U}^c_{\mathcal{I}}}$

(ii) The subfunctor $\mathcal{Q}^{[\ell,\ell]}_{\mathcal{I},\mathrm{Id}}$ is the functor of points of an affine subscheme of the affine space $\mathbb{A}^{p(\ell)\cdot q(\ell)}$.

Proof. $\langle \mathcal{U}^c_{\mathcal{I}}\rangle$ is stable under the made hypotheses by Proposition 2.5. Item (i) is a straightforward consequence of Proposition 9.2. Item (ii) follows from (i) and Theorem 4.1. ☐

Proposition 9.4. *Let U be a saturated quasi-stable module with Hilbert polynomial $p(z)$ and $\mathrm{reg}(U) \leqslant \ell$. We denote by $\mathcal{I}^{[s]}$ the set of indices defining the module $U \cap \mathbb{T}^m_s$. Let $F = F^{\mathrm{sat}}$ be a module in $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$. For any $s, s' \geqslant \ell$, the truncation $\langle F_s \rangle$ belongs to $\mathcal{Q}^{[\ell,s]}_{\mathcal{I}^{[s]},\mathrm{Id}}$, if and only if the truncation $\langle F_{s'} \rangle$ belongs to $\mathcal{Q}^{[\ell,s']}_{\mathcal{I}^{[s']},\mathrm{Id}}$.*

Proof. By Proposition 9.3 (i), our claim is equivalent to $\underline{\mathbf{Mf}}^m_{\mathcal{U}^c_{\mathcal{I}^{[s]}}} \simeq \underline{\mathbf{Mf}}^m_{\mathcal{U}^c_{\mathcal{I}^{[s']}}}$. For this isomorphism, we can repeat the arguments given in the proof of [22, Theorem 3.4 (i)]: indeed, all the arguments given in [22] apply also in the stable case and both $U \cap \mathbb{T}^m_s$ and $U \cap \mathbb{T}^m_{s'}$ are stable by Lemma 2.4 (iii). □

We now prove that the modules in $\mathbb{QS}^\ell_{p(z)}$ are sufficient to cover $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ refining the result given in Proposition 9.1.

Theorem 9.5. *Consider $\ell \leqslant s$.*

(i) Let $U = \langle U_s \rangle$ be a quasi-stable module in $\mathbb{QS}^\ell_{p(z)}$ and let $\mathcal{I}^{[s]}$ be as in Proposition 9.4. Then $\mathcal{Q}^{[\ell,s]}_{\mathcal{I}^{[s]},g} = \mathcal{Q}^{[\ell,r]}_{\mathcal{I}^{[r]},g} = \mathcal{Q}^{[r,r]}_{\mathcal{I}^{[r]},g}$ as subfunctors of $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$.

(ii) The collection of subfunctors

$$\left\{ \mathcal{Q}^{[\ell,s]}_{\mathcal{I},g} \mid g \in \mathrm{PGL},\ \mathcal{I} \subset \{1,\dots,N_m(s)\} \ s.t\ \langle \mathcal{U}^c_{\mathcal{I}} \rangle \in \mathbb{QS}^\ell_{p(z)} \right\} \tag{9.1}$$

covers the Quot functor with bounded regularity.

Proof. (i) The equality between $\mathcal{Q}^{[\ell,r]}_{\mathcal{I}^{[r]},g}$ and $\mathcal{Q}^{[r,r]}_{\mathcal{I}^{[r]},g}$ follows from Corollary 5.12. We obtain the other equality by Proposition 9.4.

(ii) By item (i), we can take $s = \ell$. As the question is local, it suffices to consider the case that the ring A is a field. Then Lemma 6.5 applies.

□

Corollary 9.6. *The Quot functor with bounded regularity $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ is an open subfunctor of $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}$.*

Definition 9.7. The *quasi-stable cover* of $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ is the collection of the open subfunctors (9.1) of Theorem 9.5.

Remark 9.8. We constructed a cover of $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ by using the quasi-stable modules in $\mathbb{QS}^\ell_{p(z)}$ and suitable deterministic changes of coordinates. There also exists a change of coordinates to reach a Borel-fixed position depending on $p = \mathrm{char}(\mathbb{k})$ (for short, p-Borel fixed position). Therefore, we could repeat the statements and proofs of the present section in order to prove (constructively) the existence of a p-Borel cover of the Quot functor, which is in general more sparse than the quasi-stable cover of Definition 9.7. However, we prefer to consider the quasi-stable cover because this cover is independent of the characteristic and the algorithm to reach the stable position is cheaper.

