EFFECTIVE GENERICITY, δ-REGULARITY AND STRONG NOETHER POSITION

WERNER M. SEILER

ABSTRACT. We show that the concept of strong Noether position for a polynomial ideal $\mathcal{I} \lhd \mathcal{P}$ is equivalent to δ-regularity and thus related to Pommaret bases. In particular, we provide explicit Pommaret bases for two of the ideal sequences used in Hashemi's definition of strong Noether position and alternative proofs for a number of his statements. Finally, we show that one consequence of δ-regularity is that any Pommaret basis contains a system of parameters and we present an algorithm for checking whether the factor ring $\mathcal{P}/\mathcal{I}$ is Gorenstein via a socle computation.

1. INTRODUCTION

Bermejo and Gimenez [2] introduced the concept of a strong Noether normalisation in their work on the Castelnuovo-Mumford regularity and related it to a particular class of monomial ideals which they termed ideals of nested type. This concept may be considered as an effective notion of genericity, as almost any linear change of variables puts an ideal into strong Noether position and Bermejo and Gimenez presented algorithmically verifiable criteria for a monomial ideal to be of nested type. Recently, Hashemi [11] introduced an alternative approach to strong Noether position for arbitrary polynomial ideals based on certain ideal sequences which is closely related to the results of Bermejo and Gimenez when restricted to monomial ideals.

In the theory of Pommaret bases [22], a special class of Gröbner bases with additional combinatorial properties, another effective notion of genericity appears, δ-regularity, and one introduces the class of quasi-stable monomial ideals related to it. It was shown in [22] that every quasi-stable ideal is of nested type and vice versa.[1] Thus, in principle, it is already clear that strong Noether position and δ-regularity are equivalent notions. The main purpose of this article is to make this connection more explicit by showing how many results of Hashemi [11] (and of Bayer and Stillman [1] on whom Hashemi's work founds) follow naturally and effectively from the theory of Pommaret bases.

In addition, we present two new applications of δ-regular variables. We first show that each homogeneous Pommaret basis contains a maximal homogeneous system of parameters. This result is a simple consequence of the fact that any Pommaret basis induces a Noether normalisation. Then we consider the question whether the factor ring is Gorenstein. It was already demonstrated in [22] that one can immediately decide with a Pommaret basis whether it is Cohen-Macaulay.

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[1] In the literature, further names for the same class of monomial ideals appear like ideals of Borel type [14] or weakly stable ideals [3].

Now we show that if this is the case one can also easily determine the dimension of the socle and thus the type of a Cohen-Macaulay ring.

Throughout this article, $\mathcal{P} = \Bbbk[X]$ denotes a polynomial ring in the variables $X = \{x_1, \dots, x_n\}$ over an infinite field $\Bbbk$ of arbitrary characteristic and $0 \neq \mathcal{I} \lhd \mathcal{P}$ a proper homogeneous ideal. When considering bases of $\mathcal{I}$, we will always assume that these are homogeneous, too. We will denote by $\mathcal{A}$ the factor ring $\mathcal{P}/\mathcal{I}$ and, if some term order has been chosen, by $\mathcal{A}'$ the monomial factor ring $\mathcal{P}/\operatorname{lt}\mathcal{I}$.

In order to be consistent with our previous works [21, 22] on Pommaret bases, we will use a non-standard convention for the degree reverse lexicographic order: given two arbitrary terms x^μ, x^ν of the same degree, we define $x^\mu \prec_{\text{degrevlex}} x^\nu$ if the first non-vanishing entry of $\mu - \nu$ is positive. Compared with the usual convention, this corresponds to a reverting of the numbering of the variables in X. Thus for a comparison of our definitions and results with the ones appearing in the literature, one must at many places perform such a reversion.

In the next section we briefly recall the basic properties of Pommaret bases and the definition of δ-regularity. Section 3 provides the relationship to Hashemi's concept of strong Noether position, whereas Section 4 is concerned with various notions of (stabilised) regularity and their interconnection. The following section discusses maximal systems of parameters and an algorithmic test whether the factor ring $\mathcal{A}$ is Gorenstein. Finally, some conclusions are given.

2. Pommaret Bases and δ-Regularity

Pommaret bases form a special class of Gröbner bases with additional combinatorial properties making them particularly useful for applications in commutative algebra and algebraic geometry. They represent a special case of the *involutive bases* introduced by Gerdt and collaborators [8, 9] (see [21] for a general survey on involutive bases and their computation). The algebraic theory of Pommaret bases was developed in [22] (see also [23, Chpts. 3-5]).

Given an exponent vector $\mu = [\mu_1, \dots, \mu_n] \neq 0$ (or the term x^μ or a polynomial $f \in \mathcal{P}$ with $\operatorname{lt} f = x^\mu$ for some fixed term order), we call $\min\{i \mid \mu_i \neq 0\}$ the *class* of μ (or x^μ or f), denoted by $\operatorname{cls}\mu$ (or $\operatorname{cls} x^\mu$ or $\operatorname{cls} f$). Then the *multiplicative variables* of x^μ or f are $X_P(x^\mu) = X_P(f) = \{x_1, \dots, x_{\operatorname{cls}\mu}\}$. We say that x^μ is an *involutive divisor* of another term x^ν, if $x^\mu \mid x^\nu$ and $x^{\nu-\mu} \in \Bbbk[x_1, \dots, x_{\operatorname{cls}\mu}]$.

Consider now a polynomial ideal $\mathcal{I} \lhd \mathcal{P}$ generated by some finite set $\mathcal{H} \subset \mathcal{P}$. The fundamental idea underlying Pommaret bases is the following "constrained generation property": if $\mathcal{H}$ is a Pommaret basis of $\mathcal{I}$, then one can obtain any element of $\mathcal{I}$ by considering only those linear combinations where each generator $h \in \mathcal{H}$ is multiplied by a polynomial in the subring $\Bbbk[X_P(h)]$.

Definition 2.1. Assume first that the set $\mathcal{H}$ consists only of terms so that $\mathcal{I}$ is a monomial ideal. We call $\mathcal{H}$ a *weak Pommaret basis* of $\mathcal{I}$, if

$$\langle \mathcal{H} \rangle_P = \sum_{h \in \mathcal{H}} \Bbbk[X_P(h)] \cdot h = \mathcal{I} \,. \tag{2.1}$$

It is a *(strong) Pommaret basis* of $\mathcal{I}$, if in addition the sum in (2.1) is direct (in this case each term $x^\nu \in \mathcal{I}$ has a unique involutive divisor $x^\mu \in \mathcal{H}$). A polynomial set $\mathcal{H}$ is a *weak Pommaret basis* of the polynomial ideal $\mathcal{I}$ for the term order $\prec$, if the leading terms $\operatorname{lt}\mathcal{H}$ form a weak Pommaret basis of the leading ideal $\operatorname{lt}\mathcal{I}$. It is a *(strong) Pommaret basis*, if all elements of $\mathcal{H}$ possess distinct leading terms and $\operatorname{lt}\mathcal{H}$ is a strong Pommaret basis of $\operatorname{lt}\mathcal{I}$.

Remark 2.2. Note that our definition of multiplicative variables implies a special relationship between the degree reverse lexicographic order $\prec_{\text{degrevlex}}$ and Pommaret bases. Given an arbitrary finite set $\mathcal{F} \subset \mathcal{P}$ of homogeneous polynomials, we always obtain the largest number of multiplicative variables, if we use the degree reverse lexicographic order. The reason lies in the well-known characterisation of this order (see e. g. [23, Lemma A.1.8] for an explicit proof) as the only degree compatible order such that for any homogeneous polynomial f and any index $1 \leq k \leq n$ we have the equivalence

$$f \in \langle x_1, \dots, x_k \rangle \Longleftrightarrow \operatorname{lt} f \in \langle x_1, \dots, x_k \rangle \ . \tag{2.2}$$

The use of the degree reverse lexicographic order and this characterisation of it is crucial for several of our results.

Pommaret bases are non-trivial already in the monomial case. Obviously, even a weak Pommaret basis is automatically a Gröbner basis for the given term order. One can show that any weak Pommaret basis can be transformed into a strong one by a simple elimination process. A key property of (strong) Pommaret bases is the characterisation via *involutive standard representations*; note that in contrast to ordinary Gröbner bases we obtain here unique representations.

