Computing Finite and Infinite Free Resolutions with Pommaret-Like Bases

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Abstract. Free resolutions are an important tool in algebraic geometry for the structural analysis of modules over polynomial rings and their quotient rings. Minimal free resolutions are unique up to isomorphism and induce homological invariants in the form of Betti numbers. It is known that Pommaret bases of ideals in the polynomial ring induce finite free resolutions and that the Castelnuovo-Mumford regularity and projective dimension can be read off already from the Pommaret basis. In this article, we generalize this construction to Pommaret-like bases, which are generally smaller. We apply Pommaret-like bases also to infinite resolutions over quotient rings. Over Clements–Lindström rings, we derive bases for the free modules in the resolution using only the Pommaret-like basis. Finally, restricting to monomial ideals over the polynomial ring, we derive an explicit formula for the differential of the induced resolution.

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1. Introduction

Involutive Bases have their origin in the works by Janet on the analysis of systems of (linear) partial differential equations [14, 15]. As in Gröbner basis theory, Janet used monomial, and thus combinatorial, structures as a tool by the means of which more complex (differential) algebraic structures can be analysed. Inspired by Janet's—and also Pommaret's [17]—works, Zharkov and Blinkov developed involutive bases for polynomial ideals [23]. Gerdt and Blinkov studied different types of involutive bases, introducing the framework of involutive divisions in the process [5]. The most wellknown involutive divisions-the Janet and Pommaret divisions-go back to Janet's works. Further involutive divisions have been studied; see, e.g., [22, 10]. As Gröbner bases, involutive bases induce free resolutions of the ideals they generate. For some types of involutive divisions, the syzygy modules in this resolution are generated by involutive bases of the same type [20]. In the case of the Pommaret division, homological invariants like projective dimension and Castelnuovo-Mumford regularity can be read off directly from the original Pommaret basis. Not every monomial ideal possesses a finite Pommaret basis; those that do are termed *quasi-stable*. For the resolution induced by the Pommaret basis of a quasi-stable monomial ideal, an explicit formula is known [20]; however, this resolution is not necessarily minimal. This formula generalizes the well-known resolution formula found by Eliahou and Kervaire [3], which only applies when the Pommaret basis coincides with the minimal generating set of the ideal. A polynomial ideal is said to be in quasi-stable position when it possesses a finite Pommaret basis for the given coordinates; moreover, this position is a generic one [19]. For a comprehensive study and applications of the theory of involutive bases to commutative algebra and the geometric theory of partial differential equations, we refer to [21].

Our contributions concern the use of relative involutive-like bases for the computation and analysis of free resolutions. For this, we focus on (relative) Pommaret and Pommaret-like bases. While Pommaret bases capture many homological properties of ideals in quasi-stable position [21], the resolutions induced by them need not be minimal, because already the basis of the ideal is not a minimal generating system. We show that Pommaret-like bases represent a significant improvement in this respect. Another aspect we investigate is the application to monomial ideals. For these, we are able to identify different classes of (relatively) quasi-stable ideals for which Pommaret-like bases induce the minimal free resolution. Even for other cases, the induced resolution has useful properties like Gröbner-reducedness in all higher syzygy modules. For a subclass of quasi-stable monomial ideals, we obtain closed formulas for the differential of the induced resolution, thereby significantly generalizing the formula by Eliahou–Kervaire [3] for stable monomial ideals. Moreover, we relate our results to a resolution formula for square-free Borel ideals in zero-dimensional Clements–Lindström rings found by Gasharov et al. [4].

The article is organized as follows. In Section 2, we recall well-known facts about involutive bases, syzygies and (infinite) free resolutions. In Section 3 we will start by analysing the resolutions induced by relative Pommaret bases. We focus on obtaining minimal Pommaret bases for the syzygy modules in each homological degree and observe phenomena that distinguish the relative situation from the case of resolutions over \mathcal{P} . Pommaret-like bases are generally smaller than their Pommaret counterparts and hence they provide better chances to obtain minimal resolutions. In Section 4 we study Pommaret-like induced resolutions over \mathcal{P} . In order to be able to carry out an analogous study over quotient rings, we introduce relative involutive-like divisions in Section 5. In Section 6 we analyse Pommaret-like induced resolutions over Clements–Lindström rings. We obtain a combinatorial formula for the bigraded Betti numbers of the induced resolutions when they are minimal. In Section 7, we obtain for some classes of monomial ideals in \mathcal{P} explicit formulas for the differential of the Pommaret-like induced resolution, generalizing for example constructions by Eliahou and Kervaire [3] and by Seiler et al. [21, 1]. Finally, some conclusions are given in Section 8

2. Preliminaries

Let $\mathcal{P} = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in *n* variables over a field \mathbb{K} , let $\mathcal{I} \leq \mathcal{P}$ be an ideal and let \mathcal{P}/\mathcal{I} denote the quotient ring defined by \mathcal{I} .

As a K-vector space, \mathcal{P} has the basis $\mathcal{T} = \{x_1^{\mu_1} \cdots x_n^{\mu_n} \mid \mu_1, \dots, \mu_n \in \mathbb{N}_0\}$ of *terms*, which are products of non-negative integer powers of the variables. If $Y = \{y_1, \dots, y_k\} \subseteq X$ is a subset of variables, then we denote by $\mathcal{T}_Y = \{y_1^{\mu_1} \cdots y_k^{\mu_k} \mid \mu_i \in \mathbb{N}_0, 1 \le i \le k\}$ the monoid of all terms in \mathcal{P} depending only on the variables in Y. To each term $t = \mathbf{x}^{\mu} = x_1^{\mu} \cdots x_n^{\mu_n} \in \mathcal{T}$ we associate its *total degree* deg $(t) = \sum_{i=1}^n \mu_i$ and its *exponent vector*, *multidegree*, or *multiindex* $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}_0^n$. We write deg_i $(t) = \mu_i$ for the degree of t in the variable x_i . For an integer $d \ge 0$, we collect the subset of all terms of degree d in the set $\mathcal{T}_d \subset \mathcal{T}$. \mathcal{T}_d generates the finite dimensional K-vector space \mathcal{P}_d of polynomials *homogeneous of degree* d: $\mathcal{P}_d := \langle \mathcal{T}_d \rangle_{\mathbb{K}}$. (Note that the zero polynomial is homogeneous of any degree.) Obviously, we have the direct sum of K-vector spaces $\mathcal{P} = \bigoplus_{d \ge 0} \mathcal{P}_d$, and $\mathcal{P}_d \cdot \mathcal{P}_e \subseteq \mathcal{P}_{d+e}$ for all $d, e \ge 0$. For a given multidegree $\mu = (\mu_1, \dots, \mu_n)$, we write $\mathcal{P}_\mu = \langle \mathbf{x}^\mu \rangle_{\mathbb{K}}$ for the one-dimensional K-vector space of *monomials* supported on the term \mathbf{x}^μ . Also for this grading, we obtain the direct sum of vector spaces $\mathcal{P} = \bigoplus_{\mu \in \mathbb{N}_0^n} \mathcal{P}_\mu$; and as before, $\mathcal{P}_\mu \cdot \mathcal{P}_\nu \subseteq \mathcal{P}_{\mu+\nu}$ for all $\mu, \nu \in \mathbb{N}_0^n$.

The main idea of an involutive division is to assign to each generator h in a basis H a subset $M_L(h, H) \subseteq X$ of multiplicative variables and to consider only \mathcal{P} -linear combinations of the generators where each generator $h \in H$ is multiplied by a coefficient depending only on the variables

in $M_L(h, H)$. In contrast to Gröbner bases, not every monomial basis of a monomial ideal is automatically an involutive basis. The rule for the assignment of the multiplicative variables is called an *involutive division*.

Definition 2.1. An *involutive division* L on $\mathcal{T} \subset \mathcal{P}$ associates to any finite set $U \subset \mathcal{T}$ of terms and any term $u \in U$ a set of *L-non-multipliers* $\overline{L}(u,U)$ given by the terms contained in a prime monomial ideal. The variables generating this prime ideal are called the *non-multiplicative variables* $\mathrm{NM}_L(u,U) \subseteq X$ of $u \in U$. The set of *L-multipliers* L(u,U) is given by the order ideal $\mathcal{T} \setminus \overline{L}(u,U)$; it is a subring of \mathcal{P} generated by the set of *multiplicative variables* $\mathrm{M}_L(u,U) = X \setminus \mathrm{NM}_L(u,U)$. For any term $u \in U$, its *involutive cone* is defined as $\mathcal{C}_L(u,U) = u \cdot L(u,U)$. For an involutive division, the involutive cones must satisfy the following conditions:

- (i) For two terms $v \neq u \in U$ with $\mathcal{C}_L(u, U) \cap \mathcal{C}_L(v, U) \neq \emptyset$, we have $u \in \mathcal{C}_L(v, U)$ or $v \in \mathcal{C}_L(u, U)$.
- (ii) If a term $v \in U$ lies in an involutive cone $C_L(u, U)$, then $L(v, U) \subset L(u, U)$.
- (iii) For any term u in a subset $V \subset U$, we have $L(u, U) \subseteq L(u, V)$.

We write $u \mid_L w$ for a term $u \in U$ and an arbitrary term $w \in \mathcal{T}$, if $w \in \mathcal{C}_L(u, U)$. In this case, u is called an *L*-involutive divisor of w and w an *L*-involutive multiple of u.

Conditions (i) and (ii) ensure that involutive cones can intersect only trivially. Condition (iii) is often called the *filter axiom*. Obviously, it suffices for defining an involutive division to say what are the (non-)multiplicative variables for each term u in a finite set U. Note that involutive divisibility $u \mid_L w$ implies ordinary divisibility, but not vice versa.

As with Gröbner bases, involutive bases are defined via monomial structures. For monomial ideals, we define involutive bases as follows.

Definition 2.2. For a finite set of terms $U \,\subset \, \mathcal{T}$ and an involutive division L on \mathcal{T} , the *involutive* span of U is the union $\mathcal{C}_L(U) = \bigcup_{u \in U} \mathcal{C}_L(u, U)$. The set U is an *L*-involutive basis of the ideal generated by it, if $\mathcal{C}_L(U) = \mathcal{T} \cdot U$ and the union is disjoint, i. e. every term in $\mathcal{C}_L(U)$ has a unique involutive divisor. An involutive division L is *Noetherian*, if every monomial ideal in \mathcal{P} possesses an *L*-involutive basis. The *L*-involutive basis H of a monomial ideal \mathcal{I} is *minimal*, if any other *L*-involutive basis H' of \mathcal{I} contains H as a subset.

For involutive divisions that are *continuous* (see [21, Def. 4.1.3]) or even *constructive* (see [21, Def. 4.1.7]), the following useful properties hold:

Proposition 2.3. [21, Prop. 4.1.4] For a continuous involutive division L, a finite set of terms $U \subset \mathcal{T}$ is an L-involutive basis of the monomial ideal $\langle U \rangle$ if and only if, for each $u \in U$ and $x \in NM_L(u, U)$, we have $xu \in C_L(U)$.

We call the criterion implied by Proposition 2.3 the criterion of *local involutivity*.

Proposition 2.4. [21, Cor. 4.2.4] For a constructive noetherian involutive division L, every monomial ideal has a unique minimal L-involutive basis.

Given a finite set H of polynomials, a monomial ordering < and an involutive division L, we call H a weak L-involutive basis of the ideal $\mathcal{I} = \langle H \rangle$, if $\operatorname{lt}(H)$ is a weak L-involutive basis of $\operatorname{lt}(\mathcal{I})$. For a (strong) L-involutive basis, we require in addition that $\operatorname{lt}(H)$ is a strong L-involutive basis and that all generators $h \in H$ have pairwise disjoint leading monomials. We assign to each polynomial $h \in H$ the multiplicative variables $M_L(\operatorname{lt}(h), \operatorname{lt}(H))$ and define the involutive cone $\mathcal{C}_{L,H,\prec}(h) := h\mathbb{K}[M_L(\operatorname{lt}(h), \operatorname{lt}(H))]$. A strong involutive basis H of an ideal \mathcal{I} induces then a disjoint decomposition $\mathcal{I} = \bigoplus_{h \in H} \mathcal{C}_{L,H,\prec}(h)$ as \mathbb{K} -linear spaces. In particular, each ideal element $f \in \mathcal{I}$ has a unique involutive standard representation $f = \sum_{h \in H} p_h \cdot h$, in which the coefficients $p_h \in \mathbb{K}[M_L(\operatorname{lt}(h), \operatorname{lt}(H))]$ additionally fulfil $\operatorname{lt}(p_h) \cdot \operatorname{lt}(h) \leq \operatorname{lt}(f)$. H is a minimal L-involutive basis of $\operatorname{lt}(\mathcal{I})$. If G is an involutive basis of the polynomial

ideal \mathcal{I} , then lt (G) is an involutive basis of the monomial ideal lt (\mathcal{I}). Thus any involutive basis is also a Gröbner basis.