Furthermore, in the next section we will show that it is possible to compute equations for the open subscheme of the Quot scheme corresponding to each quasi-stable open subfunctor. The computational cost to get such equations for open neighbourhoods of a given point of the Quot scheme can be significantly different depending on the neighbourhood we choose. Hence it is an advantage to have a relatively dense cover in order to choose the more convenient neighbourhood.

10. Representability of Quot functors and equations

From now on, we will only consider open subfunctors $\mathcal{Q}^{[\ell,s]}_{\mathcal{I},\mathrm{Id}}$ with $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}} \rangle \in \mathbb{QS}^\ell_{p(z)}$. It was proved by Grothendieck that the Grassmann and Quot functors are representable. We will now prove that the Quot functor with bounded regularity is represented by a locally closed subscheme of the Grassmann scheme using the quasi-stable open cover and the fact that the Quot functor is a Zariski sheaf.

Theorem 10.1. *The Quot functor with bounded regularity is the functor of points of a closed subscheme* $\mathrm{Quot}^{m,[\ell]}_{p(z)}$ *of* $\mathrm{Gr}^{N_m(s)}_{p(s)} \setminus L^{[\ell,s]}_{p(z)}$ *with* $L^{[\ell,s]}_{p(z)}$ *given by Definition 8.4.*

Proof. By the semicontinuity theorem for regularity, $\mathcal{Q}\mathrm{uot}^{n,m,[\ell']}_{p(z)}$ can be considered as an open subfunctor $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ for any $\ell' < \ell$. Furthermore, $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}$ is a Zariski sheaf [24, Section 5.1.3]. Hence, $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ is a Zariski sheaf, too.

By [14, Theorem VI-14], it suffices to check the representability on an open cover of $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$: we choose the one given in (9.1). By Theorem 9.5 (i) and by Proposition 9.3 (ii), we immediately conclude that the Quot functor with bounded regularity is the functor of points of a scheme. By Proposition 8.9 and by Theorem 3.13 (viii), we obtain that the scheme representing the Quot functor with bounded regularity is a closed subscheme of $\mathrm{Gr}^{N_m(s)}_{p(s)} \setminus L^{[\ell,s]}_{p(z)}$. □

Corollary 10.2. *The Quot functor* $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)}$ *is the functor of points of a scheme and it is also a closed subscheme of* $\mathrm{Gr}^{N_m(s)}_{p(s)}$.

Proof. It suffices to observe that $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)} = \mathcal{Q}\mathrm{uot}^{n,m,[r]}_{p(z)}$ and to use Theorem 10.1 noting that $L^{[r,s]}_{p(z)} = \emptyset$ where r is the Gotzmann number of $p(z)$. □

We will now give a constructive procedure to define an ideal $\mathfrak{H} \subset \Bbbk[\Delta]$ such that $\mathrm{Proj}(\Bbbk[\Delta]/\mathfrak{H}) = \mathrm{Quot}^{n,m}_{p(z)}$ where $\Bbbk[\Delta]$ is the ring of Plücker coordinates for $\mathrm{Gr}^{N_m(r)}_{p(r)}$. The construction of $\mathfrak{H}$ starts from the ideals defining the affine schemes representing the open subsets $\mathcal{Q}^{[r,r]}_{\mathcal{I},\mathrm{Id}}$ of (9.1). Since $\mathcal{Q}\mathrm{uot}^{n,m}_{p(z)} = \mathcal{Q}\mathrm{uot}^{n,m,[r]}_{p(z)}$, the cover of Definition 9.7 is in this case indexed by $\mathbb{QS}_{p(z)} = \mathbb{QS}^{r}_{p(z)}$. Furthermore, if $U \in \mathbb{QS}_{p(z)}$, then U_s is stable, since $s \geqslant r$.