Proposition 2.3 ([21, Thm. 5.4])**.** *The finite set $\mathcal{H} \subset \mathcal{I}$ is a strong Pommaret basis of the ideal $\mathcal{I} \lhd \mathcal{P}$ for the term order $\prec$, if and only if every polynomial $0 \neq f \in \mathcal{I}$ possesses a unique involutive standard representation $f = \sum_{h \in \mathcal{H}} P_h h$ where each coefficient $P_h \in \Bbbk[X_P(h)]$ satisfies $\operatorname{lt}(P_h h) \preceq \operatorname{lt}(f)$ if it does not vanish.*

The following result is an analogue to Buchberger's criterion for Gröbner bases. It provides us with a simple test whether a given set is a Pommaret basis and also builds the foundation of an algorithm for the construction of Pommaret bases.

Proposition 2.4 ([21, Cor. 7.3])**.** *Let $\mathcal{H}$ be a finite set of polynomials and $\prec$ a term order such that no leading term in $\operatorname{lt}\mathcal{H}$ is an involutive divisor of another one. The set $\mathcal{H}$ is a Pommaret basis of the ideal $\langle \mathcal{H} \rangle$ with respect to $\prec$, if and only if for every $h \in \mathcal{H}$ and every non-multiplicative index $\operatorname{cls} h < j \leq n$ the product $x_j h$ possesses an involutive standard representation with respect to $\mathcal{H}$.*

Not every ideal $\mathcal{I} \lhd \mathcal{P}$ possesses a finite Pommaret basis. A simple counter example is the monomial ideal $\mathcal{I} = \langle x_1 x_2 \rangle \lhd \Bbbk[x, y]$: all terms contained in $\mathcal{I}$ are of class 1 and hence only the infinite set $\{x_1 x_2^k \mid k \geq 1\}$ involutively generates $\mathcal{I}$. One can show [22, Sect. 2] that this is solely a problem of the used variables X; after a suitable linear change of variables $\tilde{X} = AX$ with a non-singular matrix $A \in \Bbbk^{n \times n}$ the transformed ideal $\tilde{\mathcal{I}} \lhd \tilde{\mathcal{P}} = \Bbbk[\tilde{X}]$ has a finite Pommaret basis (for the same term order). One speaks of the problem of *δ-regularity* of the used variables.[2]

Definition 2.5. The variables X are *δ-regular* for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the term order $\prec$, if $\mathcal{I}$ possesses a finite Pommaret basis for $\prec$.

[2] This is the only point where we need the assumption that $\Bbbk$ is an infinite field. In the case of a small finite field, the construction of δ-regular variables might require a field extension.

In [22] a method (originating in [7, 13]) is presented to detect effectively whether given variables are δ-singular and, if this is the case, to produce deterministically δ-regular variables. Furthermore, it is proven there that generic variables are δ-regular so that one can also employ probabilistic approaches although these are usually computationally disadvantageous.

It seems to be rather unknown that Serre implicitly presented already in 1964 a version of δ-regularity. In a letter appended to [10], he introduced the notion of a *quasi-regular* sequence and related it to Koszul homology.[3] Let $\mathcal{V}$ be a finite-dimensional vector space and $\mathcal{M}$ a finitely generated graded $S\mathcal{V}$-module. A vector $v \in \mathcal{V}$ is called quasi-regular at degree q for $\mathcal{M}$, if $vm = 0$ for an $m \in \mathcal{M}$ implies $m \in \mathcal{M}_{<q}$. A sequence $(v_1, \dots, v_k)$ of vectors $v_i \in \mathcal{V}$ is quasi-regular at degree q for $\mathcal{M}$, if each v_i is quasi-regular at degree q for $\mathcal{M}/\langle v_1, \dots, v_{i-1}\rangle\mathcal{M}$.

Given a basis X of $\mathcal{V}$, we can identify $S\mathcal{V}$ with the polynomial ring $\mathcal{P} = \Bbbk[X]$. Then it is shown in [12, Thm. 5.4] (see also [20, Thm. 5.2], [23, Thm. 6.3.2]) that the variables X are δ-regular for a homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$ and the degree reverse lexicographic order, if and only if they form a quasi-regular sequence for the factor ring $\mathcal{A}$ at degree $\operatorname{reg}\mathcal{I}$ (and not at any lower degree).

For monomial ideals it is in general useless to transform to δ-regular variables, as the transformed ideal is no longer monomial. Hence it is a special property of a monomial ideal to possess a finite Pommaret basis: such an ideal is called *quasi-stable*. The following theorem provides several purely algebraic characterisations of quasi-stability independent of Pommaret bases. It combines ideas and results from [1, Def. 1.5], [2, Prop. 3.2/3.6], [14, Prop. 2.2] and [22, Prop. 4.4].

Theorem 2.6. *Let $\mathcal{I} \lhd \mathcal{P}$ be a monomial ideal and $D = \dim \mathcal{A}$. Then the following statements are equivalent.*

- **(i)** *$\mathcal{I}$ is quasi-stable.*
- **(ii)** *The variable x_1 is not a zero divisor for $\mathcal{P}/\mathcal{I}^{\mathrm{sat}}$ and for all $1 \leq k < D$ the variable x_{k+1} is not a zero divisor for $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_k\rangle^{\mathrm{sat}}$.*
- **(iii)** *We have $\mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_D^\infty$ and for all $D < k \leq n$ an exponent $e_k \geq 1$ exists such that $x_k^{e_k} \in \mathcal{I}$.*
- **(iv)** *For all $1 \leq k \leq n$ the equality $\mathcal{I} : x_k^\infty = \mathcal{I} : \langle x_k, \dots, x_n\rangle^\infty$ holds.*
- **(v)** *For every associated prime ideal $\mathfrak{p} \in \operatorname{Ass}(\mathcal{A})$ an integer $1 \leq j \leq n$ exists such that $\mathfrak{p} = \langle x_j, \dots, x_n\rangle$.*
- **(vi)** *If $x^\mu \in \mathcal{I}$ and $\mu_i > 0$ for some $1 \leq i < n$, then for each $0 < r \leq \mu_i$ and $i < j \leq n$ an integer $s \geq 0$ exists such that $x_j^s x^\mu / x_i^r \in \mathcal{I}$.*
- **(vii)** *$\mathcal{I}$ and all its primary components are in Noether position.*

It should be noted that the properties (iii) and (iv) can be effectively verified in a straightforward manner, so that simple algorithmic tests for quasi-stability exist. Property (vi) explains the terminology "quasi-stable", as it represents a generalisation of the classical notion of a stable ideal (a monomial ideal is stable, if and only if already its minimal basis is a Pommaret basis [15, Lemma 2.13]).

δ-Regularity of the used variables implies for arbitrary polynomial ideals $\mathcal{I} \lhd \mathcal{P}$ properties which typically hold only generically. In particular, if the variables are δ-regular, then the ideal is in Noether position. Note furthermore that the following

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[3] Later, quasi-regularity was rediscovered by Schenzel et al. [17] under the name *filter-regularity*. It is amusing to note that in the same letter Serre already described the Eisenbud-Goto criterion for q-regularity (found in 1984 [5]) as a "curiosité".

results entail that in δ-regular variables $\mathcal{A}$ is Cohen-Macaulay, if and only if $\mathcal{A}'$ is Cohen-Macaulay, too.

Theorem 2.7 ([22, Cor. 3.18, Prop. 3.19, Prop. 4.1])**.** *Let the variables X be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the term order $\prec$. Let $\mathcal{H}$ be a Pommaret basis of $\mathcal{I}$ for this order.*

- **(i)** *If $D = \dim \mathcal{A}$, then $\{x_1, \dots, x_D\}$ is the unique maximal strongly independent set modulo $\mathcal{I}$.*
- **(ii)** *The restriction of the canonical map $\mathcal{P} \to \mathcal{A}$ to the subring $\Bbbk[x_1, \dots, x_D]$ defines a Noether normalisation.*
- **(iii)** *If $d = \min_{h \in \mathcal{H}} \operatorname{cls} h$ and $\prec$ is the degree reverse lexicographic order, then $x_1, \dots, x_{d-1}$ is a maximal $\mathcal{A}$-regular sequence and thus* $\operatorname{depth} \mathcal{A} = d - 1$.