Two involutive divisions are particularly important in applications: The Janet and Pommaret divisions.

The Janet division was, like the Pommaret division, already introduced by Janet [15, pp. 16-17]. Let $U \subset \mathcal{T}$ be a finite set of terms. For each sequence d_1, \ldots, d_n of non-negative integers and for each index $1 \le i \le n$, we introduce the corresponding Janet class as the subset

$$U_{[d_{i},...,d_{n}]} = \left\{ u \in U \mid \deg_{j}(u) = d_{j}, \ i \le j \le n \right\} \subseteq U .$$
(2.1)

The variable x_n is called *Janet multiplicative* (*J*-multiplicative) for the term $u \in U$, if it holds $\deg_n(u) = \max \{\deg_n(v) \mid v \in U\}$. For i < n, x_i is Janet multiplicative for $u \in U_{[d_{i+1},...,d_n]}$, if $\deg_i(u) = \max \{\deg_i(v) \mid v \in U_{[d_{i+1},...,d_n]}\}$. The Janet division is Noetherian, continuous, and constructive. We sometimes write MinJB(\mathcal{I}) for the minimal Janet basis of a given monomial ideal \mathcal{I} . We write $M_J(u, U)$ for the set of Janet multiplicative variables of a term $u \in U$, and by $NM_J(u, U)$ we denote the non-multiplicative variables.

Example 2.5. In the polynomial ring $\mathbb{K}[x_1, x_2, x_3]$, consider the ideal $\mathcal{I} = \langle x_1 x_3^2, x_2 x_3, x_1^2 x_3 \rangle$. The given minimal generating set is not a Janet basis of \mathcal{I} , but enlarging the generating set by the term $x_2 x_3^2$, we obtain the Janet basis $\{x_1 x_3^2, x_2 x_3, x_1^2 x_3, x_2 x_3^2\}$ of \mathcal{I} .

We now proceed to the Pommaret division. The *class* of a term $1 \neq x^{\mu} \in \mathcal{T}$ with $\mu = (\mu_1, \ldots, \mu_n)$ is defined as the index $\operatorname{cls}(x^{\mu}) = \min\{i \mid \mu_i \neq 0\}$. A variable x_i is Pommaret multiplicative for x^{μ} , if $i \leq \operatorname{cls}(x^{\mu})$. All variables are Pommaret multiplicative for the trivial term 1. We write $M_P(u)$ for the set of Pommaret multiplicative variables of a term $u \in \mathcal{T}$, and by $\operatorname{NM}_P(u)$ we denote the non-multiplicative variables. Note that the thus defined *Pommaret division* is global, i. e. the assignment of multiplicative variables is independent of any finite set $U \subset \mathcal{T}$. In contrast to the Janet division, the Pommaret division is not Noetherian, as e. g. the ideal $\mathcal{I} = \langle x_1 x_2 \rangle$ does not possess a finite Pommaret basis (it does not contain an element of class 2). Nevertheless, the Pommaret division is continuous and constructive. If a monomial ideal \mathcal{I} possesses a Pommaret basis, we sometimes write MinPB(\mathcal{I}) for its minimal Pommaret basis.

For sufficiently large fields \mathbb{K} , non-Noetherianity of the Pommaret division is only a problem of the used coordinates. After a generic linear change of variables any ideal $\mathcal{I} \subseteq \mathcal{P}$ admits a finite Pommaret basis [21, Thm. 4.3.15]. In this case, \mathcal{I} is said to be in *quasi-stable position*. An in-depth study of this question can be found in [11] together with a deterministic algorithm for the explicit construction of "good" coordinates for any given ideal $\mathcal{I} \subset \mathcal{P}$. For Pommaret bases, we will always consider the degree reverse lexicographical ordering < with $x_1 < \cdots < x_n$, as it is the only classrespecting term ordering [21, Lem. A.1.8]. As generally a monomial ideal does not remain monomial after a linear change of variables, Pommaret bases exist only for a special class of monomial ideals.

Definition 2.6. A monomial ideal \mathcal{I} is called *quasi-stable*, if for any term $x^{\mu} \in \mathcal{I}$ and for any index i with $\operatorname{cls}(x^{\mu}) < i \leq n$ an exponent $s \geq 0$ exists such that $x_i^s x^{\mu} / x_{\operatorname{cls}(x^{\mu})} \in \mathcal{I}$. A polynomial ideal \mathcal{I} is in *quasi-stable position*, if $\operatorname{lt}(\mathcal{I})$ is quasi-stable.

Quasi-stable ideals appear in many places (and are known under many different names like *ideals of Borel type, ideals of nested type* or *weakly stable ideals*). Besides the above combinatorial definition, they can be characterised by many algebraic properties. For our purposes, the following characterisation is relevant.

Proposition 2.7 ([21, Prop. 5.3.4]). A monomial ideal \mathcal{I} possesses a finite Pommaret basis, if and only if it is quasi-stable.

There are two important generalisations of the concept of involutive divisions: Firstly, *relative involutive divisions*, which are defined relative to a given monomial ideal $\mathcal{I} \neq \{0\}$. Given a usual

involutive division L, one can derive its relative counterpart $L_{\mathcal{I}}$, which then induces a theory of involutive bases in the quotient ring \mathcal{P}/\mathcal{I} . The details are documented in [8, Sec. 5]. Secondly, *involutive-like divisions* are defined by the assignment of non-multiplicative pure variable powers instead of non-multiplicative variables [9, Sec. 6]. The prototype of an involutive-like division is the Janet-like division [7, 6]:

Definition 2.8. Let $U \subset \mathcal{T}$ be a finite set of terms. For any term $u \in U$ and a Janet non-multiplicative variable $x_i \in \text{NM}_J(u, U)$, the power $x_i^{k_i}$ with

$$k_i = \min \{ \deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1}, \dots, d_n]}, \deg_i(v) > \deg_i(u) \}$$

is called a *non-multiplicative* power of u for the *Janet-like division*. The set of all non-multiplicative powers of $u \in U$ is denoted by NMP(u, U). The elements of the set

$$NM(u, U) = \{ v \in \mathcal{T} \mid \exists w \in NMP(u, U) : w \mid v \}$$

are called the *J*-non-multipliers for $u \in U$. The terms outside of it are the *J*-multipliers for u. An element $u \in U$ will be called a *Janet-like divisor* of $w \in \mathcal{T}$, if $w = u \cdot v$ with v a *J*-multiplier for u.

A Janet-like head autoreduced and finite set $U \subset \mathcal{T}$ is called *Janet-like basis* of the monomial ideal $\langle U \rangle$, if every term $t \in \langle U \rangle \cap \mathcal{T}$ has a Janet-like divisor in U. A finite set of polynomials $F \subset \mathcal{P} \setminus \{0\}$ is a *Janet-like basis* of $\mathcal{I} = \langle F \rangle$, if we have $\operatorname{lt}(f) \neq \operatorname{lt}(g)$ for all $f \neq g \in F$ and $\operatorname{lt}(F)$ forms a Janet-like basis for $\operatorname{lt}(\mathcal{I})$.

The Pommaret-like division was defined in [9, Def. 6.11].

Definition 2.9. The *Pommaret-like division* P assigns to each term $t \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T}$ non-multiplicative powers as follows:

- 1. The Janet non-multiplicative variables x_a with a > cls(t),
- 2. The Janet non-multiplicative powers $x_b^{p_b}$ with b > cls(t).

Note that no non-multiplicative power is assigned to any variable x_b with $b \leq \operatorname{cls}(t)$.

Let $F = {\mathbf{f}_1, \dots, \mathbf{f}_r} \subseteq \mathcal{P}^s$ be an enumerated finite subset of a finitely generated free module \mathcal{P}^s . The syzygy module of F is a submodule of \mathcal{P}^r defined by

$$\operatorname{Syz}(F) = \left\{ (g_1, \dots, g_r)^T \in \mathcal{P}^r \mid \sum_{i=1}^r g_i \mathbf{f}_i = 0 \right\}.$$

For subsets $F \subseteq (\mathcal{P}/\mathcal{I})^s$, we write $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(F)$ to emphasize that we are working over the quotient ring.

We use syzygies to construct free resolutions of homogeneous ideals $\mathcal{I} \leq \mathcal{P}$. A free resolution **F** of \mathcal{I} is given by finitely generated free \mathcal{P} -modules F_0, F_1, \ldots and homogeneous \mathcal{P} -linear maps $\delta_0, \delta_1, \delta_2, \ldots$ as in the following diagram

$$\mathbf{F}: \ \cdots \xrightarrow{\delta_{m+2}} F_{m+1} \xrightarrow{\delta_{m+1}} F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{I} \to 0,$$

such that $\operatorname{im}(\delta_0) = \mathcal{I}$ and $\operatorname{im}(\delta_{m+1}) = \operatorname{ker}(\delta_m)$ for all $m \in \mathbb{N}_0$. The collection $\{\delta_m\}_{m\geq 0}$ of maps is called the *differential* of the resolution. Leaving aside degree shifts, we can write $F_m = \mathcal{P}^{r_m}$ for $m \geq 0$. Each map δ_m is completely described by the images $\delta(\mathbf{e}_i)$, $i \in \{1, \ldots, r_m\}$; equivalently, δ_m is represented by a matrix $D_m \in \mathcal{P}^{r_{m-1} \times r_m}$, whose *i*th column is exactly $\delta_m(\mathbf{e}_i)$. (Note that we interpret the module \mathcal{I} as a submodule of \mathcal{P}^1 , so the matrix D_0 describing δ_0 is of format $(r_0 \times 1)$.) Moreover, $D_m \cdot D_{m+1} = 0$ for all m.

The above discussion now implies the next observation: $G := \{\delta_0(\mathbf{e}_1), \ldots, \delta_0(\mathbf{e}_{r_0})\}$ is a homogeneous generating set of \mathcal{I} and the columns of D_1 form a homogeneous generating set G_1 of Syz(G). Generally, the set G_m of columns of D_m is a homogeneous generating set of the iterated syzygy module $Syz^m(G)$.

Finally, we note that also for ideals $\mathcal{J} \leq \mathcal{P}/\mathcal{I}$ in a quotient ring over a homogeneous ideal \mathcal{I} , resolutions by finitely generated free \mathcal{P}/\mathcal{I} -modules exist. For these resolutions, $\delta_0(F_0) = \mathcal{J}/\mathcal{I}$ and all modules $F_m, m \geq 0$, are direct sums of copies of \mathcal{P}/\mathcal{I} . Otherwise, the terminology is the same.

Since we work with homogeneous ideals \mathcal{I} , the matrices in any free resolution of \mathcal{I} have homogeneous polynomials as entries. A free resolution is *minimal* if all entries in the matrices are either 0 or of positive degree. Up to isomorphism, there is exactly one minimal free resolution for each ideal \mathcal{I} . Since the ranks of the involved free modules F_m in a minimal free resolution are invariant under isomorphisms, they are a homological invariant of \mathcal{I} . They are called (*bigraded*) *Betti numbers* of \mathcal{I} .

Assume that in a minimal free resolution \mathbf{F} of \mathcal{I} , $F_m = \bigoplus_{d \ge 0} \mathcal{P}(-d)^{\beta_{m,d}}$ for m > 1; then the numbers $\beta_{m,d} = \beta_{m,d}(\mathcal{I})$ are the Betti numbers of \mathcal{I} . By Hilbert's syzygy theorem, the minimal free resolution of $\mathcal{I} \trianglelefteq \mathcal{P}$ is of finite length. Thus, the collection $\{\beta_{m,d}(\mathcal{I})\}_{m,d \ge 0}$ of non-zero Betti numbers of \mathcal{I} is finite. By minimality of \mathbf{F} , the sequence $(\min\{d \ge 0 \mid \beta_{m,d}(\mathcal{I}) > 0\})_{m \ge 0}$ is increasing; thus we can present the non-zero Betti numbers in a matrix $(b_{d,m})_{0 \le d \le r, 0 \le m \le s} = (\beta_{m,d+m}(\mathcal{I})) \in \mathbb{N}_0^{(r+1)\times(s+1)}$ for some positive integers $s = s(\mathcal{I}), r = r(\mathcal{I})$, such that there are neither trailing zero rows nor trailing zero columns.