For every $\mathcal{I} \subset \{1,\dots,N_m(s)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}} \rangle \in \mathbb{QS}_{p(z)}$, we denote by $\Bbbk[C_{\mathcal{I}}]/(\mathcal{R}_{\mathcal{I}})$ the quotient ring that defines the affine scheme representing $\mathcal{Q}^{[r,r]}_{\mathcal{I},\mathrm{Id}}$ (see Theorem 4.1 and Proposition 9.3). As shown in Theorem 4.1 and Proposition 9.3, $\mathrm{Spec}(\Bbbk[C_{\mathcal{I}}]/(\mathcal{R}_{\mathcal{I}}))$ is the open subset of $\mathrm{Quot}^{n,m}_{p(z)}$ corresponding to the locus where $\Delta_{\mathcal{I}}$ can be inverted. Hence, the ideal $(\mathcal{R}_{\mathcal{I}})$ is the dehomogenisation of the ideal in $\Bbbk[\Delta]$ that defines $\mathcal{Q}^{[r,r]}_{\mathcal{I},\mathrm{Id}}$ as a closed subscheme of $\mathrm{Gr}^{N_m(r)}_{p(r)} \setminus \{\Delta_{\mathcal{I}} \neq 0\}$.

We construct an ideal $\mathfrak{h}_{\mathcal{I}} \subset \Bbbk[\Delta]$ starting from $\mathcal{R}_{\mathcal{I}}$: Let $\mathfrak{G}_{\mathcal{I}} \subseteq \Bbbk[C_{\mathcal{I}}][\mathbf{x}]$ be the $\mathcal{U}^c_{\mathcal{I}}$-marked set as defined in (4.1). We consider the $p(r) \times N_m(r)$-matrix $\mathcal{M}$, with columns indexed by the terms in $\mathbb{T}^m_r$ and rows indexed by the set $\mathcal{U}_{\mathcal{I}}$. The columns of $\mathcal{M}$ contain the coefficients of the polynomials h_{ij} such that $x^{\alpha_i} e_j \xrightarrow{\mathfrak{G}^{(r)}_{\mathcal{I}}}_{*} h_{ij}$, for every $x^{\alpha_i} e_j \in \mathbb{T}^m_r$. Consider now the set of equations

$$\{\Delta_{\mathcal{J}} - \mathcal{M}_{\mathcal{J}} \mid \mathcal{J} \subset \{1,\dots,N_m(r)\}, |\mathcal{J}| = p(r)\} \tag{10.1}$$

where $\mathcal{M}_{\mathcal{J}}$ is the minor corresponding to the columns with indices $j \in \mathcal{J}$. Consider the set $\mathcal{R}_{\mathcal{I}} \subset \Bbbk[C_{\mathcal{I}}]$ and compute the complete Gröbner reduction of the set $\mathcal{R}_{\mathcal{I}}$ with respect to the set of polynomials in (10.1) using an elimination term order for the variables $C_{\mathcal{I}}$. We obtain a set of non-homogeneous polynomials in $\Bbbk[\Delta]$. We homogenise each polynomial in this set with $\Delta_{\mathcal{I}}$ and take these homogeneous polynomials as generators for an ideal in $\Bbbk[\Delta]$ that we denote by $\mathfrak{h}_{\mathcal{I}}$.

We define $\mathfrak{h} := \bigcup_{\mathcal{I} | \mathcal{U}^c_{\mathcal{I}} \in \mathbb{QS}_{p(z)}} \mathfrak{h}_{\mathcal{I}}$. Moreover, we consider for every $g \in \mathrm{PGL}$ the set of equations $\mathfrak{h}^g$ obtained by the action of g on the elements of $\mathfrak{h}$. Finally, we define the ideal

$$\mathfrak{H} := \mathfrak{P} \cup \left(\bigcup_{g \in \mathrm{PGL}} \mathfrak{h}^g \right),$$

where $\mathfrak{P}$ is the ideal $\Bbbk[\Delta]$ containing the Plücker relations, that is $\mathrm{Proj}(\Bbbk[\Delta]/\mathfrak{P}) = \mathrm{Gr}^{N_m(r)}_{p(r)}$.