We denote by $\operatorname{hilb} \mathcal{I}$ the *Hilbert regularity* of $\mathcal{I}$, i. e. the smallest degree from which on the Hilbert function and the Hilbert polynomial of $\mathcal{I}$ coincide, and by $\operatorname{reg} \mathcal{I}$ the *Castelnuovo-Mumford regularity* of $\mathcal{I}$. As usually, the *saturation* of $\mathcal{I}$ is the ideal $\mathcal{I}^{\mathrm{sat}} = \mathcal{I} : \mathfrak{m}^\infty$ with $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ and the *satiety* $\operatorname{sat} \mathcal{I}$ is the lowest degree from which on the ideals $\mathcal{I}$ and $\mathcal{I}^{\mathrm{sat}}$ coincide. Note that all these quantities are invariant under linear transformations and hence in the sequel we may assume without loss of generality that we are in δ-regular variables for the chosen term order. Then most of them can be easily read off from a Pommaret basis (for any finite set $\mathcal{F} \subset \mathcal{P}$, we denote by $\deg \mathcal{F}$ the maximal degree of an element of the set).

Theorem 2.8 ([22, Thm. 9.2, Prop. 10.1, Cor. 10.2])**.** *Let the variables X be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and $\prec_{\mathrm{degrevlex}}$ and let $\mathcal{H}$ be the corresponding Pommaret basis. We denote by $\mathcal{H}_1 = \{h \in \mathcal{H} \mid \operatorname{cls} h = 1\}$ the subset of generators of class* 1*.*

- **(i)** $\operatorname{reg} \mathcal{I} = \operatorname{reg} \operatorname{lt} \mathcal{I} = \deg \mathcal{H}$.
- **(ii)** *Let $\tilde{\mathcal{H}}_1 = \{h / x_1^{\deg_{x_1} \operatorname{lt} h} \mid h \in \mathcal{H}_1\}$. Then the set $\bar{\mathcal{H}} = (\mathcal{H} \setminus \mathcal{H}_1) \cup \tilde{\mathcal{H}}_1$ is a weak Pommaret basis of $\mathcal{I}^{\mathrm{sat}}$. Thus $\mathcal{I}^{\mathrm{sat}} = \mathcal{I} : x_1^\infty$ and the ideal $\mathcal{I}$ is saturated, if and only if $\mathcal{H}_1 = \emptyset$.*
- **(iii)** $\operatorname{sat} \mathcal{I} = \deg \mathcal{H}_1$.

The first equality in Part (i) and the last sentence in Part (ii) are statements which were already independently proven by Bayer and Stillman [1, Thm. 2.4(b)] and Bermejo and Gimenez [2, Cor. 2.4], respectively. We obtain them here as simple by-products of the properties of a Pommaret basis for $\prec_{\mathrm{degrevlex}}$.

3. Strong Noether Position

Following Hashemi [11], we introduce three sequences of ideals associated with a given polynomial ideal $\mathcal{I} \lhd \mathcal{P}$. We first set $\mathcal{I}^{(0)} = \bar{\mathcal{I}}^{(0)} = \hat{\mathcal{I}}^{(0)} = \mathcal{I}$ and then define for $k = 1, \dots, n$ the ideals

$$\mathcal{I}^{(k)} = \mathcal{I} + \langle x_1, \dots, x_k \rangle \trianglelefteq \mathcal{P}\,, \tag{3.1a}$$

$$\bar{\mathcal{I}}^{(k)} = \mathcal{I}|_{x_1 = \cdots = x_k = 0} \cap \mathcal{P}_k \trianglelefteq \mathcal{P}_k\,, \tag{3.1b}$$

$$\hat{\mathcal{I}}^{(k)} = \left(\hat{\mathcal{I}}^{(k-1)} : x_k^\infty\right)_{x_k = 0} \cap \mathcal{P}_k \trianglelefteq \mathcal{P}_k\,, \tag{3.1c}$$

where $\mathcal{P}_k = \Bbbk[x_{k+1}, \dots, x_n]$.

Lemma 3.1. *Let the variables X be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and $\prec_{\mathrm{degrevlex}}$ and let $\mathcal{H}$ be the corresponding Pommaret basis of $\mathcal{I}$. Then the variables $\{x_{k+1}, \dots, x_n\}$*

are δ-regular for the ideals $\bar{\mathcal{I}}^{(k)}, \hat{\mathcal{I}}^{(k)} \trianglelefteq \mathcal{P}_k$ and $\prec_{\text{degrevlex}}$. A strong Pommaret basis of $\bar{\mathcal{I}}^{(k)}$ is given by the set

$$\bar{\mathcal{H}} = \bigl\{ h|_{x_1=\cdots=x_k=0} \mid h \in \mathcal{H} \wedge \operatorname{cls} h > k \bigr\} \tag{3.2}$$

and a weak Pommaret basis of $\hat{\mathcal{I}}^{(k)} \subseteq \mathcal{P}_k$ by the set

$$\begin{aligned} \hat{\mathcal{H}} &= \bigl\{ h|_{x_1=\cdots=x_k=0} \mid h \in \mathcal{H} \wedge \operatorname{cls} h > k+1 \bigr\} \\ &\cup \bigl\{ (h/x_k^{\deg_{x_k} \operatorname{lt} h})|_{x_1=\cdots=x_k=0} \mid h \in \mathcal{H} \wedge \operatorname{cls} h = k \bigr\} \,. \end{aligned} \tag{3.3}$$

Proof. For the ideals $\bar{\mathcal{I}}^{(k)}$, the assertions follow immediately from the leading terms of the generators in $\mathcal{H}$ and the fact that involutive standard representations remain valid after setting some variables to zero so that Proposition 2.4 can still be applied. For the ideals $\hat{\mathcal{I}}^{(k)}$, the assertions are a consequence of Theorem 2.8. □

In general, the ideals $\mathcal{I}^{(k)}$ for $k > 0$ do not possess finite Pommaret bases, as the addition of a generator of class 1 to an ideal with a depth greater than 2 makes automatically the variables δ-singular for the augmented ideal. An exception are zero-dimensional ideals $\mathcal{I}$ for which any set of variables is δ-regular. This fact also implies together with Theorem 2.7 that for $k \geq \dim \mathcal{A}$ a finite Pommaret basis exists (namely $x_1, \dots, x_k$ together with all generators of a class greater than k).

Bermejo and Gimenez [2, Prop. 3.6] introduced for monomial ideals the notion of a strong Noether normalisation as a simultaneous normalisation of the ideal itself and all its primary components and showed that it is equivalent to quasi-stability (cf. Part (vii) of Theorem 2.6). Hashemi [11, Def. 2.4-6, Thm. 2.19] proposed a seemingly very different definition applicable to arbitrary ideals.

Definition 3.2. The ideal $\mathcal{I} \lhd \mathcal{P}$ is in *strong Noether position*, if one of the following three equivalent conditions is satisfied:

- **(i)** $(\mathcal{I}^{(k)})^{\text{sat}} = \mathcal{I}^{(k)} : x_{k+1}^{\infty}$ for $0 \leq k < \dim \mathcal{A}$;
- **(ii)** $(\bar{\mathcal{I}}^{(k)})^{\text{sat}} = \bar{\mathcal{I}}^{(k)} : x_{k+1}^{\infty}$ for $0 \leq k < \dim \mathcal{A}$;
- **(iii)** $(\hat{\mathcal{I}}^{(k)})^{\text{sat}} = \hat{\mathcal{I}}^{(k)} : x_{k+1}^{\infty}$ for $0 \leq k < \dim \mathcal{A}$.

Hashemi [11, Prop. 2.11] proved that an ideal $\mathcal{I}$ is in strong Noether position, if and only if its leading ideal $\operatorname{lt} \mathcal{I}$ with respect to the degree reverse lexicographic order is in strong Noether position. As the definition of a Pommaret basis is also based on the leading ideal, it suffices for the proof of the following result to consider only monomial ideals.

Proposition 3.3. *The ideal $\mathcal{I} \lhd \mathcal{P}$ is in strong Noether position, if and only if the used variables are δ-regular for it and the degree reverse lexicographic term order.*

Proof. As explained above, it suffices to consider the monomial case. Assume first that the variables are δ-regular for $\mathcal{I}$. Lemma 3.1 provides us with Pommaret bases for all ideals $\bar{\mathcal{I}}^{(k)}$ and $\hat{\mathcal{I}}^{(k)}$. It follows immediately from Part (ii) of Theorem 2.8 that the Conditions (ii) and (iii) of Definition 3.2 are satisfied.