Consider a homogeneous ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ and a finimal free resolution of it, yielding the numbers $r(\mathcal{I})$ and $s(\mathcal{I})$ of rows and columns in its Betti table. Then $\operatorname{reg}(\mathcal{I}) := r(\mathcal{I})$ is the *Castelnuovo-Mumford regularity*, or simply *regularity*, of \mathcal{I} , and $\operatorname{projdim}(\mathcal{I}) := s(\mathcal{I})$ is its *projective dimension*.

The minimal \mathcal{P}/\mathcal{I} -free resolutions of homogeneous ideals $\mathcal{J} \leq \mathcal{P}/\mathcal{I}$ are in general infinite in the sense that inifinitely many non-zero Betti numbers exist. Thus, ideals in \mathcal{P}/\mathcal{I} in general do not have finite regularity or projective dimension. As a succinct way of writing the infinitely many Betti numbers, we use *Poincaré series*. They are formal power series in two independent variables—u and s, say—such that the coefficient of a term $u^m s^d$ is given by the Betti number $\beta_{m,d}(\mathcal{J})$.

Consider an *L*-involutive basis $H \,\subset\, \mathcal{P}$ of a polynomial ideal $\mathcal{I} = \langle H \rangle$ with respect to a continuous involutive division *L*. The set lt(H) is a strong *L*-involutive basis of the leading ideal $lt(\mathcal{I})$. One can construct an acyclic directed graph, the *L*-graph, with node set lt(H) and arrows from $lt(h_i)$ to $lt(h_j)$ whenever there is a non-multiplicative variable $x \in \mathrm{NM}_L(\mathrm{lt}(h_i), \mathrm{lt}(H))$ such that $lt(h_j)$ is an *L*-divisor of $x \, \mathrm{lt}(h_i)$ [21, Lem. 5.4.5]. Now consider the following method of enumerating lt(H): As first element $lt(h_1)$, take any leading term whose node in the *L*-graph is not the target of any arrow. Deleting $lt(h_1)$ and its associated arrows from the graph, we obtain another acyclic graph, and as the second element $lt(h_2)$ in the enumeration we take a leading term whose node is not the target of any arrow in the modified graph. Continuing in this manner, we obtain an *L*-ordering of lt(H).

Adapting a construction due to Schreyer [18], one can use the *L*-involutive basis *H*, ordered according to an *L*-ordering, to construct a Gröbner basis G_{Syz} of Syz(H) that has as leading terms exactly the module terms xe_i , where $x \in NM_L(lt(h_i), lt(H))$. If *L* is of *Schreyer type* [21, Def. 5.4.8], then G_{Syz} is again an *L*-involutive basis, and the construction can be iterated to yield a linear, but generally non-minimal, free resolution of $\langle H \rangle$. The Pommaret and Janet divisions are of Schreyer type [21, Lem. 5.4.9]. We will use Schreyer-type constructions also for relative involutive bases.

The resolution induced by the Pommaret basis of a homogeneous ideal \mathcal{I} in quasi-stable position can be used to determine the Castelnuovo-Mumford regularity and projective dimension of \mathcal{I} without computing the minimal free resolution of \mathcal{I} . The Castelnuovo-Mumford regularity is simply the largest degree of a generator in the Pommaret basis; the projective dimension is the maximal number of non-multiplicative variables that an element of the Pommaret basis can have. For further details, see [21, Sec. 5.5].

For a quasi-stable monomial ideal \mathcal{I} , we refer to [21, Thm. 5.4.18] for an explicit formula for the differential of the resolution induced by the monomial Pommaret basis. It is immediate from [21, Eq. (5.53)] that the resolution is minimal if and only if \mathcal{I} is stable. The formula can be read off from

the weighted *P*-Graph of the basis, which includes for each arrow $h_i \rightarrow h_j$ not only the variable $x \in NM_P(h_i)$ with $xh_i \in C_P(h_j)$, but also the cofactor $t \in \mathbb{K}[M_P(h_j)]$ such that $xh_i = th_j$.

3. Resolutions induced by relative Pommaret bases

Let us recall the definition from [8] of the concept of an involutive division $L_{\mathcal{I}}$ relative to a monomial ideal \mathcal{I} of *Schreyer type*. As our aim is to define a related notion better suited to the computation of free resolutions, we repeat it here for the reader's convenience.

Definition 3.1. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal and $L_{\mathcal{I}}$ an involutive division relative to $\operatorname{lt}(\mathcal{I})$ induced by a continuous involutive division L on \mathcal{T} . We say that $L_{\mathcal{I}}$ is of *Schreyer type* if, whenever H is a strong $L_{\mathcal{I}}$ -involutive basis of $\langle H \rangle + \mathcal{I}$ relative to \mathcal{I} and G is a strong L-involutive basis of \mathcal{I} , we have that for all $x^{\mu} \in \operatorname{lt}(H)$ the monomial set

$$B = \left(\left\{ \frac{\operatorname{lcm}(x^{\nu}, x^{\mu})}{x^{\mu}} \mid x^{\nu} \in \operatorname{lt}(G) \right\} \setminus \operatorname{lt}(\mathcal{I}) \right) \cup \left(\operatorname{NM}_{L_{\mathcal{I}}}(x^{\mu}, \operatorname{lt}(H)) \right)$$
(3.1)

is an $L_{\mathrm{lt}(\mathcal{I})}$ -involutive basis of the ideal $\langle B \rangle + \mathrm{lt}(\mathcal{I})$ relative to $\mathrm{lt}(\mathcal{I})$.

The following example shows that Definition 3.1 is not optimal:

Example 3.2. In $\mathcal{P} = \mathbb{K}[x, y]$, consider the following ideals (see [16, Ex. 5.2]): $\mathcal{I} = \langle x^3, y^3 \rangle$, $\mathcal{J} = \langle x^2, xy, y^2 \rangle$. Note that the monomial ideals \mathcal{I} and \mathcal{J} are quasi-stable. The minimal Pommaret basis of \mathcal{I} is $G = \{x^3, x^3y, x^3y^2, y^3\}$ and the minimal Pommaret basis of \mathcal{J} relative to \mathcal{I} is $H = \{x^2, xy, y^2\}$. We can now apply [8, Prop. 5.14] to obtain a Pommaret basis for $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H)$ —note that G and H are already ordered according to a $P_{\mathcal{I}}$ ordering. Precisely, the enumerations are $g_1 = x^3, g_2 = x^3y, g_3 = x^3y^2, g_4 = y^3$ and $h_1 = x^2, h_2 = xy, h_3 = y^2$.

Using [8, Thm. 5.13], we compute a Pommaret basis of the first syzygy module of H relative to \mathcal{I} , being a subset of the free \mathcal{P}/\mathcal{I} -module $(\mathcal{P}/\mathcal{I})^3$ with the canonical basis $\{\mathbf{e}_1^{(1)}, \mathbf{e}_2^{(1)}, \mathbf{e}_3^{(1)}\}$ (the superscript encodes the homological degree.) We underline the leading module terms.

- As A-syzygies, we obtain $\mathbf{A}_1 = \underline{x} \mathbf{e}_1^{(1)}$, $\mathbf{A}_2 = \underline{x} y \mathbf{e}_1^{(1)}$, and $\mathbf{A}_3 = \underline{x} y^2 \mathbf{e}_1^{(1)}$ for h_1 , $\mathbf{A}_4 = \underline{x}^2 \mathbf{e}_2^{(1)}$, $\mathbf{A}_5 = x^2 y \mathbf{e}_2^{(1)}$, and $\mathbf{A}_6 = y^2 \mathbf{e}_2^{(1)}$ for h_2 , as well as $\mathbf{A}_7 = \underline{y} \mathbf{e}_3^{(1)}$ for h_3 .
- As syzygies from non-multiplicative prolongations, we obtain $\mathbf{S}_1 = \underline{y} \mathbf{e}_1^{(1)} x \mathbf{e}_2^{(1)}$ for h_1 and $\mathbf{S}_2 = y \mathbf{e}_2^{(1)} x \mathbf{e}_3^{(1)}$ for h_2 .

We notice immediately that the relative Pommaret basis $\{A_1, \ldots, A_7, S_1, S_2\}$ is not minimal, because the leading terms of the syzygies A_2 , A_3 , A_5 , A_6 are redundant.

As seen in Example 3.2, for relative Pommaret divisions, Definition 3.1 implies that the relative Pommaret bases computed for the syzygy modules of an \mathcal{P}/\mathcal{I} -free resolution are generally non-minimal. What is more, relative Janet divisions are in general *not* of Schreyer type if one applies Definition 3.1 — see Example 5.15 of [8]. While it is true that the definition ensures that relative divisions of Schreyer type are suitable for the computation of free resolutions, the construction is not optimal. The reason for this is that the set *B* in Equation (3.1) is not chosen optimally. Indeed, it is in general not autoreduced with respect to classical (non-restricted) division, because, the multipliers of the form $lcm(x^{\nu}, x^{\mu})/x^{\mu}$, which are needed for the *A*-syzygies, may be divisible by non-multiplicative variables. Thus, it is natural to propose the following adapted definition:

Definition 3.3. Let \mathcal{I} and $L_{\mathcal{I}}$ be as in Definition 3.1. Then $L_{\mathcal{I}}$ is of *strong Schreyer type* if, whenever H is a strong $L_{\mathcal{I}}$ -involutive basis of $\langle H \rangle + \mathcal{I}$ relative to \mathcal{I} and G is a strong L-involutive basis for \mathcal{I} , then for all $x^{\mu} \in \text{lt}(H)$, the set

$$M(x^{\mu}, \operatorname{lt}(H), \operatorname{lt}(G)) \cup \operatorname{NM}_{L_{\mathcal{I}}}(x^{\mu}, \operatorname{lt}(H))$$

is a strong $L_{\mathcal{I}}$ -involutive basis for the monomial ideal it generates relative to $\operatorname{lt}(\mathcal{I})$ where the set of *multiplicative A-multipliers* $M(x^{\mu}, \operatorname{lt}(H), \operatorname{lt}(G))$ is defined by

$$M(x^{\mu}, \operatorname{lt}(H), \operatorname{lt}(G)) = \left\{ \frac{\operatorname{lcm}(x^{\nu}, x^{\mu})}{x^{\mu}} \mid x^{\nu} \in \operatorname{lt}(G) \right\} \setminus \left(\operatorname{lm}(\mathcal{I}) + \left\langle \operatorname{NM}_{L_{\mathcal{I}}}(x^{\mu}, \operatorname{lt}(H)) \right\rangle \right).$$
(3.2)

Note that the set $M(x^{\mu}, \operatorname{lt}(H), \operatorname{lt}(G)) \cup \operatorname{NM}_{L_{\mathcal{I}}}(x^{\mu}, \operatorname{lt}(H))$ from Definition 3.3 is a subset of the set *B* defined in Equation (3.1).

Proposition 3.4. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal in quasi-stable position and P the Pommaret division on \mathcal{T} . Then the relative involutive division $P_{lt(\mathcal{I})}$ induced by P is of strong Schreyer type.

Proof. Let G be the strong Pommaret basis of \mathcal{I} and let H be a strong Pommaret basis of the ideal $\langle H \rangle + \mathcal{I}$ relative to \mathcal{I} . For each $x^{\mu} \in lt(H)$, we have to show that the set $B' := M(x^{\mu}, lt(H), lt(G)) \cup$ $NM_{P_{lt(\mathcal{I})}}(x^{\mu})$ is a strong Pommaret basis of the ideal it generates relative to $lt(\mathcal{I})$. We know that $B' \subseteq B$, where B is defined as in Equation (3.1). Moreover, from the definitions, it is easy to see that $\langle B, \operatorname{lt}(\mathcal{I}) \rangle = \langle B', \operatorname{lt}(\mathcal{I}) \rangle$. We still have to show that B' is a strong relative Pommaret basis. Note that B is a weak Pommaret basis and each term in $t \in B \setminus B'$ is divisible by a variable x_i with $j > \operatorname{cls}(x^{\mu})$, i.e., by a non-multiplicative variable for x^{μ} . Assume that j is the maximal index having this property. Then, $t \in \mathcal{C}_P(x_i)$. We can deduce that B' is also a weak Pommaret basis. Also, it is clear that the Pommaret cones $C_P(x_i)$ and $C_P(t)$, where $x_i \in NM_P(x^{\mu})$ and $t \in \mathbb{K}[M_P(x^{\mu})] \cap B'$, have empty intersection (look at the x_i -degrees). Finally, we need to show that all Pommaret cones $\mathcal{C}_P(t), \mathcal{C}_P(s)$, where $s \neq t \in \mathbb{K}[M_P(x^{\mu})] \cap B'$, have empty intersection. For this, first note that $\operatorname{cls}(s) = \operatorname{cls}(\overline{s})$ and $\operatorname{cls}(t) = \operatorname{cls}(\overline{t})$, where $\overline{s}, \overline{t} \in \operatorname{lt}(G)$ are the terms inducing the multipliers s, t for x^{μ} . Indeed, s and t are Pommaret multiplicative for x^{μ} , and so the only indices i for which $\deg_i(\bar{s}) > \deg_i(x^{\mu})$ is possible are indices $i \leq \operatorname{cls}(x^{\mu})$. And there must be an index with this property, as otherwise $\overline{s}|x^{\mu}$, which is impossible. The minimal such index then obviously is just $cls(\bar{s})$, and similarly for \bar{t} . Hence, a non-empty intersection of the Pommaret cones of s and t would imply a non-empty intersection of the Pommaret cones \overline{s} and \overline{t} . This is impossible, because $\overline{s}, \overline{t}$ are elements of the strong Pommaret basis of the ideal $lt(\mathcal{I})$.