Theorem 10.3. *Let $p(z)$ be an admissible Hilbert polynomial for submodules of $A[\mathbf{x}]^m$. The homogeneous ideal $\mathfrak{H}$ in the ring of Plücker coordinates $\Bbbk[\Delta]$ of the Plücker embedding $\mathrm{Gr}^{N_m(r)}_{p(r)} \hookrightarrow \mathbb{P}^{\binom{N_m(r)}{p(r)}}$ defines $\mathrm{Quot}^{n,m}_{p(z)}$ as a closed subscheme of $\mathrm{Gr}^{N_m(r)}_{p(r)}$.*

Proof. We follow the lines of the proof of [7, Theorem 6.5] on the Hilbert scheme. For convenience, we denote by $\mathcal{Z}$ the subscheme of $\mathrm{Gr}^{N_m(r)}_{p(r)}$ defined by $\mathfrak{H}$ and by $\mathfrak{D}$ the saturated ideal in $\Bbbk[\Delta]$ that defines $\mathrm{Quot}^{n,m}_{p(z)}$. We will show that $Z = \mathrm{Quot}^{n,m}_{p(z)}$, although in general $\mathfrak{H} \neq \mathfrak{D}$.

As equality of subschemes is a local property, we can check the equality locally. The proof is divided in two steps.

Step 1: For every $\mathcal{I} \subset \{1,\dots,N_m(r)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}_{p(z)}$, the ideal generated by $\mathfrak{h}_{\mathcal{I}}$ defines the affine scheme representing $\mathcal{Q}^{[r,r]}_{\mathcal{I},\mathrm{Id}}$ as closed subscheme of the scheme $\mathrm{G}^{[r]}_{\mathcal{I},\mathrm{Id}}$ of Corollary 8.8, representing $\mathcal{G}^{[r]}_{\mathcal{I},\mathrm{Id}}$.

Step 2: For every (closed) point F of $\mathrm{Gr}^{N_m(r)}_{p(r)}$, $\mathcal{Z}$ and $\mathrm{Quot}^{n,m}_{p(z)}$ coincide on a neighbourhood of $\langle F\rangle$.

Proof of Step 1. We have to prove that for every $\mathcal{I} \subset \{1,\dots,N_m(r)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}_{p(z)}$ and for every $\Bbbk$-algebra A and F belonging to $\mathcal{G}r^{N_m(r)}_{p(r)}(A)$, $\langle F\rangle \subset A[\mathbf{x}]^m$ belongs to $\mathcal{Q}^{[r,r]}_{\mathcal{I},\mathrm{Id}}$, if and only if the polynomials in $\mathfrak{H}$ vanish when evaluated at $\langle F\rangle$. Referring to Proposition 9.2, Theorem 4.1 and Proposition 9.3, it suffices to observe that the vanishing at $\langle F\rangle$ of the polynomials in $\mathfrak{H}$ is equivalent to the vanishing at $\langle F\rangle$ of the polynomials in $(\mathcal{R}_{\mathcal{I}})$.

Proof of Step 2. Both ideals $\mathfrak{H}$ and $\mathfrak{D}$ are invariant under the action of PGL: $\mathfrak{H}$ by construction and $\mathfrak{D}$ because $\mathrm{Quot}^{n,m}_{p(z)}$ is. Since $\Bbbk[\Delta]$ is a Noetherian ring, we can choose generators $h_1,\dots,h_d$ of the ideal $\mathfrak{H}$. More precisely, we denote by g_i the element in PGL such that $h_i \in \mathfrak{h}^{g_i}$. Hence, $\mathfrak{h}^{g_1}\cup\dots\cup\mathfrak{h}^{g_d} = \mathfrak{H}$. Since $\mathfrak{H}$ is nvariant under the action of PGL, we get for every $g \in$ PGL

$$\mathfrak{h}^{gg_1}\cup\dots\cup\mathfrak{h}^{gg_d} = (\mathfrak{h}^{g_1}\cup\dots\cup\mathfrak{h}^{g_d})^g = \mathfrak{H}^g = \mathfrak{H}.$$