For the converse, let $\mathcal{I}$ satisfies Condition (i) of Definition 3.2. Obviously, it implies that x_1 is not a zero divisor for $\mathcal{P}/\mathcal{I}^{\text{sat}}$ and that for all $1 \leq k < \dim \mathcal{A}$ the variable x_{k+1} is not a zero divisor for $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_k \rangle^{\text{sat}}$. According to Part (ii) of Theorem 2.6, this property is equivalent to the variables being δ-regular for $\mathcal{I}$. □

Thus for monomial ideals being in strong Noether position is equivalent to being quasi-stable. Hashemi [11] proposed to exploit Condition (iv) of Theorem 2.6 for an effective test of quasi-stability; Bermejo and Gimenez [2] recommended in their work on the Castelnuovo-Mumford regularity Condition (iii). Indeed, as already mentioned above, both conditions are straightforward to verify and the required computations are not overly expensive. More precisely, Hashemi [11, Algo. 1] gives an efficiency improved test requiring even less computations. Adapting it to our conventions (and correcting some typos in [11]), we obtain Algorithm 1.

Algorithm 1 Test for quasi-stability of monomial ideal $\mathcal{I}$

Input: minimal basis $\mathcal{B}$ of $\mathcal{I}$
Output: dimension d of $\mathcal{A}$ if $\mathcal{I}$ quasi-stable; $-\infty$ otherwise
$q \leftarrow \deg \mathcal{B}$
$d \leftarrow \min\{\ell \mid \forall \ell \leq j \leq n : x_j^q \in \mathcal{I}\} - 1$
for all $x^\mu \in \mathcal{B}$ such that $k = \operatorname{cls} \mu \leq d$ **do**
  for $j = k+1$ **to** d **do**
    if $x_j^q \cdot x^\mu / x_k^{\mu_k} \notin \mathcal{I}$ **then**
      return $-\infty$
    end if
  end for
end for
return d

The main point in Algorithm 1 is that for analysing $\mathcal{I} : x_k^\infty$ we do not consider all generators contained in $\mathcal{B}$ but only those of class k which, of course, considerably reduces the costs. However, Hashemi [11] does not provide any proof for the correctness of this improvement. The next result closes this gap and implies that Algorithm 1 is indeed correct.

Lemma 3.4. *Let $\mathcal{I}$ be a monomial ideal, $\mathcal{B}$ its minimal basis and $q = \deg \mathcal{B}$. The ideal $\mathcal{I}$ is quasi-stable, if and only if for each term $x^\mu \in \mathcal{B}$ and for each variable x_j with $j > \ell = \operatorname{cls} \mu$ the term $x_j^q \cdot x^\mu / x_k^{\mu_\ell}$ lies in the ideal $\mathcal{I}$.*

Proof. One direction is of course trivial: if the given ideal $\mathcal{I}$ is quasi-stable, then by Part (iii) of Theorem 2.6 $\mathcal{I} : x_k^\infty = \mathcal{I} : \langle x_k, \dots, x_n\rangle^\infty$ implying our claim, as obviously ∞ can here be replaced by q.

For the other direction we must show the following: let $x^\mu \in \mathcal{B}$ be a minimal generator of class ℓ; then for any $j > k > \ell$ the term $t = x_j^q \cdot x^\mu / x_k^{\mu_k}$ lies in $\mathcal{I}$ under the made assumptions. Indeed, then $\mathcal{I} : x_k^\infty \subseteq \mathcal{I} : \langle x_k, \dots, x_n\rangle^\infty$. By assumption, we know that $x_j^q \cdot x^\mu / x_\ell^{\mu_\ell} \in \mathcal{I}$. Hence, $\mathcal{B}$ contains a term x^ν with $\operatorname{cls} \nu = \tilde{\ell} > \ell$ such that $\nu_i \leq \mu_i$ for all $i \neq j$. If $\nu_j = 0$, then $x^\nu \mid t$ and hence $t \in \mathcal{I}$ as required. Otherwise, our assumption implies the existence of a generator $x^\lambda \in \mathcal{B}$ dividing $x_j^q \cdot x^\nu / x_{\tilde{\ell}}^{\nu_{\tilde{\ell}}}$ and satisfying $\operatorname{cls} \lambda > \tilde{\ell}$ and $\lambda_i \leq \nu_i$ for all $i \neq j$. Again we are done if $\lambda_j = 0$ and otherwise we iterate. As we consider each time a generator of higher class, we must find after a finite number of steps a divisor. $\square$

[22, Sect. 2] presents another approach to the question of δ-regularity based on a comparison of the multiplicative variables obtained with the above mentioned rule used for Pommaret bases and an other rule used for Janet bases (see [21, Sect. 2] for the definition of the Janet division): it signals problems with the δ-regularity, if there are more Janet than Pommaret multiplicative variables. This approach is not only very cheap, its main advantage lies in the fact that it provides us with concrete information how the variables should be transformed, if they are not δ-regular.

The following result relates indirectly the different approaches. [22, Prop. 4.8] contains a proof using Condition (iii) of Theorem 2.6 and thus providing a link to the work by Bermejo and Gimenez [2]. Here we present an alternative proof based on Condition (ii) and thus make explicit the relationship to Hashemi's results.

Proposition 3.5. *Let $\mathcal{I} \lhd \mathcal{P}$ be a monomial ideal and $\mathcal{B}$ a finite monomial basis of it where no generator is an involutive divisor of another one. If the ideal $\mathcal{I}$ is not quasi-stable, then for at least one generator in the basis $\mathcal{B}$ a variable exists which is Janet but not Pommaret multiplicative.*

Proof. If $\mathcal{I}$ is not quasi-stable, then by Lemma 3.4 the basis $\mathcal{B}$ contains terms x^ν with the property: if $k = \operatorname{cls}\nu$, then $x^\nu/x_k^{\nu_k} \in (\mathcal{I} : x_k^\infty) \setminus (\mathcal{I} : \langle x_k, \dots, x_n\rangle^\infty)$. Let $x^\mu \in \mathcal{B}$ be among these a generator of minimal class k. If some $j > k$ is Janet multiplicative for x^μ, then we are done.

Otherwise, there must exist for every $j > k$ a generator $x^{\nu^{(j)}} \in \mathcal{B}$ with $\nu_\ell^{(j)} = \mu_\ell$ for all $\ell > j$ and $\nu_j^{(j)} > \mu_j$ in order to render x_j not Janet multiplicative for x^μ. If several generators with this property appear in $\mathcal{B}$, then we choose one with the maximal value of $\nu_j^{(j)}$. Because of this choice, the variable x_j is Janet multiplicative for $x^{\nu^{(j)}}$ and if $\operatorname{cls}\nu^{(j)} < j$, then we are done, as now the generator $x^{\nu^{(j)}}$ has the required properties. If $\operatorname{cls}\nu^{(j)} = j$, then we reach a contradiction, as in this case $x^{\nu^{(j)}}$ divides the term $x^\mu/x_k^{\mu_k}$ entailing that it even lies in $\mathcal{I}$. $\square$

4. Regularities

We already mentioned above three different notions of regularity for a homogeneous ideal $\mathcal{I}\lhd\mathcal{P}$: the Hilbert regularity $\operatorname{hilb}\mathcal{I}$, the Castelnuovo-Mumford regularity $\operatorname{reg}\mathcal{I}$ and the satiety $\operatorname{sat}\mathcal{I}$. In this section we study some relations between these notions. The first result is well-known. We provide here a novel proof of it using Pommaret bases.[4]

Proposition 4.1. *Any ideal $\mathcal{I} \lhd \mathcal{P}$ satisfies* $\operatorname{reg}\mathcal{I} = \max\{\operatorname{sat}\mathcal{I}, \operatorname{reg}\mathcal{I}^{\mathrm{sat}}\}$.