Remark 3.5. The Janet division $J_{\mathcal{I}}$ relative to a monomial ideal \mathcal{I} is not of strong Schreyer type, even when \mathcal{I} is quasi-stable: Consider $\mathcal{I} = \langle x_1^2 x_3^2, x_3^3, x_2 x_3^2 \rangle = \langle G \rangle$ and $H = \{x_1, x_2\}$. \mathcal{I} is a stable ideal, and H is a Janet basis relative to \mathcal{I} . Consider the term $x^{\mu} = x_1$. The set $M(x^{\mu}, H, G) \cup$ $\mathrm{NM}_{J_{\mathcal{I}}}(x^{\mu}, H)$ is given by $\{x_1 x_3^2\} \cup \{x_2\}$. This is not a Janet basis relative to \mathcal{I} , because the variable x_3 is non-multiplicative for x_2 in this set, but $x_2 x_3$ does not possess a Janet divisor in the same set.

Proposition 3.4 ensures that we get minimal Pommaret bases in each step of the resolution computation. We use Schreyer orderings for these Pommaret bases, which depend on P-orderings (i.e., orderings adapted to the Pommaret-involutive structure). There is an easy procedure by which P-orderings can be obtained automatically for the next syzygy module. Indeed, for any given generator of the current module, we need to take first the multiplicative A-syzygies in the order that is induced by the ordering on G. Then we take the non-multiplicative variables in ascending order. We do this for each generator sequentially, and we obtain a minimal Pommaret basis, already P-ordered, for the next syzygy module.

Example 3.6. As in Example 3.2 consider $\mathcal{I} = \langle x^2, xy, y^2 \rangle$, $\mathcal{J} = \langle x^3, y^3 \rangle = \langle x^3, x^3y, x^3y^2, y^3 \rangle$. Applying Proposition 3.4 repeatedly, we obtain relative Pommaret bases for the next iterated syzygy modules as follows. Note that that the columns of D_k represent the minimal relative Pommaret basis for the *k*-th iterated syzygy module.

Remark 3.7. Example 3.6 shows that constants can appear in some homological degree i of the Pommaret-induced resolution even if there are no constants in the differential at the previous homological degree i - 1. This is for instance the case for the homological degree 4 in Example 3.6: the matrix D_4 contains constants even though D_3 does not. This behaviour of the induced resolution is new, compared to the Pommaret-induced resolutions for \mathcal{P} -modules. Compare [21, Lem. 5.5.1], where it is shown that a Pommaret-induced resolution over \mathcal{P} is minimal if and only if the first differential does not contain any constant terms.

Example 3.8. In $\mathcal{P} = \mathbb{K}[x, y, z]$, consider the ideals $\mathcal{I} = \langle z^3 \rangle$, $\mathcal{J} = \langle xyz, y^2z, yz^2, \mathcal{I} \rangle$. With the usual notation, we verify by computation that the Pommaret-induced resolution is the minimal \mathcal{P}/\mathcal{I} -free resolution of the \mathcal{P}/\mathcal{I} -module \mathcal{J} :

$$D_{0} = \begin{pmatrix} xyz & y^{2}z & yz^{2} \end{pmatrix},$$

$$D_{1} = \begin{pmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

$$D_{2} = \begin{pmatrix} z & 0 & 0 & 0 \\ -y & z^{2} & 0 & 0 \\ x & 0 & z^{2} & 0 \\ 0 & xz & yz & z^{2} \end{pmatrix},$$

$$D_{3} = \begin{pmatrix} z^{2} & 0 & 0 & 0 \\ y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

$$D_{4} = D_{2}.$$

4. Resolutions induced by Pommaret-like bases

In order to analyse free resolutions induced by Pommaret-like bases, a necessary first step is to understand resolutions over the polynomial ring \mathcal{P} of quasi-stable monomial ideals \mathcal{I} generated by a Pommaret-like basis H. If H is also a Pommaret basis, then the structure of the induced resolution is known; for results about this, see [21]. In the case that \mathcal{I} is a monomial ideal, the resolution is minimal if and only if H is the minimal generating set of \mathcal{I} , see [21]. Note that if \mathcal{I} is not monomial, this result does not hold in general, see e.g. [21, Ex. 5.5.9]. As a first step to a similar result for Pommaret-like bases, some combinatorial characterization of monomial ideals whose minimal generating set is also a Pommaret-like basis may be helpful.

Remark 4.1. If a Pommaret-like basis H of a monomial ideal \mathcal{I} is given, then ordering the elements ascendingly with respect to the lexicographic ordering with $x_1 < \cdots < x_n$ gives a P-ordering, because for each $h \in H$ and $x_j^{p_j} \in \text{NMP}_P(h, H)$, the Pommaret-like divisor $u \in H$ of $x_j^{p_j} \cdot h$ fulfills $\deg_j(u) = \deg_j(h) + p_j$ and $\deg_\ell(u) = \deg_\ell(h)$ for $\ell > j$. Thus, $h <_{\text{lex}} u$. From this P-ordering, one can derive a Schreyer ordering in the syzygy module which has non-multiplicative powers as leading terms.

A Pommaret-like basis H of an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ in quasi-stable position induces a free resolution of \mathcal{I} over \mathcal{P} , and at each homological degree, the corresponding syzygy module is generated by a Pommaret-like basis [9]. There are special classes of ideals for which this induced resolution is in fact the minimal free resolution. One class of ideals for which this is true is the class of *componentwise linear ideals* (provided that the ideal is in componentwise quasi-stable position [11, Thm. 19]). We can apply [21, Thm. 5.5.2] to see this, even though that result is concerned with Pommaret bases, because Pommaret bases are a special kind of Pommaret-like bases. Moreover, for *stable* monomial ideals the induced Pommaret-like resolution is also minimal because the Pommaret resolution is [21, Prop. 5.5.6]. The following result shows that the class of monomial ideals for which the Pommaretlike resolution is minimal is larger than the class of ideals for which the Pommaret resolution is minimal i.e., stable monomial ideals:

Theorem 4.2. Let $\mathcal{I} \leq \mathcal{P}$ be a quasi-stable monomial ideal (with $\mathcal{I} \notin \{\{0\}, \mathcal{P}\}$) generated by the minimal Pommaret-like basis $H \subset \mathcal{I} \cap \mathcal{T}$. Assume that H is simultaneously the minimal monomial generating set of \mathcal{I} . Moreover, assume that for each Pommaret-like non-multiplicative power $x_j^{p_j}$ of t with respect to the set H, it holds $(t/x_{\operatorname{cls}(t)})x_j^{p_j} \in \mathcal{I}$. Then the free resolution of \mathcal{I} over \mathcal{P} induced by the basis H is the minimal free resolution of \mathcal{I} over \mathcal{P} .

Before we prove Theorem 4.2, we need the following lemma.

Lemma 4.3. Let $H \subset \mathcal{T}$ be a minimal Pommaret-like basis generating the ideal $\mathcal{I} = \langle H \rangle$. The condition $(x_j^{p_j} \cdot t)/x_{\operatorname{cls}(t)} \in \mathcal{I}$ in Theorem 4.2 is equivalent to the statement that the unique Pommaret-like divisor $s \in H$ of $x_j^{p_j} \cdot t$ fulfills $\operatorname{cls}((x_j^{p_j} \cdot t)/s) \leq \operatorname{cls}(s)$.

Proof. Let $k := \operatorname{cls}(t)$ and let s be the unique Pommaret-like divisor in H of $(x_j^{p_j} \cdot t)$. Note that j > k. Moreover, since H is minimal, $f := (x_j^{p_j} \cdot t)/s \neq 1$. By the definition of Pommaret-like non-multiplicative powers, it is clear that $(x_j^{p_j} \cdot t)|_{x_1=\cdots=x_{j-1}=1} = s|_{x_1=\cdots=x_{j-1}=1}$. Hence we have $\operatorname{cls}(f) < j$. Now, if $\operatorname{cls}(f) \leq \operatorname{cls}(s)$, then $k = \operatorname{cls}(f)$, and x_k is multiplicative for s. From this we see $(x_j^{p_j} \cdot t)/x_k \in \mathcal{I}$. Conversely, if $\operatorname{cls}(f) > \operatorname{cls}(s)$, then $s \cdot f$ is an element of the minimal Pommaret basis of \mathcal{I} , and $k = \operatorname{cls}(s)$. Thus $(s \cdot f)/x_k \notin \mathcal{I}$ and consequently $(x_j^{p_j} \cdot t)/x_k \notin \mathcal{I}$.

Proof of Theorem 4.2. We need to show that no non-zero constant terms appear in the matrices describing the differential of the induced resolution. Write the resolution as

$$\mathbf{F}:\cdots \xrightarrow{d_3} \mathcal{P}^{b_2} \xrightarrow{d_2} \mathcal{P}^{b_1} \xrightarrow{d_1} \mathcal{P}^{|H|} \xrightarrow{d_0} \mathcal{I} \to 0.$$

The matrix D_0 describing d_0 consists of one row containing the elements of H as entries. Hence, no constant terms appear there. As the next step, we show that in the matrix D_1 describing d_1 there are no constant terms. By construction and using the fact that \mathcal{I} is a monomial ideal, each column of D_1 contains only two non-zero entries: $x_a^{p_a}$ (a non-multiplicative power of a term $t \in H$) and a cofactor $f \in \mathcal{T}$ such that

$$t \cdot x_a^{p_a} = s \cdot f, \tag{4.1}$$

where $s \in H$ is the unique term such that $t \cdot x_a^{p_a} \in C_P(s, H)$. Since the set H is by assumption the minimal monomial generating set of \mathcal{I} , we have $f \neq 1$. Hence, no column of D_1 contains any constant term and the whole matrix D_1 is free of constant terms.

The columns of D_1 represent a minimal Pommaret-like basis of the first syzygy module $\operatorname{Syz}(H) \subset \mathcal{P}^{|H|}$ of H. The leading module terms $x^{\mu} \mathbf{e}_i^{(1)}$ of this syzygy module are exactly of the form $x_a^{p_a} \mathbf{e}_i^{(1)}$ where $x_a^{p_a}$ is a non-multiplicative power of the i^{th} element of H. They are found in the i^{th} row of D_1 . There may be other non-zero entries in the said row, but they are cofactors f as given in Equation (4.1). Moreover, in the situation of Equation (4.1), it is clear that $\operatorname{cls}(t) \leq \operatorname{cls}(s)$ and hence, by Lemma 4.3,

$$\operatorname{cls}(f) \le \operatorname{cls}(s).$$
 (4.2)

From this it follows that $cls(f) < cls(x_b^{p_b})$ for all non-multiplicative powers $x_b^{p_b}$ of s. We will use this property in the next step.