Using the invariance of $\mathfrak{D}$ under the action of PGL and by Step 1, we see that if we restrict to the open subset $\mathrm{G}^{[r]}_{\mathcal{I},gg_1}\cap\dots\cap\mathrm{G}^{[r]}_{\mathcal{I},gg_1}$, then the ideals $\mathfrak{H}$ and $\mathfrak{D}$ define the same scheme, hence

$$\mathrm{Quot}^{n,m}_{p(z)} \cap \left(\mathrm{G}^{[r]}_{\mathcal{I},gg_1}\cap\dots\cap\mathrm{G}^{[r]}_{\mathcal{I},gg_1}\right) = \mathcal{Z}\cap\left(\mathrm{G}^{[r]}_{\mathcal{I},gg_1}\cap\dots\cap\mathrm{G}^{[r]}_{\mathcal{I},gg_1}\right).$$

It only remains to prove that for every $F \in \mathrm{Gr}^{N_m(r)}_{p(r)}$, there is $\mathcal{I} \subset \{1,\dots,N_m(r)\}$ such that $\langle \mathcal{U}^c_{\mathcal{I}}\rangle \in \mathbb{QS}$ and there is $g \in$ PGL such that $\langle F\rangle \in \mathrm{G}^{[r]}_{\mathcal{I},gg_1}\cap\dots\cap\mathrm{G}^{[r]}_{\mathcal{I},gg_1}$.

By Proposition 8.5, there is $\overline{\mathcal{I}} \subset \{1,\dots,N_m(r)\}$ such that $\langle \mathcal{U}^c_{\overline{\mathcal{I}}}\rangle \in \mathbb{QS}$ and there is $\overline{g} \in$ PGL such that $\langle F\rangle \in \mathrm{G}^{[r]}_{\overline{\mathcal{I}},\overline{g}}$. Since $\mathrm{G}^{[r]}_{\overline{\mathcal{I}},\overline{g}}$ is an open subset of $\mathrm{Gr}^{N_m(r)}_{p(r)}$, an open subset of the orbit of $\langle F\rangle$ under the action of PGL is contained in $\mathrm{G}^{[r]}_{\overline{\mathcal{I}},\overline{g}}$: there is an open subset $\mathcal{G}$ of PGL such that for every $g' \in \mathcal{G}$, $g'^{-1} \centerdot F \in \mathrm{G}^{[r]}_{\overline{\mathcal{I}},\overline{g}}$, in other words $F \in \mathrm{G}^{[r]}_{\overline{\mathcal{I}},g'\overline{g}}$. Hence, for a general $g \in$ PGL, $gg_1\overline{g},\dots,gg_d\overline{g} \in \mathcal{G}$ and $F \in \mathrm{G}^{[r]}_{\mathcal{I},gg_1}\cap\dots\cap\mathrm{G}^{[r]}_{\mathcal{I},gg_1}$ as desired. □

Remark 10.4. We can rephrase the construction of the ideal $\mathfrak{H}$ and the statement of Theorem 10.3 in order to obtain the ideal defining the scheme representing the functor $\mathcal{Q}\mathrm{uot}^{n,m,[\ell]}_{p(z)}$ as a closed subscheme of $\mathrm{Gr}^{N_m(s)}_{p(s)} \setminus L^{[\ell,s]}_{p(z)}$: it suffices to use in the construction the set of quasi-stable modules in $\mathbb{QS}^{\ell}_{p(z)}$.

11. An example: Computations on Quot^2_2 on $\mathbb{P}^1$

We consider $p(z) = 2$, $n = 1$ and $m = 2$. This is the very first example one can think about in order to consider a non-trivial Quot scheme which is not a Hilbert scheme. Nevertheless, even if this is the simplest case on which we can test our methods, to our knowledge nothing is known about this Quot scheme. In this section, we describe the construction of the ideals $\mathcal{R}_{\mathcal{I}}$

defining the open cover of of Definition 9.7 and the global equations defining Quot_2^2. A detailed description on the geometry of Quot_2^2 can be found in [6].

We consider the scheme Quot_2^2 parameterising the saturated submodules of $A[x_0,x_1]^2$ with constant Hilbert polynomial 2. The Gotzmann number is $r=2$, hence we will study this Quot scheme under the embedding in Gr_2^6. We will also study $\mathrm{Quot}_2^{2,[1]}$, which is the subscheme of $\mathrm{Gr}_2^6 \setminus L_2^{1,2}$ whose functor of points is $\mathcal{Q}\mathrm{uot}_2^{2,[1]}$. Consider

$$U_1 = (x_1^2)e_1 \oplus (1)e_2, \quad U_2 = (x_1)e_1 \oplus (x_1)e_2, \quad U_3 = (1)e_1 \oplus (x_1^2)e_2.$$

We have $\mathbb{QS}_2 = \{U_1, U_2, U_3\}$ and $\mathbb{QS}_2^1 = \{U_2\}$.