Proof. Without loss of generality, we may assume that we are using δ-regular variables for $\prec_{\mathrm{degrevlex}}$ and thus the existence of a Pommaret basis $\mathcal{H}$ for this order. Theorem 2.8 then immediately implies (with the there introduced notations) the inequalities

$$\operatorname{sat}\mathcal{I} = \deg\mathcal{H}_1 \leq \deg\mathcal{H} = \operatorname{reg}\mathcal{I}\ , \tag{4.1}$$

$$\operatorname{reg}\mathcal{I}^{\mathrm{sat}} \leq \deg\bar{\mathcal{H}} \leq \deg\mathcal{H} = \operatorname{reg}\mathcal{I}\ . \tag{4.2}$$

This proves $\operatorname{reg}\mathcal{I} \geq \max\{\operatorname{sat}\mathcal{I}, \operatorname{reg}\mathcal{I}^{\mathrm{sat}}\}$.

For the converse inequality, we first note that in δ-regular variables all involved quantities are already determined by the leading ideal and thus we may restrict to a monomial ideal $\mathcal{I}$. If $\operatorname{sat}\mathcal{I} = \operatorname{reg}\mathcal{I}$, then we are done. Therefore assume

[4]In [22] it was falsely claimed that the equality follows immediately from Theorem 2.8. However, as one can see in our proof, this argument yields only an inequality. Now we close this gap by also proving the converse inequality.

that $\operatorname{sat}\mathcal{I} < \operatorname{reg}\mathcal{I}$: we will show that it is not possible that $\operatorname{reg}\mathcal{I}^{\mathrm{sat}} < \operatorname{reg}\mathcal{I}$. Consider an element $h_{\max} \in \mathcal{H}$ with $\deg h_{\max} = \deg \mathcal{H}$; by assumption, we must have $\operatorname{cls} h_{\max} > 1$. The inequality $\operatorname{reg}\mathcal{I}^{\mathrm{sat}} < \operatorname{reg}\mathcal{I}$ may only hold, if $\mathcal{H}_1$ contains an element $h_1 = x_1^\ell \bar{h}_1$ with $\bar{h}_1$ an involutive divisor of $h_{\max}$: otherwise $h_{\max}$ was still a member of the Pommaret basis of $\mathcal{I}^{\mathrm{sat}}$ (by Theorem 2.8) and thus $\operatorname{reg}\mathcal{I}^{\mathrm{sat}} = \operatorname{reg}\mathcal{I}$.

It follows from our assumptions that $\bar{h}_1$ is a proper divisor of $h_{\max}$. Thus we can find a variable x_j with $j > 1$ which divides $h_{\max}/\bar{h}_1$. Since x_j is non-multiplicative for h_1, the Pommaret basis $\mathcal{H}$ must contain an involutive divisor h_2 of $x_j h_1$. If $\operatorname{cls} h_2 = 1$, then either $\deg h_2 > \deg h_1$ (if $h_2 = x_j h_1$) or $\deg_{x_1} h_2 < \deg_{x_1} h_1$. In both cases we may replace h_1 by h_2 and start again; after a finite number of such restarts we will have $\operatorname{cls} h_2 > 1$ by degree reasons. But if $\operatorname{cls} h_2 > 1$, then $h_2 \mid x_1^\ell h_{\max}$ implies $h_2 \mid h_{\max}$ and thus the iteration yields an involutive divisor $h_2 \in \mathcal{H}$ of $h_{\max}$ in contradiction to the definition of a strong involutive basis. Therefore we must have $\operatorname{reg}\mathcal{I}^{\mathrm{sat}} = \operatorname{reg}\mathcal{I}$. □

For zero-dimensional ideals one obtains a stronger statement which was already given by Bayer and Stillman [1, Lemma 1.7] with a proof using local cohomology.

Proposition 4.2. *Let $\mathcal{I} \lhd \mathcal{P}$ be a zero-dimensional ideal. Then for any $q \in \mathbb{N}$*

$$\operatorname{sat}\mathcal{I} \leq q \iff \operatorname{reg}\mathcal{I} \leq q \iff \mathcal{I}_q = \mathcal{P}_q \,. \tag{4.3}$$

Consequently, we find the equalities

$$\operatorname{reg}\mathcal{I} = \operatorname{reg}\operatorname{lt}\mathcal{I} = \operatorname{sat}\mathcal{I} = \operatorname{sat}\operatorname{lt}\mathcal{I} \,. \tag{4.4}$$

Proof. For a zero-dimensional ideal $\mathcal{I}$ any variables are δ-regular. Theorem 2.8 implies that the equalities $\operatorname{reg}\mathcal{I} = \operatorname{reg}\operatorname{lt}\mathcal{I}$ and $\operatorname{sat}\mathcal{I} = \operatorname{sat}\operatorname{lt}\mathcal{I}$ hold for *any* ideal in δ-regular variables. Again by Theorem 2.8 (and with the notations used there), all other assertions boil down to the simple statement $\deg \mathcal{H} = \deg \mathcal{H}_1$.

The assumption that $\mathcal{I}$ is zero-dimensional implies that $\mathcal{H}_1 \neq \emptyset$ and that $\operatorname{lt}\mathcal{H}_1$ contains a term $x_1^{q_1}$. Let $x^\mu \in \operatorname{lt}\mathcal{H}$ be a term of maximal degree and assume that $k = \operatorname{cls}\mu > 1$. As the term $x^{\bar{\mu}} = x_1^{q_1} \cdot x^\mu / x_k$ obviously lies in $\operatorname{lt}\mathcal{I}$, it must have an involutive divisor $x^\nu \in \mathcal{H}$. If $\operatorname{cls}\nu > 1$, then x^ν is also an involutive divisor of x^μ which contradicts the assumption that $\mathcal{H}$ is a strong Pommaret basis. Hence $\operatorname{cls}\nu = 1$ (and thus $x^\nu \in \operatorname{lt}\mathcal{H}_1$) implying that $x^\nu = x_1^\ell \cdot x^\mu / x_k$ for some $\ell \geq 1$. But this form immediately entails that $\deg \mathcal{H}_1 \geq \deg x^\nu \geq \deg x^\mu = \deg \mathcal{H} \geq \deg \mathcal{H}_1$ and therefore the claimed equality. □

Bayer and Stillman [1, Def. 1.5] call a linear form $y \in \mathcal{P}_1$ *generic*, if it is not a zero divisor on $\mathcal{P}/\mathcal{I}^{\mathrm{sat}}$. Comparing with Theorem 2.6(ii), we see that this property is equivalent to the fact that y may be used as x_1 in a δ-regular set of variables. They further proved via local cohomology the following statement [1, Lemma 1.8] (according to Bayer and Stillman, it is already implicitly contained in [5]).

Proposition 4.3. *Let $y \in \mathcal{P}_1$ be generic for the ideal $\mathcal{I} \lhd \mathcal{P}$. Then for any $q \in \mathbb{N}$*

$$\operatorname{reg}\mathcal{I} \leq q \iff \operatorname{sat}\mathcal{I} \leq q \;\wedge\; \operatorname{reg}\langle \mathcal{I}, y\rangle \leq q \,. \tag{4.5}$$

Proof. We choose variables with $x_1 = y$ and which are δ-regular for $\prec_{\text{degrevlex}}$. Let $\mathcal{H}$ be the corresponding Pommaret basis of $\mathcal{I}$. Then it follows from Theorem 2.8 that $\operatorname{reg}\mathcal{I} = \deg \mathcal{H}$ and $\operatorname{sat}\mathcal{I} = \deg \mathcal{H}_1$ (resp. $\operatorname{sat}\mathcal{I} = 0$ if $\mathcal{H}_1 = \emptyset$). We claim that $\operatorname{reg}\langle \mathcal{I}, y\rangle = \deg(\mathcal{H} \setminus \mathcal{H}_1)$ which implies our assertion.

If $\mathcal{H}_1 \neq \emptyset$, then the set $(\mathcal{H} \setminus \mathcal{H}_1) \cup \{x_1\}$ is the Pommaret basis of $\langle \mathcal{I}, y \rangle$ and Theorem 2.8 entails our claim. Note that in this case we may have $\operatorname{reg}\langle \mathcal{I}, y \rangle < \operatorname{reg}\mathcal{I}$, if all elements of maximal degree in $\mathcal{H}$ are of class 1.