The matrix D_2 has as many rows as D_1 has columns. Each column of D_2 contains at least the non-zero entry x_c^{pc} , a non-multiplicative power of a generator of the leading module of Syz(H). Since this leading term module is generated by module terms whose monomial parts are the nonmultiplicative powers of the set H, also the non-multiplicative powers of this leading term module will have monomial parts of the same form. These non-multiplicative powers are obviously not constants. The further non-zero entries of a column of D_2 result from the involutive-like standard representation of the vector $x_c^{p_c} \cdot \mathbf{c}$, where c is a column of D_1 , with respect to the set of all columns of D_1 . We focus on the possible non-zero entries that can be generated by the cancellations which happen in the *i*-th row. During the involutive-like reduction process, it can happen that an intermediate result has a non-zero entry there, but this entry will be of the form $f \cdot p$, where p is some polynomial and f is a term with the properties given in (4.2). In the column of D_2 encoding the involutive-like reduction we are studying at present, a non-zero entry (other than the one already analysed) can be created in row j only if the jth column \mathbf{c}_j of D_1 has as its leading module term $x_b^{p_b} \mathbf{e}_i^{(1)}$, where $x_b^{p_b}$ is as studied in Equation (4.2). The class condition given in Equation (4.2) now guarantees that the non-zero entry generated in the j^{th} row of the column of D_2 will be free of constant terms. What is more, all terms in the support of this entry will have class less or equal to cls(f). Now, since the indices i and j in the discussion above were arbitrary, we have proved that also the matrix D_2 does not contain any constant terms.

The last thing we need to prove is that, also in D_2 , we have a condition on the classes of terms analogous to that given in Equation (4.2). If we can show this, then an iteration of the arguments used for the analysis of D_2 can be applied to all successive matrices in the resolution.

To prove this class condition, again consider the j^{th} row of D_2 , where a non-zero entry q with $\operatorname{cls}(q) \leq \operatorname{cls}(f)$ is located resulting from a step in an involutive-like reduction which uses the leading module term $x_b^{p_b} \mathbf{e}_i^{(1)}$ of the j^{th} column of D_1 . We need to compare this class with the classes of all leading module terms of $\operatorname{Syz}^2(H)$ of the form $u \cdot \mathbf{e}_j^{(2)}$. But these leading module terms arise from non-multiplicative powers of the leading module term $x_b^{p_b} \mathbf{e}_i^{(1)}$ in the leading module of $\operatorname{Syz}(H)$, and hence $u = x_d^{p_d}$ for some index $b < d \le n$. Using Equation (4.2), it is now clear that $\operatorname{cls}(q) \le \operatorname{cls}(f) < \operatorname{cls}(x_b^{p_b}) < \operatorname{cls}(x_d^{p_d})$, i.e., the class condition we need is fulfilled.

We continue by giving two examples for minimal free resolutions induced by Pommaret-like bases.

Example 4.4. In this example, we show that the class of monomials satisfying the conditions of Theorem 4.2 is larger than the class of monomial ideals for which we can construct minimal free resolutions as proved in [21]. Let $a, b, c \ge 1$ be any three positive integers and let $\mathcal{I} = \langle x^a, y^b, z^c \rangle$ be an irreducible monomial ideal given by its minimal generating system $H = \{x^a, y^b, z^c\}$, which is easily seen to be also a Pommaret-like basis. Moreover, H satisfies the additional assumptions of Theorem 4.2. Hence, it induces a minimal Pommaret-like free resolution of \mathcal{I} . The matrices of the differentials are given as follows:

$$D_0 = \begin{pmatrix} x^a & y^b & z^c \end{pmatrix}, \ D_1 = \begin{pmatrix} y^b & z^c & 0 \\ -x^a & 0 & z^c \\ 0 & -x^a & -y^b \end{pmatrix}, \ D_2 = \begin{pmatrix} z^c \\ -y^b \\ x^a \end{pmatrix}.$$

Example 4.5. In the polynomial ring $\mathbb{K}[w, x, y, z]$ with $w \prec x \prec y \prec z$, consider the monomial ideal $\mathcal{I} = \langle H \rangle$ with

$$H = \{w^9x^3y^2z^2, x^5y^2z^2, w^7y^4z^2, x^3y^4z^2, y^6z^2, x^3y^2z^4, y^4z^4, z^8\}.$$

(The elements have been ordered lexicographically from lowest to highest.) One can verify that H is simultaneously the minimal generating system of \mathcal{I} and a Pommaret-like basis satisfying the additional assumptions of Theorem 4.2. Hence, it induces a minimal Pommaret-like free resolution of \mathcal{I} . The matrices of the differentials are given as follows:

We shall notice that Theorem 4.2 does not completely cover the class of quasi-stable monomial ideals whose Pommaret-like bases induce minimal free resolutions. In other words, there exist quasi-stable monomial ideals that do not satisfy the theorem's assumptions but whose Pommaret-like bases nevertheless induce the minimal free resolution:

Example 4.6. In the polynomial ring $\mathbb{K}[x, y, z]$, consider the monomial ideal \mathcal{I} with minimal generating set $G = \{xy, y^3, xz, y^2z, z^2\}$. As one can check, G is also a Pommaret-like basis. The generator t = xy has the non-multiplicative powers y^2 and z. While it is true that $(t/x) \cdot y^2 = y^3 \in \mathcal{I}$, we have $(t/x) \cdot z = yz \notin \mathcal{I}$. Only if we increase the exponent of the variable z to 2, i.e., higher than the non-multiplicative power, we reach the term $yz^2 \in \mathcal{I}$.

The Pommaret-like basis G induces a minimal free resolution with differential represented by the following matrices:

$$D_{0} = \begin{pmatrix} xy & y^{3} & xz & y^{2}z & z^{2} \end{pmatrix}, D_{1} = \begin{pmatrix} y^{2} & z & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ 0 & -y & 0 & y^{2} & z & 0 \\ 0 & 0 & -y & -x & 0 & z \\ 0 & 0 & 0 & 0 & -x & -y^{2} \end{pmatrix}, D_{2} = \begin{pmatrix} z & 0 \\ -y^{2} & 0 \\ x & 0 \\ -y & z \\ 0 & -y^{2} \\ 0 & x \end{pmatrix}.$$

We finish this section with a result that is useful for relating resolutions induced by Pommaretlike bases to other free resolutions.

Proposition 4.7. Let \mathcal{I} with $\{0\} \neq \mathcal{I} \neq \mathcal{P}$ be a polynomial ideal in quasi-stable position and let H be its minimal Pommaret-like basis. Then the free resolution induced by H consists of reduced Gröbner bases for all syzygy modules $\operatorname{Syz}^m(H)$, $m \ge 1$. In other words, in each homological degree $m \ge 1$, the set of columns of the matrix describing the differential is the unique reduced Gröbner basis of $\operatorname{Syz}^m(H)$ for the chosen module term order.

Proof. For $m \ge 1$ let D_m be the matrix in the induced free resolution that represents the differential map δ_m . By [9, Thm. 7.7], the set of columns of D_1 , say $C(D_1)$, is a Pommaret-like basis of Syz(H). The leading monomials of this basis are given by $Z := \{x_a^{p_a} \cdot \mathbf{e}_i \mid h_i \in H \land x_a^{p_a} \in$ $\text{NMP}_P(h_i, H)\}$, and $|C(D_1)| = |Z|$. Since Z is the minimal generating set for the module it generates then $C(D_1)$ is a minimal Gröbner basis. It is reduced because all non-leading module monomials arise as coefficients in an involutive-like reduction computation. More precisely, if the first syzygy generators induced by H do not form a reduced set, then there exists a Pommaret-like multiplicative term t for $x^{\mu_i} \in H$ which is divisible by a Pommaret-like non-multiplicative term s for x^{μ_i} , leading to a contradiction. Thus we have shown the claim for m = 1.

Note that $C(D_1)$ is in particular again a strong Pommaret-like basis. The claim now follows by induction on m, using [9, Thm. 7.7] in the induction step.

5. Relative involutive-like divisions

In this section, we present a generalization of the concepts of relative involutive divisions and involutive-like divisions. For a detailed explanation of relative involutive divisions, see [8]. For the definition and properties of involutive-like divisions, see [9, Sec. 6].

Definition 5.1. Let $\{0\} \neq \mathcal{I} \leq \mathcal{P}$ be a non-zero monomial ideal. An *involutive-like division* $L_{\mathcal{I}}$ relative to \mathcal{I} associates to any finite set $U \subset \mathcal{T} \setminus \mathcal{I}$ of terms and any term $u \in U$ a set of $L_{\mathcal{I}}$ -non-multipliers $\overline{L_{\mathcal{I}}}(u, U)$ given by the terms contained in an irreducible monomial ideal. The powers generating this irreducible ideal are called the *non-multiplicative powers* and the set of these powers is denoted by $\mathrm{NMP}_{L_{\mathcal{I}}}(u, U)$. The set of $L_{\mathcal{I}}$ -multipliers $L_{\mathcal{I}}(u, U)$ is given by the order ideal $\mathcal{T} \setminus \overline{L_{\mathcal{I}}}(u, U)$. For any term $u \in U$, its relative involutive-like cone is defined as $\mathcal{C}_{L_{\mathcal{I}}}(u, U) = u \cdot L_{\mathcal{I}}(u, U) \setminus \mathcal{I}$. For a

 \wedge

relative involutive-like division, the relative involutive-like cones must satisfy the following conditions:

- 1. For two terms $v \neq u \in U$ with $\mathcal{C}_{L_{\mathcal{I}}}(u,U) \cap \mathcal{C}_{L_{\mathcal{I}}}(v,U) \neq \emptyset$, we have $u \in \mathcal{C}_{L_{\mathcal{I}}}(v,U)$ or $v \in \mathcal{C}_{L_{\mathcal{I}}}(u,U)$.
- 2. If a term $v \in U$ lies in an involutive cone $\mathcal{C}_{L_{\mathcal{I}}}(u, U)$, then $L_{\mathcal{I}}(v, U) \subset L_{\mathcal{I}}(u, U)$.

Remark 5.2. Following [9, Def. 6.1], Definition 5.1 does not require a filter axiom.

It is straightforward to prove that from an involutive-like division L on \mathcal{T} , one can derive a relative involutive-like division $L_{\mathcal{I}}$ by using the same rule for the assignment of non-multiplicative powers as for L and merely adapting the cones to make them subsets of $\mathcal{T} \setminus \mathcal{I}$. One can do this, in particular, for the important special case of the Janet-like division J:

Definition 5.3. Let $\mathcal{I} \triangleleft \mathcal{P}$ be a non-zero monomial ideal and let $U \subseteq \mathcal{T} \setminus \mathcal{I}$ be a finite set of terms. Let $u \in U$ be a term. Then the non-multiplicative powers of u with respect to U, \mathcal{I} and the relative Janet-like division $J_{\mathcal{I}}$ are defined as follows:

$$\mathrm{NMP}_{J_{\mathcal{I}}}(u, U) = \mathrm{NMP}_{J}(u, U) \setminus (\mathcal{I} : u).$$
(5.1)

Therefore, and in other words, the relative Janet-like division uses the same rule for the assignment of non-multiplicative powers as the Janet-like division J, but it excludes variable powers that form part of the ideal quotient associated with the term u in question.

If x_a is a variable for which a relative Janet-like non-multiplicative power for u exists, then we write the exponent of this power as

$$p(J_{\mathcal{I}}, u, U, a).$$

Remark 5.4. The relative Janet-like division $J_{\mathcal{I}}$ is an involutive-like division relative to \mathcal{I} . This fact can be easily proven by using the properties of the Janet-like division J. Also, other properties like the continuity of the Janet-like division J are inherited by $J_{\mathcal{I}}$.

There are a number of different options how one could define the Pommaret-like division $P_{\mathcal{I}}$ relative to a monomial ideal \mathcal{I} . One possibility is using the same assignment of non-multiplicative powers that the Pommaret-like division P also employs. However, this is not an optimal choice for the definition. The definition should aim to guarantee that the following properties are fulfilled:

- 1. Cones should of course be disjoint if they are not contained in each other.
- 2. For $u \in U$, no non-multiplicative powers should be assigned for variables $x_1, \ldots, x_{cls(u)}$.
- For A ⊃ I, a relative Pommaret-like bases should exist if and only if A is quasi-stable relative to I.
- 4. A unique minimal relative Pommaret-like basis should exist for any monomial ideal A that is quasi-stable relative to I.
- 5. The minimal relative Pommaret-like basis should be as small as possible.

The following definition is designed to guarantee the above enumerated properties:

Definition 5.5. Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{P}$ be a non-zero monomial ideal. The *Pommaret-like division* $P_{\mathcal{I}}$ *relative to* \mathcal{I} assigns to each term $u \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T} \setminus \mathcal{I}$ non-multiplicative powers as follows: For each x_a with $a > \operatorname{cls}(u)$, if $x_a \in \operatorname{NM}_J(u, U)$, then set $p(P_{\mathcal{I}}, u, U, a) =$ $p(J_{\mathcal{I}}, u, U, a)$. If $x_a \in \operatorname{M}_J(u, U)$ and there does not exist any exponent $s \in \mathbb{N}$ with $u \cdot x_a^s \in \mathcal{I}$, set $p(P_{\mathcal{I}}, u, U, a) = 1$. No other variable gets assigned a non-multiplicative power with respect to the relative Pommaret-like division $P_{\mathcal{I}}$. In particular, no variable x_b with $b \leq \operatorname{cls}(u)$ is assigned a relative Pommaret-like non-multiplicative power for the term u.