11.1. **Global equations for Quot_2^2 in $\mathbb{P}^{14}$.** We keep the notation U_i for the embedding of the quasi-stable module U_i in Gr_2^6 for $i=1,2,3$. By the procedure of Section 4, and in particular by Theorem 4.1, we explicitely construct the affine scheme representing $\mathbf{Mf}_{\mathcal{P}(U_i)}$, $i\in\{1,2,3\}$.

We construct the marked scheme on $\mathcal{P}(U_1)$ starting from the following marked set:

$$f_1 = (x^2 - C_{1,1}xy - C_{1,2}y^2)e_1, \quad f_2 = -(C_{2,1}xy + C_{2,2}y^2)e_1 + x^2e_2,$$
$$f_3 = -(C_{3,1}xy + C_{3,2}y^2)e_1 + xye_2, \quad f_4 = (-C_{4,1}xy - C_{4,2}y^2)e_1 + y^2e_2.$$

We compute the ideal $\mathcal{R}_1$ obtaining

$$\mathcal{R}_1 = (-C_{1,2}C_{3,1} + C_{2,2}, C_{1,1}C_{4,1} - C_{3,1} + C_{4,2}, -C_{1,1}C_{3,1} - C_{1,2}C_{4,1} + C_{2,1}, -C_{1,2}C_{4,1} + C_{3,2})$$

In the same way, we can construct the ideals $\mathcal{R}_2$ and $\mathcal{R}_3$ which define the schemes representing $\mathbf{Mf}_{\mathcal{P}(U_2)}$ and $\mathbf{Mf}_{\mathcal{P}(U_3)}$. Each of these three ideals has four generators and each of them allows the elimination of a variable (in the sense of Gröbner theory). Hence for every $i\in\{1,2,3\}$, $\mathbf{Mf}_{\mathcal{P}(U_i)} \simeq \mathbb{A}^4$.

Following the construction outlined in Section 10, we can compute the ideal $\mathfrak{h}$ in the polynomial ring $\Bbbk[\Delta]$ where Δ is the set of Plücker coordinates of Gr_2^6, $|\Delta| = 15$. We then repeatedly apply some random elements $g_i \in \mathrm{PGL}$ on the ideal $\mathfrak{h}$ until for some t

$$\mathfrak{h}^{g_1} \cup \cdots \cup \mathfrak{h}^{g_t} \cup \mathfrak{h}^{g_{t+1}} = \mathfrak{h}^{g_1} \cup \cdots \cup \mathfrak{h}^{g_t}.$$

By noetherianity, such a t exists and for this specific example $t=4$.

Adding the Plücker relations, we obtain the ideal defining Quot_2^2 as a closed subscheme of $\mathbb{P}^{14}$. We can exhibit a set of generators consisting of 61 polynomials of degree 2, 3 and 4. This Quot scheme has the Hilbert polynomial

$$\frac{11}{12}z^4 + \frac{11}{3}z^3 + \frac{67}{12}z^2 + \frac{23}{6}z + 1. \tag{11.1}$$

Hence it is a fourfold in $\mathbb{P}^{14}$ of degree 22.

11.2. **Global equations for $\mathrm{Quot}_2^{2,[1]}$ in $\mathbb{P}^{14}$.** By Theorem 9.5 (i), $\mathrm{Quot}_2^{2,[1]}$ embeds in Gr_2^4, which embeds in $\mathbb{P}^5$. In this case, $\mathrm{Quot}_2^{2,[1]}$ is simply the open subset $\mathbb{A}^4 \simeq \mathrm{Gr}_2^4 \setminus L_2^{1,1}$. Indeed, in this case $L_2^{1,1}$ is defined by the ideal $(\Delta_{3,4})$.