If $\mathcal{H}_1 = \emptyset$, then it follows immediately from Buchberger's first criterion that $\mathcal{H}|_{x_1=0} \cup \{x_1\}$ is a Gröbner basis of $\langle \mathcal{I}, y \rangle$. We write $\mathcal{H}|_{x_1=0} = \{h_1, \dots, h_r\}$ and set $h_0 = x_1$. Denote by $\mathbf{e}_0, \dots, \mathbf{e}_r$ the standard basis of $\mathcal{P}^{r+1}$. We obtain by Schreyer's theorem [18] a Gröbner basis of the first syzygy module $\operatorname{Syz}(h_0, \dots, h_r)$ consisting of the syzygies $\mathbf{S}_{ij}$ induced by the S-polynomials $S(h_i, h_j)$. The syzygies stemming from h_0 are all of the form $\mathbf{S}_{0i} = x_1\mathbf{e}_i - h_i\mathbf{e}_0$ and these are the only ones containing a term in position $\mathbf{e}_0$. According to the induced term order used in the Schreyer theorem, we find $\operatorname{lt}\mathbf{S}_{0i} = \operatorname{lt}(h_i)\mathbf{e}_0$ and hence these syzygies do not affect the further construction of a free resolution of $\langle \mathcal{I}, y \rangle$. Hence, beyond the first syzygy module, $\mathcal{I}$ and $\langle \mathcal{I}, y \rangle$ have isomorphic resolutions, as setting $x_1 = 0$ does not affect any leading terms in the resolution. But this implies in our case that $\operatorname{reg}\mathcal{I} = \operatorname{reg}\langle \mathcal{I}, y \rangle$. □

The following results are also well-known; compared with [11, Lemma 5.5/6], we provide simpler proofs. The first one asserts that the Hilbert regularity shows the same behaviour with respect to saturation as the Castelnuovo-Mumford regularity.

Proposition 4.4. *Any ideal* $\mathcal{I} \lhd \mathcal{P}$ *satisfies* $\operatorname{hilb}\mathcal{I} = \max\{\operatorname{sat}\mathcal{I}, \operatorname{hilb}\mathcal{I}^{\mathrm{sat}}\}$.

Proof. We have $\mathcal{I}_q = \mathcal{I}^{\mathrm{sat}}_q$ for any degree $q \geq \operatorname{sat}\mathcal{I}$ and a strict inclusion for any lower degree. If $\operatorname{hilb}\mathcal{I} \leq \operatorname{sat}\mathcal{I}$, then trivially $\operatorname{hilb}\mathcal{I}^{\mathrm{sat}} = \operatorname{sat}\mathcal{I}$, and if $\operatorname{hilb}\mathcal{I} \geq \operatorname{sat}\mathcal{I}$, then obviously $\operatorname{hilb}\mathcal{I}^{\mathrm{sat}} = \operatorname{hilb}\mathcal{I}$. □

Proposition 4.5. *Any ideal* $\mathcal{I} \lhd \mathcal{P}$ *satisfies* $\operatorname{hilb}\mathcal{I} \leq \operatorname{reg}\mathcal{I}$.

Proof. The claim follows trivially from the fact that the Hilbert function of the truncated ideal $\mathcal{I}_{\geq \operatorname{reg}\mathcal{I}}$ is polynomial[5] which in turn is a simple consequence of the fact that $\mathcal{I}_{\geq \operatorname{reg}\mathcal{I}}$ possesses a linear resolution [5] (see [22, Thm. 9.10] for a proof based on Pommaret bases). □

Hashemi [11, Def. 4.5] introduced *stabilised regularities* as the maximal regularities of the ideals $\mathcal{I}^{(k)}$. As all the considered regularities actually depend on the factor ring $\mathcal{P}/\mathcal{I}^{(k)} \cong \mathcal{P}_k/\bar{\mathcal{I}}^{(k)}$, we can equivalently define them via the ideals $\bar{\mathcal{I}}^{(k)}$. This approach has the advantage that for the latter ideals, Pommaret bases are given by Lemma 3.1 and we can thus directly apply the results of [22]. It will turn out that Hashemi's main result arises now as a simple corollary.

Definition 4.6. The *stabilised Hilbert regularity*, *stabilised Castelnuovo-Mumford regularity* and the *stabilised satiety*, respectively, of the homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$ of dimension D are given by

$$\overline{\operatorname{hilb}}\,\mathcal{I} = \max\{\operatorname{hilb}\bar{\mathcal{I}}^{(k)} \mid 0 \leq k \leq D\} \tag{4.6a}$$

$$\overline{\operatorname{reg}}\,\mathcal{I} = \max\{\operatorname{reg}\bar{\mathcal{I}}^{(k)} \mid 0 \leq k \leq D\} \tag{4.6b}$$

$$\overline{\operatorname{sat}}\,\mathcal{I} = \max\{\operatorname{sat}\bar{\mathcal{I}}^{(k)} \mid 0 \leq k \leq D\} \tag{4.6c}$$

The following result is to a large extent just [11, Thm. 4.17]. However, we do not only provide a much shorter proof, but based on the results in [22] we can also replace the one inequality left by Hashemi by an equality.

[5] In the context of differential equations an explicit expression is e. g. given in [19].

Theorem 4.7. *Let the variables X be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and $\prec_{\text{degrevlex}}$ and let $\mathcal{H}$ be the corresponding Pommaret basis. Then*

$$\deg \mathcal{H} = \operatorname{reg} \operatorname{lt} \mathcal{I} = \operatorname{reg} \mathcal{I} = \overline{\operatorname{reg}}\, \mathcal{I} = \overline{\operatorname{sat}}\, \mathcal{I} = \overline{\operatorname{hilb}}\, \mathcal{I}\ . \tag{4.7}$$

Proof. The first two equalities are just Theorem 2.8(i). From the Pommaret bases given in Lemma 3.1 and again from Theorem 2.8, it follows that

$$\begin{aligned} \operatorname{reg} \bar{\mathcal{I}}^{(k)} &= \max \{\deg h \mid h \in \mathcal{H} \wedge \operatorname{cls} h \geq k+1\}\ , \\ \operatorname{sat} \bar{\mathcal{I}}^{(k)} &= \max \{\deg h \mid h \in \mathcal{H} \wedge \operatorname{cls} h = k+1\}\ , \end{aligned} \tag{4.8}$$

which implies all equalities except the last one. However, Propositions 4.4 and 4.5 entail for any ideal $\mathcal{J}$ the inequalities $\operatorname{sat} \mathcal{J} \leq \operatorname{hilb} \mathcal{J} \leq \operatorname{reg} \mathcal{J}$. Since we know already $\overline{\operatorname{sat}}\, \mathcal{I} = \overline{\operatorname{reg}}\, \mathcal{I}$, also $\overline{\operatorname{hilb}}\, \mathcal{I}$ must have the same value. □

5. Systems of Parameters

Recall that if D is the dimension of $\mathcal{I} \lhd \mathcal{P}$, then a *maximal system of parameters* consists of $c = n - D$ elements $f_1, \dots, f_c \in \mathcal{I}$ such that the ideal $\tilde{\mathcal{I}} = \langle f_1, \dots, f_c \rangle$ has the same dimension. Obviously, this condition is equivalent to $f_1, \dots, f_c$ forming a $\mathcal{P}$-regular sequence in $\mathcal{I}$ and $\tilde{\mathcal{I}}$ is then a complete intersection. Such systems of parameters are important for a number of computational tasks in commutative algebra like the determination of a primary decomposition of $\mathcal{I}$ (see e. g. [4]). The usual approach to their construction consists of taking generic linear combinations of ideal generators and thus faces similar problems as the determination of δ-regular variables via random transformations. The following result shows that a Pommaret basis always contains a system of parameters.

Proposition 5.1. *Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal of codimension c. Assume that the used variables X are δ-regular for $\mathcal{I}$ and some term order*[6] *$\prec$ and let $\mathcal{H}$ be the corresponding Pommaret basis. Then $\mathcal{H}$ contains elements h_i for $i = 1, \dots, c$ such that $\operatorname{lt} h_i = x_{n-i+1}^{e_i}$ for some integers $e_i \geq 1$ and these elements define a maximal system of parameters for $\mathcal{I}$.*

Proof. The first assertion follows immediately from Theorem 2.6(iii), since, by definition of a Pommaret basis for the ideal $\mathcal{I}$, the leading ideal $\operatorname{lt} \mathcal{I}$ is a quasi-stable ideal with Pommaret basis $\operatorname{lt} \mathcal{H}$.