Proposition 5.6. *The relative Pommaret-like division* $P_{\mathcal{I}}$ *is a relative involutive-like division.*

Proof. Let $u \neq v \in U$ be two terms in the finite subset $U \subset \mathcal{T} \setminus \mathcal{I}$. Let $k = \max\{\operatorname{cls}(u), \operatorname{cls}(v)\}$. If $k = n = \operatorname{cls}(u) = \operatorname{cls}(v)$, the disjointness of the relative Pommaret-like cones is easily seen, as also in the case where k = n but one of $\operatorname{cls}(u), \operatorname{cls}(v)$ is less than n. If k < n and the projections $u|_{x_1=\cdots=x_k=1}$, $v|_{x_1=\cdots=x_k=1}$ are equal, then either disjointness or containment of the relative Pommaret-like cones is also easily seen. It remains the case when k < n but the projections on the subring $\mathbb{K}[x_{k+1}, \ldots, x_n]$ are not equal. There, note that from any two elements u' = su and v' = tv, where s and t are $P_{\mathcal{I}}$ -multiplicative terms, we get in the subring that the projections of s and t are Janet-like multipliers of the projections of u and v. Hence the projections of the relative Pommaret-like cones of u and v on the same subring are either contained one in the other or they are disjoint. If they are disjoint, the same also holds true for the full cones in the whole ring \mathcal{P} . If they are contained one in the other, then checking the k-degrees of u and v will yield that the full cones are either disjoint or contained. A containment will hold if and only if the term with larger class, without loss of generality v, has a smaller or equal x_k -degree compared to that of the other term and the projection of the cone of v in the subring is a superset of the other cone projection.

Definition 5.7. Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{P}$ be a non-zero monomial ideal and let $\mathcal{A} \supset \mathcal{I}$ be a further monomial ideal in \mathcal{P} . Let $L_{\mathcal{I}}$ be an involutive-like division relative to \mathcal{I} . A finite set of terms $H \subset \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ is called a *weak* $L_{\mathcal{I}}$ -*involutive-like basis of* \mathcal{A} *relative to* \mathcal{I} if every term $t \in \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ has an $L_{\mathcal{I}}$ involutive-like divisor in the set H. Such a basis H is called *strong* if every term $t \in \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ has a unique $L_{\mathcal{I}}$ involutive-like divisor in the set H.

We recall the following definition from [8, Def. 7.1].

Definition 5.8. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. We say that \mathcal{J} is *quasi-stable relative to* \mathcal{I} , if for all terms $x^{\mu} \in \mathcal{J} \setminus \mathcal{I}$ and for all indices i with $\operatorname{cls}(x^{\mu}) < i \leq n$ there exists an exponent $s \geq 0$ such that either $x_i^s x^{\mu} \in \mathcal{I}$ or $x_i^s x^{\mu} / x_{\operatorname{cls}(x^{\mu})} \in \mathcal{J}$.

Theorem 5.9. Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. Then there exists a Pommaret-like basis of \mathcal{J} relative to \mathcal{I} if and only if \mathcal{J} is quasi-stable relative to \mathcal{I} .

Proof. First assume that \mathcal{J} is quasi-stable relative to \mathcal{I} . By [8, Prop. 7.4], we know that there exists a relative Pommaret basis H of \mathcal{J} . Since relative Pommaret-like cones are always supersets of relative Pommaret cones, the set H is also a relative Pommaret-like basis of \mathcal{J} . (It need not be a minimal relative Pommaret-like basis, though.)

Now assume that there exists a Pommaret-like basis $H \,\subset\, \mathcal{T} \cap (\mathcal{J} \setminus \mathcal{I})$ of \mathcal{J} relative to \mathcal{I} . Arguing by *reductio ad absurdum*, suppose that \mathcal{J} is not quasi-stable relative to \mathcal{I} . In particular, H is a generating set of \mathcal{J} relative to \mathcal{I} . Since \mathcal{J} is not quasi-stable relative to \mathcal{I} , there is a term $1 \neq h \in H$ and an index $j > k =: \operatorname{cls}(h)$ such that for every exponent $s \in \mathbb{N}$ we have $(h/x_k)x_j^s \notin \mathcal{J}$ and $hx_j^s \notin \mathcal{I}$. Consider the Janet class $C := H_{[\deg_{j+1}(h),\ldots,\deg_n(h)]}$. Among the terms in C, there is one with maximal x_j -degree. Let this degree be denoted by d.

Now, since H is a relative Pommaret-like basis, the term $h \cdot x_j^{d-\deg_j(h)}$ has a $P_{\mathcal{I}}$ -divisor u in H. By definition of the $P_{\mathcal{I}}$ -like division, u must be an element of the Janet class $H_{[d,\deg_{j+1}(h),\ldots,\deg_n(h)]}$. Moreover, it must be a divisor (in the non-involutive sense) of $h \cdot x_j^{d-\deg_j(h)}$, so there is a term $t \in \mathcal{T}$ with $h \cdot x_j^{d-\deg_j(h)} = u \cdot t$. Now, if there were an exponent $e \in \mathbb{N}$ with $u \cdot x_j^e \in \mathcal{I}$, then also $u \cdot t \cdot x_j^e \in \mathcal{I}$ and hence $h \cdot x_j^{e+d-\deg_j(h)} \in \mathcal{I}$, in contradiction to the assumptions made for h. Hence, such an exponent e does not exist. Moreover, by construction, $x_j \in M_J(u, H)$. Additionally, it is not possible that $\operatorname{cls}(u) \geq j$, because otherwise u would be a divisor of $(h/x_k)x_j^{d-\deg_j(h)}$, again in contradiction to the assumptions made for h.

By the statements just shown, and by Definition 5.5, one $P_{\mathcal{I}}$ -non-multiplicative power of u with respect to the set H is x_i^1 . Now, $u \cdot x_j \in \mathcal{J} \setminus \mathcal{I}$, and it cannot have any $P_{\mathcal{I}}$ -divisor in the set

H, since such a divisor would be an element of a Janet class $H_{[d+1,\deg_j(h),\ldots,\deg_n(j)]}$. But this Janet class is empty by the maximality property of *d*. All in all, we have shown that there is a term in $\mathcal{J} \setminus \mathcal{I}$ which has no $P_{\mathcal{I}}$ -like divisor in the set *H*. This contradicts the assumption that *H* is a relative Pommaret-like basis of \mathcal{J} .

Example 5.10. Consider the ideals $\mathcal{I} = \langle x^6, y^6, z^6 \rangle$ and $\mathcal{J} = \langle \mathcal{I}, xy, yz \rangle$. These ideals are taken from [4, Ex. 4.12]. Here, the set $H = \{xz, yz\}$ is a Pommaret-like basis of \mathcal{J} relative to \mathcal{I} .

To see this, use as a first step the Janet division to see that $M_J(yz, H) = \{x, y, z\}$ and $M_J(xz, H) = \{x, z\}$. The set of Janet-like non-multiplicative powers of xz is $\text{NMP}_J(xz, H) = \{y\}$. Note that for each term $h \in H$ and for each variable in the ring, by multiplying h with a high enough power of that variable, we get a term in \mathcal{I} . Hence, the relative Pommaret-like non-multiplicative powers of the terms in H are $\text{NMP}_{P_T}(yz, H) = \emptyset$, $\text{NMP}_{P_T}(xz, H) = \{y\}$.

Now, it is clear that all non-multiplicative multiples of xz are in the relative Pommaret-like cone of yz and H is a relative Pommaret-like basis as claimed.

Example 5.10 can be generalized. In [4], Gasharov et al. studied the (infinite) free resolution of a special type of squarefree ideal in a particular quotient ring, so-called Clements–Lindström ring. In the rest of this paper, by applying relative Pommaret-like bases, we give an alternative approach to construct such a resolution. As the first step, in this setting, in Proposition 5.13, we show that the minimal generating set of a square-free Borel monomial ideal is a relative minimal Pommaret-like basis. In this direction, we need two definitions. The first is taken from [4]. The second goes back essentially to [2]. We adapt both to our conventions on variable orderings. Below, for a term s, we refer to supp(s) as the set of all variables appearing in s. Furthermore, for a given monomial ideal \mathcal{I} , Min(\mathcal{I}) stands for its minimal generating set of terms.

Definition 5.11. We call a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ generated by squarefree terms *squarefree Borel*, if for any (necessarily squarefree) term $s \in Min(\mathcal{I})$ the following holds: For any variable $x_i \in supp(s)$ and any index j with $i < j \le n$ such that $x_j \notin supp(s)$, it holds $(s/x_i) \cdot x_j \in \mathcal{I}$.

Definition 5.12. We say that an irreducible, non-zero monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ is *Clements–Lindström*, if $Min(\mathcal{I})$ is of the form $\{x_i^{a_i}, x_{i+1}^{a_{i+1}}, \ldots, x_n^{a_n}\}$ with $2 \le a_n \le a_{n-1} \le \cdots \le a_{i+1} \le a_i$. We call \mathcal{P}/\mathcal{I} a *Clements–Lindström ring*.

Proposition 5.13. Let \mathcal{I} be a zero-dimensional Clements–Lindström ideal and let H be the minimal generating set of a square-free Borel monomial ideal. Then the ideal $\mathcal{J} = \langle \mathcal{I}, H \rangle$ is quasi-stable relative to \mathcal{I} and the set H is the minimal Pommaret-like basis of \mathcal{J} relative to \mathcal{I} .

Proof. As a zero-dimensional Clements–Lindström ideal, $\mathcal{I} = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$ with $a_1 \geq \cdots \geq a_n \geq 2$. The square-free minimal generating set H of \mathcal{J} is disjoint from \mathcal{I} . Hence, it is the minimal generating set of \mathcal{J} relative to \mathcal{I} . Exclude in the following the trivial special case $H = \{1\}$. H fulfils the square-free Borel property. Hence, for any term $1 \neq h \in H$ and any index $j > k = \operatorname{cls}(h)$ such that $x_j \notin \operatorname{supp}(h)$, there is another term $u_1 \in H$ dividing $(h/x_k)x_j$. Since H is a minimal generating set, we have $\operatorname{cls}(u_1) \leq j$. If either $\operatorname{cls}(u_1) = j$ or

$$\operatorname{supp}(u_1) \supseteq \{x_i\} \cup \{x_a \in \operatorname{supp}(h) \mid a > j\}$$

then we have are done and obtained the desired term u_1 . Otherwise, we repeat this process, by finding $u_2 \in H$ such that u_2 divides $(h/x_{\operatorname{cls}(u_1)})x_\ell$ with $\ell > j$ and $x_\ell \in \operatorname{supp}(h)$. We know that for i, the constructed term u_i fulfils the condition $\operatorname{cls}(u_i) \leq j$. It is clear to see that the sequence of u_1, u_2, \ldots is finite. Assume that u_m is the last constructed term. If $\operatorname{supp}(u_m) \supseteq \{x_j\} \cup \{x_a \in \operatorname{supp}(h) \mid a > j\}$ then we are done. Otherwise, we have $\operatorname{cls}(u_m) = j$ and $\operatorname{supp}(u_m) \supseteq \{x_j\} \cup \{x_a \in \operatorname{supp}(h) \mid a > j\}$. Repeating the mentioned process on u_m leads to a contradiction to the minimality of H. Thus, at the end, for each index j with $j > \operatorname{cls}(h)$, we arrive at a term v such that $\operatorname{supp}(v) \cap \{x_{\operatorname{cls}(v)}, \ldots, x_n\} = \{x_j\} \cup \{x_a \in \operatorname{supp}(h) \mid a \ge \operatorname{cls}(v)\}$.