We can also compute the equations defining $\mathrm{Quot}_2^{2,[1]}$ as an open subscheme of $\mathrm{Gr}_2^6 \setminus L_2^{1,2}$. It is sufficient to consider only one marked scheme, the one defined by the ideal $\mathcal{R}_2$ and use the procedure described in Section 10. After homogenising the generators of $\mathcal{R}_2$ in $\Bbbk[\Delta]$, we obtain the ideal $\mathfrak{h}'$. We apply four times random elements $g_i \in \mathrm{PGL}$ on $\mathfrak{h}'$ obtaining the ideal $\mathfrak{H}'$ defining $\mathrm{Quot}_2^{2,[1]}$ as a subscheme of $\mathrm{Gr}_2^6 \setminus L_2^{1,2}$ in $\mathbb{P}^{14}$.

In this case, $L_2^{1,2} = (\Delta_{3,6}, \Delta_{1,4})$ and the closed scheme defined by $\mathfrak{H}'$, which contains the scheme $\mathrm{Quot}_2^{2,[1]}$, has the Hilbert polynomial

$$\frac{11}{12}z^4 + 5z^3 + \frac{67}{12}z^2 + \frac{7}{2}z + 1. \tag{11.2}$$

The construction of $\mathfrak{H}'$ is faster than that of $\mathfrak{H}$ due to the fact that we have only one open subset in the open cover of $\mathrm{Quot}_2^{2,[1]}$ up to the action of PGL. Nevertheless, the ideal $\mathfrak{H}'$ is, by construction, contained in the ideal $\mathfrak{H}$ that defines Quot_2^2 and the Hilbert polynomial (11.2) of $\mathrm{Proj}(\Bbbk[\Delta]/\mathfrak{H}')$ is smaller than the one computed for Quot_2^2 in (11.1). Hence $\mathrm{Proj}(\Bbbk[\Delta]/\mathfrak{H}') \supset$

Quot_2^2. Indeed, the ideal $\mathfrak{H}'$ defines a closed scheme that strictly contains $\mathrm{Quot}_2^{2,[1]}$. We have that $\mathrm{Proj}(\Bbbk[\Delta]/\mathfrak{H}') \setminus \mathrm{Quot}_2^{2,[1]} \subset L_2^{1,2}$.

I would say that (11.2) is *larger* than (11.1)?

12. Conclusions

In this paper, we defined and investigated properties of marked bases over a quasi-stable monomial module $U \subseteq A[\mathbf{x}]_{\mathbf{d}}^m$. The family of all modules generated by a marked basis over $\mathcal{P}(U)$ possesses a natural structure as an affine scheme (Theorem 4.1). In particular, we proved that the quasi-stable module U provides upper bounds on some homological invariants of any module generated by a $\mathcal{P}(U)$-marked basis such as Betti numbers, regularity or projective dimension (Corollary 5.8).

We exploited these properties and constructions to obtain local and global equations of Quot schemes and of special loci of them, namely those given by an upper bound on the Castelnuovo-Mumford regularity of a module. Indeed, we proved that we have an open cover of a Quot scheme (resp. of its locus defined by an upper bound on the regularity) whose open subsets are suitable marked schemes over a quasi-stable module (Theorem 9.5). Starting from this open cover, we obtained global equations defining a Quot scheme (resp. its locus defined by an upper bound on the regularity) as a closed (resp. locally closed) subscheme of a suitable projective space (Theorem 10.3).

In the future, inspired by Corollary 5.8, we intend to investigate other loci of a Quot scheme, given by bounds on other numerical invariants of a module, such as projective dimension or extremal Betti numbers. In order to obtain similar results to those for the locus with bounded regularity, we will need also other tools, since a preliminary study showed, for instance, that the locus given by a bound on projective dimension is in general not an open subset of a Quot scheme.

We will also investigate some explicit examples of Quot schemes, as we are doing in [6], in order to have a better comprehension of the geometry of a Quot scheme using explicit equations defining it, locally or globally.

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References

[1] Edoardo Ballico, Cristina Bertone, and Margherita Roggero. The locus of points of the Hilbert scheme with bounded regularity. *Comm. Algebra*, 43(7):2912–2931, 2015.

[2] Isabel Bermejo and Philippe Gimenez. Saturation and Castelnuovo-Mumford regularity. *J. Algebra*, 303(2):592–617, 2006.