For the second assertion, we note that obviously the leading terms of $h_1, \dots, h_c$ are pairwise relatively prime. Hence, by Buchberger's first criterion, these polynomials form a Gröbner basis of the ideal $\mathcal{J} \subseteq \mathcal{I}$ they generate. Furthermore, by Schreyer's theorem, the syzygy module $\operatorname{Syz}(h_1, \dots, h_c)$ is generated by the trivial syzygies $\mathbf{S}_{ij} = h_j \mathbf{e}_i - h_i \mathbf{e}_j$.

Assume now that for some index $1 \leq k \leq c$ a polynomial $0 \neq f \in \mathcal{P}$ exists such that $f h_k \in \langle h_1, \dots, h_{k-1} \rangle$. This relation induces a syzygy $\mathbf{S} \in \operatorname{Syz}(h_1, \dots, h_k) \subseteq \operatorname{Syz}(h_1, \dots, h_c)$ with a component $f \mathbf{e}_k$. By the observation above, $\operatorname{Syz}(h_1, \dots, h_k)$ is generated by the syzygies $\mathbf{S}_{ij}$ with $1 \leq i < j \leq k$ implying that $f \in \langle h_1, \dots, h_{k-1} \rangle$. But this fact entails that h_k is not a zero divisor on $\mathcal{P}/\langle h_1, \dots, h_{k-1} \rangle$ and the sequence $h_1, \dots, h_c$ is $\mathcal{P}$-regular. □

[6] We exploit here that in the homogeneous case $\mathcal{I}$ and $\operatorname{lt} \mathcal{I}$ always have the same dimension. The proposition remains true for non-homogeneous ideals, if we restrict to degree compatible orders.

Remark 5.2. One should note that even if $\mathcal{I}$ itself is already a complete intersection, the ideal $\mathcal{J} = \langle h_1, \dots, h_c\rangle$ is in general still a proper subset of $\mathcal{I}$. As a simple example consider the ideal $\mathcal{I} = \langle x_2^2 - x_1^2, x_1x_2\rangle$. It is a complete intersection and we obtain a Pommaret basis for it by adding the generator x_1^3. Hence the construction above yields $\mathcal{J} = \langle x_2^2 - x_1^2, x_1^3\rangle \subsetneq \mathcal{I}$.

Remark 5.3. Proposition 5.1 may be considered as a variation of a result by Eisenbud and Sturmfels [6, Thm. 1.3]. It is much simpler because of the use of δ-regular variables. In the notations of [6], we partition the Pommaret basis $\mathcal{H} = \mathcal{F}$ into the disjoint sets $\mathcal{F}_k = \{h \in \mathcal{H} \mid \operatorname{cls} h = k\}$. Then all terms contained in $\operatorname{lt} \mathcal{F}_k$ have the variable x_k in common. For the construction of the maximal system of parameters, if suffices to consider the sets $\mathcal{F}_k$ with $k \geq D = \dim \mathcal{A}$. However, in contrast to [6, Thm. 1.3], there is no need to take linear combinations within the sets $\mathcal{F}_k$; we simply choose the above described polynomial $h_k \in \mathcal{F}_k$.

Eisenbud and Sturmfels were concerned with obtaining a system of parameters as sparse as possible. They showed that the construction of an optimal solution is in general NP-hard and thus not feasible in practice. In our approach, we have no direct influence on the sparsity of the obtained system of parameters, as it depends on the sparsity of the transformation to δ-regular variables. In a comparison one should keep in mind that δ-regularity implies much more than just a simple system of parameters and hence one has less freedom in the search for an optimal solution.

In principle, one could try to turn the argument around and to use their results for improvements in the construction of δ-regular variables. However, in the few examples we tried, we did not really obtain an improvement. As a concrete example consider the monomial ideal $\mathcal{I} = \langle x_1x_2, x_1x_3, x_2x_3, x_4^2\rangle \lhd \Bbbk[x_1, x_2, x_3, x_4]$ [6, Ex. 1.1]. The used variables are not δ-regular for $\mathcal{I}$. An analysis of the Janet multiplicative variables suggests to transform the variable x_1 into $x_1 + x_2 + x_3$. This transformation yields directly δ-regular variables and we obtain the new transformed ideal $\tilde{\mathcal{I}} = \langle x_2x_3, x_1x_2 + x_2^2, x_1x_3 + x_3^2, x_4^2\rangle$ where the last three generators define a system of parameters. In the original variables, this system of parameters is given by the three polynomial $x_1x_2 - x_2x_3$, $x_1x_2 - x_1x_3$ and x_4^2.

Eisenbud and Sturmfels proposed the partition $\{x_1x_2\}$, $\{x_2x_3, x_1x_3\}$ and $\{x_4^2\}$ and thus as system of parameters x_1x_2, $\lambda x_2x_3 + \mu x_1x_3$ and x_4^2 for a generic choice of λ, μ. Compared with our system of parameters it contains one term less. As a first step towards δ-regular variables one should look for a transformation such that the first set contains a polynomial with leading term x_2^2 and the second set one with leading term x_3^2. Obviously, the simplest way to achieve this goal consists of transforming x_1 into $x_1 + x_2 + x_3$ which is exactly the same transformation as suggested by our method.

In general, an analysis of the partition obtained with the methods of Eisenbud and Sturmfels does not suffice for the construction of δ-regular variables. A closer inspection of the proof of Proposition 5.1 shows that it does not really require a Pommaret basis. For the purposes of the proof, it suffices, if the basis induces a quasi-Rees decomposition of the factor ring $\mathcal{A}$, as then one can invoke [22, Lemma 3.16] (which is a variant of [16, Lemma 14]) for proving the first assertion. The point is that the existence of such a decomposition implies that the ideal is in Noether position [22, Prop. 4.1].

According to Part (ii) of Theorem 2.7, the variables $x_1, \dots, x_D$ induce a Noether normalisation in the δ-regular case. Eisenbud and Sturmfels [6, Sect. 2] also treat the question of constructing a sparse Noether normalisation considered as computing a maximal system of parameters modulo an ideal. Here it again seems that our approach via the Janet multiplicative variables is at least comparable in efficiency. In fact, for their concrete example we even obtain a higher sparsity than their greedy algorithm. They consider the ideal

$$\mathcal{I} = \langle x_1x_2, x_1x_3, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5, x_4x_6, x_5x_6\rangle \lhd \Bbbk[x_1, \dots, x_6] \tag{5.1}$$

which from the point of view of δ-regularity is very "bad", as no pure powers are present in the given basis. An analysis of the Janet multiplicative variables suggests a number of transformations. Two of them, $x_1 \mapsto x_1 + x_2 + x_3$ and $x_4 \mapsto x_4 + x_5 + x_6$, together with a subsequent swap $x_2 \leftrightarrow x_4$ lead directly to δ-regular variables. Since the ideal $\mathcal{I}$ is two-dimensional, we obtain with Pommaret bases as Noether normalisation in the original variables:

$$y_1 = x_1 + x_2 + x_3\ , \quad y_2 = x_4 + x_5 + x_6\ . \tag{5.2}$$

By contrast, the algorithm of Eisenbud and Sturmfels yields

$$y_1 = x_1 - x_6\ , \quad y_2 = x_2 + x_3 + x_4 + x_5 + x_6 \tag{5.3}$$

and thus one term more.

Consider now the factor ring $\mathcal{A}$ and assume that the elements $a_1, \dots, a_D \in \mathcal{A}$ define a system of parameters in the sense that $\dim\big(\mathcal{A}/\langle a_1, \dots, a_D\rangle\big) = 0$. Recall that the *socle* of a $\mathcal{P}$-module $\mathcal{M}$ is defined as $\operatorname{Soc}\mathcal{M} = 0 : \mathfrak{m} = \{m \in \mathcal{M} \mid \mathfrak{m}\cdot m = 0\}$. If $\mathcal{A}$ is Cohen-Macaulay (which can be easily checked using Theorem 2.7), then the *type* of $\mathcal{A}$ is defined as $t = \dim_{\Bbbk}\operatorname{Soc}\big(\mathcal{A}/\langle a_1, \dots, a_D\rangle\big)$ (one can show that t is independent of the chosen system of parameters) and $\mathcal{A}$ is *Gorenstein*, if $t = 1$.