Now, for a given index $j > \operatorname{cls}(h)$, assume that the term v satisfies the above equality. Both h and v are in the Janet class $H_{[\deg_{j+1}(h),\ldots,\deg_n(h)]}$ and this shows that $x_j \in \operatorname{NM}_J(h, H)$ and x_j is a $P_{\mathcal{I}}$ -non-multiplicative power of h with respect to H. On the other hand, we know that each variable $x_a \in \operatorname{supp}(h)$ with $a > \operatorname{cls}(h)$ lies in $M_J(h, H)$, because H is square-free and $\deg_a(h) = 1$. Additionally, for each such variable x_a , of course there is an exponent $e \in \mathbb{N}$ such that $hx_a^e \in \mathcal{I}$, because \mathcal{I} is zero-dimensional. Hence, for such variable x_a , no $P_{\mathcal{I}}$ -non-multiplicative power exists for h. These arguments imply that $x_j \cdot h \in C_{P_{\mathcal{I}}}(v)$. Applying a local involution argument, we see that H is a relative Pommaret-like basis of \mathcal{J} .

Remark 5.14. A close inspection of the proof of Proposition 5.13 shows that the proposition holds also under slightly weaker conditions. Let $k \in \{1, ..., n\}$ be defined as $\min\{\operatorname{cls}(h) \mid h \in H\}$. Then the proposition also holds if \mathcal{I} is an irreducible quasi-stable ideal for which some power of x_{k+1} is a minimal generator.

6. Pommaret-like free resolutions over Clements-Lindström rings

Since relative Pommaret-like bases are a special kind of relative Gröbner bases, they induce free resolutions via the relative involutive Schreyer Theorem [8]. If we assume that the ambient quotient ring is \mathcal{P}/\mathcal{I} , where \mathcal{I} is a quasi-stable monomial ideal, and if we complete the relative Pommaret-like basis to a relative Pommaret basis, then the induced resolution will consist of Pommaret bases for the syzygy modules in each homological degree. In this section, we will show that if we restrict to the class of *irreducible* quasi-stable monomial ideals, then we can skip the completion step from Pommaret-like basis to Pommaret basis: The relative Pommaret-like basis will then induce a free resolution which consists of Pommaret-like bases for each syzygy module. Up to a permutation of coordinates, the class of irreducible quasi-stable monomial ideals is equivalent to the class of Clements–Lindström ideals. We will formulate our results in the most general form possible, but for simplicity one can think of the ring in which computations take place as a Clements–Lindström ring $\mathcal{P}/\langle x_k^{a_k}, \ldots, x_n^{a_n} \rangle$ with $a_k \geq \cdots \geq a_n \geq 2$. We shall note that a variant of the next proposition in the non-relative setting was given in [9, Thm. 7.7].

Proposition 6.1. Let \mathcal{I} be an irreducible quasi-stable monomial ideal. We work in the ring \mathcal{P}/\mathcal{I} . Let H be a Pommaret-like basis relative to \mathcal{I} of the (polynomial) ideal $\mathcal{J} \supset \mathcal{I}$. If H is ordered according to a P-ordering for its set of leading terms, then a Pommaret-like basis of $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H)$ is given by the S-polynomials of H induced by non-multiplicative multiples of the leading terms and the A-polynomials induced by multiplicatively annihilating leading terms of H modulo \mathcal{I} . Iteration of this result implies that a free resolution is induced consisting of relative Pommaret-like bases in each homological degree.

Sketch of Proof. The proof is similar to that of the corresponding results in [8, Thm. 5.13 and Prop. 5.14], where relative Pommaret bases are treated. As in the proof of [8, Prop. 5.14], quasi-stability of \mathcal{I} is needed to ensure the relative quasi-stability of the leading module of the first syzygy module $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H)$; the irreducibility of \mathcal{I} implies that the leading module terms have pure powers as their polynomial parts, from which it is easily seen that the set of these leading terms forms a relative Pommaret-like basis. Moreover, the proof uses continuity of the Pommaret-like division [9, Prop. 6.15] (for the *P*-ordering) and the relative Schreyer Theorem [8, Thm. 3.12]. Note that "gaps" can appear in the lists of leading module terms in some syzygy module components. These gaps appear for the variables where one can reach \mathcal{I} by multiplying by a power of that variable. That one can reach \mathcal{I} implies that relative quasi-stability is not destroyed by these gaps.

Remark 6.2. Only those A-polynomials whose annihilating factor is not identical to a generator of \mathcal{I} contribute non-zero syzygies [8, Cor. 4.9].

As in the non-relative case, we are interested in a description of at least a part of the class of monomial ideals $\mathcal{J} \supset \mathcal{I}$ quasi-stable relative to \mathcal{I} whose relative Pommaret-like bases induce *minimal* free resolutions by the process of Proposition 6.1. Recall that an estimate for the classes of "tail" terms compared to the classes of leading module terms was central to the proof of Theorem 4.2. In order to be able to use a similar argument, we need to impose even stricter assumptions on the relative Pommaret-like basis generating \mathcal{J} than we had to impose in Theorem 4.2. The reason for this is that in the relative case, the contributions of A-polynomials have an effect which amounts to a "non-increasing" property for the classes of leading module terms in the resolution.

Example 6.3. Continuing Example 4.5, in the ring $\mathbb{K}[w, x, y, z]/\langle x^{10}, y^{10}, z^{10} \rangle$, we consider the monomial ideal $\mathcal{J} = \langle H \rangle$ minimally generated by the relative Pommaret-like basis

$$H = \{w^9x^3y^2z^2, x^5y^2z^2, w^7y^4z^2, x^3y^4z^2, y^6z^2, x^3y^2z^4, y^4z^4, z^8\}$$

Note that $y^2 \cdot x^5 y^2 z^2 - x^2 \cdot x^3 y^4 z^2 = 0$ (compare the fourth column of the matrix D_1 in Example 4.5); moreover, $x^5 \cdot x^5 y^2 z^2 = 0 = x^7 \cdot x^3 y^4 z^2$. We obtain the three elements $\mathbf{S}_1 = y^2 \mathbf{e}_2 - x^2 \mathbf{e}_4$, $\mathbf{S}_2 = x^5 \mathbf{e}_2$, and $\mathbf{S}_3 = x^7 \mathbf{e}_4$ of the first syzygy module of H. Observe that y^2 is a Pommaret-like nonmultiplicative power of \mathbf{S}_2 . Multiplying and reducing, we see that $y^2 \mathbf{S}_2 - x^5 \mathbf{S}_1 - \mathbf{S}_3 = 0$. Thus, a constant appears in the Pommaret-like induced resolution, which is consequently not minimal.

Theorem 6.4. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be an irreducible quasi-stable monomial ideal (with $\mathcal{I} \notin \{\{0\}, \mathcal{P}\}$) and let $\mathcal{J} \supset \mathcal{I}$ be a larger monomial ideal generated by the minimal Pommaret-like basis $\{1\} \neq H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$ relative to \mathcal{I} . Assume that H is simultaneously the minimal monomial generating set of \mathcal{J} relative to \mathcal{I} . Moreover, let H be such that for each $t \in H$ and $x_a^{p_a} \in \text{NMP}_{P_{\mathcal{I}}}(t, H)$, the unique $P_{\mathcal{I}}$ -divisor $s \in H$ of $t \cdot x_a^{p_a}$ is of greater class than t i.e. $\operatorname{cls}(s) > \operatorname{cls}(t)$. Then the free resolution of \mathcal{J} over \mathcal{P}/\mathcal{I} induced by the basis H is minimal.

Moreover, for each $m \ge 1$, the set of columns of the matrix D_m describing the differential consists of the unique relative reduced Gröbner basis of $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}^m(H)$ for the chosen module monomial ordering.

Proof. We need to show that the matrices describing the differential do not contain any constant terms. By assumption, $H \neq \{1\}$ and hence it does not contain any constant. We now analyse the matrices D_1, D_2, \ldots iteratively. For every $h \in H$, the matrix D_1 contains as leading module terms the non-multiplicative powers of h as well as, for $k = \operatorname{cls}(h)$, a factor $x_k^{d_k-\operatorname{deg}_k(h)}$ if $x_k^{d_k}$ is a minimal generator of \mathcal{I} . The tail terms in D_1 arise by division of terms $h \cdot x_j^{p_j}$, where $x_j^{p_j}$ is a non-multiplicative power, by their unique Pommaret-like divisor s in H:

$$h \cdot x_j^{p_j} = s \cdot u. \tag{6.1}$$

Since *H* is the minimal relative generating set of \mathcal{A} , these tail terms are not constant. Moreover, by assumption, we have $\operatorname{cls}(s) > \operatorname{cls}(h)$ and in turn $\operatorname{cls}(u) = \operatorname{cls}(h)$. A tail term *u* will be found in the row corresponding to the generator $s \in H$, and the leading terms in that row will be of $\operatorname{class} \ge \operatorname{cls}(s)$, and so *u* has strictly smaller class than the leading terms in the same row. Note that columns of D_1 belonging to annihilating factors do not have any tail term. Summarizing, D_1 does not contain any constant terms and all tail terms have a strictly smaller class than the leading terms in the leading terms in the same row.

It is now straightforward to proceed analogously as in the proof of Theorem 4.2, showing by induction on homological degree that no constant terms appear in the resolution, and thus to show its minimality.

Central to this induction proof is the fact that tail terms always have strictly smaller class than leading terms in the same row. The reducedness of the Gröbner bases in each degree is an obvious consequence. $\hfill \Box$

As a consequence, we see below that using this theorem and Proposition 5.13, we are able to describe minimal free resolutions for the class of monomial ideals considered in Proposition 5.13.

Corollary 6.5. Let \mathcal{I} be a zero-dimensional Clements–Lindström ideal and let H be the minimal generating set of a square-free Borel monomial ideal. Then the free resolution induced by H is minimal.

Proof. A close inspection of the proof of Proposition 5.13 shows that the elements of H fulfil the class condition imposed in Theorem 6.4 and this completes the proof.

Example 6.6. Let us continue Example 5.10 by considering the ideals $\mathcal{I} = \langle x^6, y^6, z^6 \rangle$ and $\mathcal{A} = \langle \mathcal{I}, xz, yz \rangle$. The set $H = \{xz, yz\}$ is the minimal generating system of the ideal \mathcal{A} relative to \mathcal{I} , and it is simultaneously a relative Pommaret-like basis, as proven in Example 5.10. Since cls(xz) < cls(yz), the additional conditions of Theorem 6.4 are also fulfilled. Hence, H induces an infinite minimal free resolution of \mathcal{A} over \mathcal{P}/\mathcal{I} , with the first differential matrices given by:

$$D_{0} = \begin{pmatrix} xz & yz \end{pmatrix}, \quad D_{1} = \begin{pmatrix} x^{5} & y & z^{5} & 0 & 0 \\ 0 & -x & 0 & y^{5} & z^{5} \end{pmatrix},$$
$$D_{2} = \begin{pmatrix} x & y & z^{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^{5} & 0 & y^{5} & z^{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & -x^{5} & 0 & -y & z & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & y & z^{5} & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & -y^{5} & z \end{pmatrix}.$$

Remark 6.7. The minimal free resolution constructed by Gasharov et al. [4, Cons. 4.4] for square-free borel ideals relative to a zero-dimensional Clements–Lindström ring is necessarily isomorphic to the Pommaret-like resolution of the same ideal, since both resolutions are minimal.

In fact, one can always find an isomorphism that consists only of permutations of bases. One can prove this by assigning leading terms to the syzygies defined in [4, Eqn. 4.10]. This assignment can be done in such a way that the leading terms for each homological degree will coincide with the leading terms in the Pommaret-like resolution. The sets of leading terms being equal, we can conclude that the syzygies of [4, Eqn. 4.10] form Gröbner bases in each homological degree; the reducedness can then be shown in a straightforward manner using a basic result on Borel monomial ideals.

The uniqueness of the reduced Gröbner basis then shows that the resolution of [4, Cons. 4.4] and the Pommaret-like resolution coincide. This also gives an explicit formula for the differential, depending only on the data contained in the first two matrices D_0 and D_1 .