[3] Cristina Bertone. Quasi-stable ideals and Borel-fixed ideals with a given Hilbert polynomial. *Appl. Algebra Engrg. Comm. Comput.*, 26:507–525, 2015.

[4] Cristina Bertone, Francesca Cioffi, Paolo Lella, and Margherita Roggero. Upgraded methods for the effective computation of marked schemes on a strongly stable ideal. *J. Symbolic Comput.*, 50:263–290, 2013.

[5] Cristina Bertone, Paolo Lella, and Margherita Roggero. A Borel open cover of the Hilbert scheme. *J. Symbolic Comput.*, 53:119 – 135, 2013.

[6] Cristina Bertone, Margherita Roggero, and Roy Skjelnes. An explicit description of Quot_2^2. in preparation, 2018.

[7] Jerome Brachat, Paolo Lella, Bernard Mourrain, and Margherita Roggero. Extensors and the Hilbert scheme. *Ann. Sc. Norm.*, XVI:65–96, 2016.

[8] Giulio Caviglia and Enrico Sbarra. Characteristic-free bounds for the Castelnuovo-Mumford regularity. *Compos. Math.*, 141(6):1365–1373, 2005.

[9] Michela Ceria, Teo Mora, and Margherita Roggero. Term-ordering free involutive bases. *J. Symbolic Comput.*, 68(part 2):87–108, 2015.
[10] Francesca Cioffi and Margherita Roggero. Flat families by strongly stable ideals and a generalization of Gröbner bases. *J. Symbolic Comput.*, 46(9):1070–1084, 2011.
[11] Roger Dellaca. Gotzmann regularity for globally generated coherent sheaves. *J. Pure Appl. Algebra*, 220(4):1576–1587, 2016.
[12] David Eisenbud. *Commutative Algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
[13] David Eisenbud. *The Geometry of Syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.
[14] David Eisenbud and Joe Harris. *The Geometry of Schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
[15] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010.
[16] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.
[17] A. Hashemi, M. Schweinfurter, and W.M. Seiler. Deterministic genericity for polynomial ideals. *J. Symb. Comput.*, 86:20–50, 2018.
[18] Jürgen Herzog, Dorin Popescu, and Marius Vladoiu. On the Ext-modules of ideals of Borel type. In L.L. Avramov, M. Chardin, M. Morales, and C. Polini, editors, *Commutative Algebra: Interactions with Algebraic Geometry*, Contemp. Math. 331, pages 171–186. Amer. Math. Soc., Providence, 2003.
[19] Heather A. Hulett. A generalization of Macaulay's theorem. *Communications in Algebra*, 23(4):1249–1263, 1995.
[20] Steven L. Kleiman. The Picard scheme. In *Fundamental Algebraic Geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.
[21] Martin Kreuzer and Lorenzo Robbiano. *Computational Commutative Algebra 1*. Springer-Verlag, Berlin, 2000.
[22] Paolo Lella and Margherita Roggero. On the functoriality of marked families. *J. Commut. Algebra*, 8(3):367–410, 2016.
[23] Daniel Mall. On the relation between Gröbner and Pommaret bases. *Appl. Algebra Engrg. Comm. Comput.*, 9(2):117–123, 1998.
[24] Nitin Nitsure. Construction of Hilbert and Quot schemes. In *Fundamental Algebraic Geometry*, volume 123 of *Math. Surveys Monogr.*, pages 105–137. Amer. Math. Soc., Providence, RI, 2005.
[25] Alyson Reeves and Bernd Sturmfels. A note on polynomial reduction. *J. Symbolic Comput.*, 16(3):273–277, 1993.
[26] Michael Schweinfurter. *Deterministic Genericity and the Computation of Homological Invariants*. PhD thesis, Universität Kassel, 2016.
[27] Werner M. Seiler. A combinatorial approach to involution and δ-regularity. I. Involutive bases in polynomial algebras of solvable type. *Appl. Algebra Engrg. Comm. Comput.*, 20(3-4):207–259, 2009.
[28] Werner M. Seiler. A combinatorial approach to involution and δ-regularity. II. Structure analysis of polynomial modules with Pommaret bases. *Appl. Algebra Engrg. Comm. Comput.*, 20(3-4):261–338, 2009.
[29] Edoardo Sernesi. *Deformations of Algebraic Schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

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