Theorem 5.4. *Let the chosen variables X be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the degree reverse lexicographic order $\prec_{\text{degrevlex}}$ and let $\mathcal{H}$ be the corresponding Pommaret basis. Then $[x_1], \dots, [x_D]$ form a system of parameters for $\mathcal{A}$. If we denote by $\mathcal{H}_d = \{h \in \mathcal{H} \mid \operatorname{cls} h = d\}$ where $d = \min\{\operatorname{cls} h \mid h \in \mathcal{H}\}$ all elements of minimal class and collect in the subset $\tilde{\mathcal{H}}_d$ all those elements $h \in \mathcal{H}_d$ such that for no $k > d$ the involutive standard representation of x_kh contains a non-vanishing constant coefficient, then the set $\big\{[h/x_d] \mid h \in \tilde{\mathcal{H}}_d\big\}$ is a basis of the socle $\operatorname{Soc}\big(\mathcal{A}/\langle[x_1], \dots, [x_D]\rangle\big)$.*

Before we present the proof of this assertion, we should briefly discuss the meaning of the expression $[h/x_d]$. Although the leading term of h is by construction of class d and thus divisible by x_d, this is not necessarily true for the remaining terms: we only know that their classes are at most d because of the use of the degree reverse lexicographic order (cf. Remark 2.2). Hence the quotient h/x_d makes generally no sense in the polynomial ring $\mathcal{P}$. However, since $\mathcal{A}$ is by assumption Cohen-Macaulay, we have $D = d - 1$ and thus for defining the residue class $[h] \in \mathcal{A}/\langle[x_1], \dots, [x_D]\rangle$ we may ignore all terms in h of class less than d. All remaining terms are divisible by x_d and we may speak of the residue class $[h/x_d]$.

Proof. According to Part (ii) of Theorem 2.7, the variables $x_1, \dots, x_D$ induce a Noether normalisation of $\mathcal{A}$ and thus a system of parameters. Without loss of generality, we may assume that already $D = 0$ and thus that $d = 1$. Then our claim is true, if we can prove the following equality:

$$\mathcal{I} : \mathfrak{m} = \mathcal{I} + \left\langle \frac{h}{x_1} \mid h \in \tilde{\mathcal{H}}_1 \right\rangle . \tag{5.4}$$

If $f \in \mathcal{I} : \mathfrak{m}$, then in particular $x_1 f \in \mathcal{I}$ and hence there exists an involutive standard representation $x_1 f = \sum_{h\in\mathcal{H}} P_h h$. A constant term can only appear in one of the coefficients P_h where $h \in \mathcal{H}_1$, as otherwise we had on the right hand side a summand not divisible by x_1 whereas on the left hand side everything is divisible by x_1 (since we are using the degree reverse lexicographic term order). Thus for $\operatorname{cls} h > 1$ we can write $P_h = x_1\tilde{P}_h$ and for $\operatorname{cls} h = 1$ we find $P_h = c_h + x_1\tilde{P}_h \in \Bbbk[x_1]$ with $c_h \in \Bbbk$. Consider now the polynomial $\tilde{f} = f - \sum_{h\in\mathcal{H}} \tilde{P}_h h$; obviously, $\tilde{f}$ differs from f only by an element of the ideal $\mathcal{I}$ and it is a $\Bbbk$-linear combination of the polynomials h/x_1 with $h \in \mathcal{H}_1$.

Conversely take a generator $\hat{h} \in \mathcal{H}_1$. For any $1 < k \leq n$ we have an involutive standard representation $x_k\hat{h} = \sum_{h\in\mathcal{H}} Q_h h$. Again a constant term can only appear in one of the coefficients Q_h where $h \in \mathcal{H}_1$. But if this happens then $x_k\hat{h}/x_1$ cannot be contained in $\mathcal{I}$ and hence $\hat{h}/x_1 \notin \mathcal{I} : \mathfrak{m}$. Thus only the elements of $\tilde{\mathcal{H}}_1$ induce socle generators. □

Example 5.5. Consider the monomial ideal $\mathcal{I} = \langle x^2, xy^2, y^3\rangle \lhd \Bbbk[x, y]$. Its minimal Pommaret basis is $\mathcal{H} = \{x^2, x^2y, xy^2, y^3\}$. For the three generators of class 1 we find $y \cdot x^2 = x^2y$, $y \cdot x^2y = x \cdot xy^2$ and $y \cdot xy^2 = x \cdot y^3$. Hence a basis of $\operatorname{Soc}\mathcal{A}$ is induced by the two monomials xy, y^2 and $\mathcal{A}$ is not Gorenstein.

As a second example we consider the ideal $\mathcal{I} = \langle z^2 - xy, yz, y^2, xz, x^2\rangle \lhd \Bbbk[x, y, z]$ for which we obtain a Pommaret basis by adding the monomial x^2y. Again we find three generators of class 1 in $\mathcal{H}$. However, $z\cdot xz = x\cdot(z^2 - xy) + x^2y$ and $y\cdot x^2 = x^2y$. Hence the socle of $\mathcal{A}$ is generated by the single monomial $[xy]$ and $\mathcal{A}$ is Gorenstein. One easily checks that here the socle of the monomial factor ring $\mathcal{A}'$ is generated by the two monomials $[xy]$, $[z]$ and thus $\mathcal{A}'$ is *not* Gorenstein. This observation is in marked contrast to the fact that in many other respects the algebras $\mathcal{A}$ and $\mathcal{A}'$ have identical properties, if δ-regular variables are used (see the discussion in the Conclusions of [22]). In particular, we mentioned already above that in the δ-regular case $\mathcal{A}$ is Cohen-Macaulay, if and only if $\mathcal{A}'$ is Cohen-Macaulay.

6. Conclusions

We showed that Hashemi's approach to strong Noether position provides yet another algebraic characterisation of the concept of δ-regularity arising in the theory of Pommaret bases. This observation allows to put it into a larger context and to relate it with many other results. In particular, it becomes much more constructive, as deterministic methods are known for achieving δ-regularity.

Our proofs in Section 4 show that the results of Hashemi on the stabilised regularities become trivial from the point of view of Pommaret bases. For $\overline{\operatorname{sat}}\,\mathcal{I}$ and $\overline{\operatorname{reg}}\,\mathcal{I}$ they are simple consequences of Theorem 2.8 relating satiety and Castelnuovo-Mumford regularity to the degrees in the Pommaret basis. The situation is slightly different for $\overline{\operatorname{hilb}}\,\mathcal{I}$: while Pommaret bases allow us to determine straightforwardly the Hilbert regularity of any concrete ideal via the Hilbert series, $\operatorname{hilb}\mathcal{I}$ is not naturally related to some invariant of the basis. However, here we can resort to the simple inequality $\operatorname{sat}\mathcal{I} \leq \operatorname{hilb}\mathcal{I} \leq \operatorname{reg}\mathcal{I}$.

Compared with other methods to check whether a given factor ring $\mathcal{A}$ is Cohen-Macaulay or Gorenstein, the approach via Pommaret bases (for $\prec_{\text{degrevlex}}$) is much more efficient, as it requires hardly any computations. Dimension and depth can be directly read off the basis and our construction of a socle basis requires only a few involutive standard representations (which are anyway computed during the construction of the Pommaret basis). By contrast, standard methods determine either (large parts of) a minimal resolution or a number of Ext-groups, both requiring several Gröbner bases computations. Again this result depends crucially on the fact that δ-regular variables induce a Noether normalisation.

Of course, all these applications of Pommaret bases require the construction of δ-regular variables. As mentioned above, deterministic algorithms for this task exist and in many cases succeed in finding fairly sparse transformations. As a Pommaret basis (for $\prec_{\text{degrevlex}}$) contains so much information about the ideal it generates, we believe that it is worth while spending this extra bit of computation time. If one is only interested in certain invariants, then one may also resort to the approach of Bermejo and Gimenez [2] where the leading ideal in δ-regular variables is considered separately from the ideal itself (which is not transformed). However, other applications like the here presented socle construction require that the full basis is known in δ-regular variables.

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Institut für Mathematik, Universität Kassel, 34132 Kassel, Germany
E-mail address: `seiler@mathematik.uni-kassel.de`