The next example demonstrates that our construction covers many elementary cases:

Example 6.8. Let a_1, \ldots, a_n be positive integers. By fixing $i \in \{1, \ldots, n\}$, let $1 \le b_i < a_i$ be another integer. Then, relative to the irreducible monomial ideal $\mathcal{I} = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$, the set $H = \{x_i^{b_i}\}$ is a relative Pommaret-like basis of $\mathcal{A} = \langle H, \mathcal{I} \rangle$ and the induced resolution over \mathcal{P}/\mathcal{I} is the obvious 2-periodic minimal free resolution with differentials described by the following matrices:

$$D_0 = (x_i^{b_i}), \ D_1 = (x_i^{a_i-b_i}), \ D_2 = (x_i^{b_i}) = D_0.$$

A final, more or less "generic" example shows the general behaviour of the construction:

Example 6.9. If $\mathcal{I} = \langle y^4, z^5 \rangle$ and $\mathcal{A} = \langle \mathcal{I}, x^2y^3, xy^2z^2, y^3z^2, z^3 \rangle$, then $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$ is the minimal relative generating set of \mathcal{A} , and it is simultaneously a relative Pommaret-like basis satisfying the additional conditions of Theorem 6.4. Hence, it induces a minimal free resolution of \mathcal{A} over \mathcal{P}/\mathcal{I} , with the first maps of the differential represented by the following matrices:

$$D_0 = \begin{pmatrix} x^2 y^3 & x y^2 z^2 & y^3 z^2 & z^3 \end{pmatrix}, \quad D_1 = \begin{pmatrix} y & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 & 0 \\ 0 & -x^2 & -x & 0 & y & z & 0 \\ 0 & 0 & 0 & -x y^2 & 0 & -y^3 & z^2 \end{pmatrix},$$

In the remainder of this section, we will derive formulas for the Betti numbers of minimal free resolutions induced by Pommaret-like bases over factor rings of the form \mathcal{P}/\mathcal{I} , where \mathcal{I} is an irreducible quasi-stable monomial ideal. We understand this to include also the case $\mathcal{I} = \{0\}$, and hence all resolutions induced by Pommaret-like bases over the ordinary polynomial ring \mathcal{P} = $\mathcal{P}/\{0\} = \mathbb{K}[x_1, \dots, x_n]$. The results can also be applied to non-minimal free resolutions induced by Pommaret-like bases, but then one gets only formulas for the ranks of the free modules in these nonminimal resolutions. These ranks can be understood as *pseudo*-Betti numbers of these resolutions. They yield, degree by degree, upper bounds for the true Betti numbers of the resolved ideals.

We need the following definition.

Definition 6.10. Let $\mathcal{I} = \langle x_k^{a_k}, x_{k+1}^{a_{k+1}}, \dots, x_n^{a_n} \rangle$ be an irreducible quasi-stable monomial ideal. Then we write $\operatorname{cls}(\mathcal{I}) \coloneqq k$ and $\operatorname{supp}(\mathcal{I}) \coloneqq \{x_k, x_{k+1}, \dots, x_n\}$.

Let now $\mathcal{J} \supseteq \mathcal{I}$ be any homogeneous polynomial ideal in quasi-stable position relative to \mathcal{I} with respect to the degrevlex term ordering, and let H be its minimal relative Pommaret-like basis. We construct a basis for the bigraded free *R*-module supporting the induced resolution, only using the Pommaret-like basis of the leading ideal of \mathcal{J} relative to \mathcal{I} .

The resolution induced by H is supported on free R-modules. The first free R-module M_0 has a basis that we enumerate as $\{\mathbf{e}_{\alpha} \mid h_{\alpha} \in H\}$. Write $t_{\alpha} = \operatorname{lt}(h_{\alpha})$ for each $h_{\alpha} \in H$. As always, we order H according to a P-order. The next free module M_1 has a basis whose cardinality equals that of the minimal Pommaret-like basis of Syz(H) with respect to the Schreyer module term order—note that this is a reduced Gröbner basis. Hence, the free basis of M_1 is in bijection with the elements of this Gröbner basis; in other words, it is in bijection with the leading module terms of the Gröbner basis. These leading module terms are given as follows (cf. Proposition 6.1):

- $x_a^{p_a} \cdot \mathbf{e}_{\alpha}$, where $x_a^{p_a} \in \text{NMP}_{P_{\mathcal{I}}}(t_{\alpha}, \text{lt}(H))$, $x_i^{a_i \deg_i(t_{\alpha})} \cdot \mathbf{e}_{\alpha}$, where $x_i \in \text{supp}(t_{\alpha}) \cap \text{supp}(\mathcal{I})$ and there is no $P_{\mathcal{I}}$ -non-multiplicative power for t_{α} at x_i . (If $\ell = \operatorname{cls}(t_{\alpha}) \ge \operatorname{cls}(\mathcal{I})$, then this case will always include $x_{\ell}^{a_{\ell} - \operatorname{deg}_{\ell}(t_{\alpha})} \cdot \mathbf{e}_{\alpha}$.)

Since the two cases are mutually exclusive, and each concerns leading module terms whose polynomial parts are pure variable powers, we can identify each leading module term by its position and the variable involved. Thus, a free basis of M_1 can be enumerated as

$$\{\mathbf{e}_{\alpha,x_i} \mid x_i \ge \operatorname{cls}(t_\alpha) \land (x_i \in \operatorname{NM}_{P_{\mathcal{T}}}(t_\alpha,\operatorname{lt}(H)) \lor x_i \in \operatorname{supp}(t_\alpha) \cap \operatorname{supp}(\mathcal{I}))\}.$$

We keep the condition " $x_i \ge cls(t_\alpha)$ " for clarity, even though it could be omitted, being implicit in the other conditions. At this stage, it is useful to introduce notation for the leading ideals in each module component of M_1 , because we can use them to describe, by an iteration, all further leading terms in the resolution. Set

$$\mathcal{J}_{\alpha} = \langle x_i^{d_i} \mid x_i^{d_i} \cdot \mathbf{e}_{\alpha} \in \mathrm{lt}(\mathrm{Syz}(H)) \rangle$$

These ideals are irreducible and we will use the notation $\operatorname{supp}(\mathcal{J}_{\alpha})$ for the set of variables appearing in their respective generating sets.

Consider now the leading terms of the Pommaret-like basis of $Syz^2(H)$, which are in bijection to a free basis of the next module in the resolution, M_2 . Each of them is induced by a leading term of the basis of Syz(H). Such a leading term, $x_i^{d_i} \cdot \mathbf{e}_{\alpha}$, say, induces exactly the following leading terms in $\operatorname{Syz}^2(H)$:

- $x_j^{d_j} \cdot \mathbf{e}_{\alpha, x_i}$, where $x_j \in \text{supp}(\mathcal{J}_{\alpha})$ and j > i, $x_i^{a_i d_i} \cdot \mathbf{e}_{\alpha, x_i}$, if $x_i \in \text{supp}(\mathcal{I})$.

Note that the polynomial part of the new leading term will be supported on a variable whose index is not less than that of the polynomial part of the term which induces it. We can now list the free basis of M_2 : Leading terms induced as in the first case correspond to basis elements e_{α,x_ix_i} , whereas leading terms induced as in the second case correspond to basis elements $\mathbf{e}_{\alpha,x_i^2}$.

We can iterate this construction. For the rth module in the resolution, M_r , it yields a basis consisting of elements of the form $e_{\alpha,x^{\mu}}$, where x^{μ} is a term of degree r with $cls(x^{\mu}) \ge cls(t_{\alpha})$. Moreover, x^{μ} is supported on supp (\mathcal{J}_{α}) , and if for each variable $x_i \in \text{supp}(\mathcal{I})$ we substitute 1 into x^{μ} , we get a squarefree term supported on $\operatorname{supp}(\mathcal{J}_{\alpha}) \setminus \operatorname{supp}(\mathcal{I})$.

From this description of the free bases, we obtain the following formula for the total Betti numbers of the resolution, where we write S_{α} for $\operatorname{supp}(\mathcal{J}_{\alpha})$ and S for $\operatorname{supp}(\mathcal{I})$: For r = 0, $\operatorname{rank}(M_0) = |H|; \text{ for } r \ge 1,$

$$\operatorname{rank}(M_{r}) = \sum_{\substack{h_{\alpha} \in H \\ S_{\alpha} \cap S \neq \emptyset}} \sum_{j=0}^{\min\{r, |S_{\alpha} \setminus S|\}} \binom{|S_{\alpha} \setminus S|}{j} \cdot \binom{|S_{\alpha} \cap S| + r - j - 1}{|S_{\alpha} \cap S| - 1} + \sum_{\substack{h_{\alpha} \in H \\ S_{\alpha} \cap S = \emptyset}} [r \leq |S_{\alpha}|] \binom{|S_{\alpha}|}{r},$$
(6.2)

where the product of binomial coefficients counts the number of terms x^{μ} of degree r supported on \mathcal{J}_{α} with the additional restriction of being squarefree outside $\operatorname{supp}(\mathcal{I})$. Moreover, the term $[r \leq |\mathcal{S}_{\alpha}|]$ is defined as having values in $\{0,1\}$, yielding 1 exactly when the statement enclosed in the square brackets is true.

We now turn to the bigraded Betti numbers, which we will compute in the form of a Poincaré series, which is a formal power series in two variables, which we name s and u. The first variable encodes homological degrees and the second encodes degrees as given by the ordinary grading of the polynomial ring \mathcal{P} . Recall that each basis element $\mathbf{e}_{\alpha,x^{\mu}}$ has homological degree $\deg(x^{\mu})$. Its polynomial degree is the sum of $deg(t_{\alpha})$ (recall $t_{\alpha} = lt(h_{\alpha})$) and the degrees of the polynomial parts of all leading module terms involved in the building up of the syzygy $\mathbf{S}_{\alpha,x^{\mu}} \in \operatorname{Syz}^{\deg(x^{\mu})}(H)$. These polynomial parts are pure powers of variables from $\operatorname{supp}(\mathcal{J}_{\alpha})$. Moreover, their indices form a nondecreasing sequence. There can be repeated indices in this sequence, and if an index j is repeated, it means that the next syzygy is formed from the annihilation of the current leading term. So if a module term with polynomial part $x_i^{c_j}$ is to annihilate, the next leading term will have polynomial part $x_i^{a_j-c_j}$ (recall that \mathcal{I} is generated by the $x_j^{a_j}$). More repetitions of the same index will cause the involved leading terms to have polynomial parts oscillating between $x_j^{c_j}$ and $x_j^{a_j-c_j}$. This means that the contribution of x_j -terms to the overall polynomial degree of $S_{\alpha,x^{\mu}}$ depends, on one hand, on the parity of μ_j , and the remaining part is just $h_j \cdot \lfloor \mu_j/2 \rfloor$. Since \mathcal{J}_{α} is generated by terms $x_j^{d_j}$, we get the following formula for the Poincaré series of our resolution, where we write S_{α} for supp (\mathcal{J}_{α}) and S for supp(\mathcal{I}):

$$\sum_{h_{\alpha} \in H} \left(u^{\deg(t_{\alpha})} \cdot \left(1 + \sum_{B \subseteq \mathcal{S}_{\alpha}} \binom{|\mathcal{S}_{\alpha}|}{|B|} \right) s^{|B|} \prod_{x_{b} \in B} u^{d_{b}} \prod_{x_{j} \in \mathcal{S}_{\alpha} \cap \mathcal{S}} \frac{1}{1 - s^{2} u^{a_{j}}} \right) \right).$$
(6.3)

Example 6.11. Let us continue Example 6.9 where $\mathcal{I} = \langle y^4, z^5 \rangle$ and $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$. We will use Equation (6.2) to compute the Betti numbers of the ideal generated by H relative to \mathcal{I} and then compare it with the results of Example 6.9.

We write $h_{\alpha} = x^2 y^3$, $h_{\beta} = x y^2 z^2$, $h_{\gamma} = y^3 z^2$, and $h_{\delta} = z^3$. An analysis of the Pommaret-like non-multiplicative powers of these generators shows that $\mathcal{J}_{\alpha} = \{y, z^2\}, \mathcal{J}_{\beta} = \{y, z\}, \mathcal{J}_{\gamma} = \{y, z\}, \mathcal{J}_{\gamma}$

and $\mathcal{J}_{\delta} = \{z^2\}$. Since $\operatorname{supp}(\mathcal{I}) = \{x, y\}$, we have $\operatorname{supp}(\mathcal{J}_{\alpha}) = \operatorname{supp}(\mathcal{J}_{\alpha}) \cap \operatorname{supp}(\mathcal{I})$, and the same equality holds also for the other indices. Thus, Equation (6.2) reduces to:

$$\operatorname{rank}(M_r) = \sum_{h_{\alpha} \in H} \binom{|\operatorname{supp}(\mathcal{J}_{\alpha})| + r - 1}{|\operatorname{supp}(\mathcal{J}_{\alpha})| - 1},$$

and this gives, since we have three generators with $|\operatorname{supp}(\mathcal{J}_{\bullet})| = 2$ and one generator with $|\operatorname{supp}(\mathcal{J}_{\bullet})| = 1$, the formula

$$\operatorname{rank}(M_r) = 3\binom{1+r}{1} + \binom{r}{0} = 4 + 3r$$

which is for $r \in \{1, 2, 3\}$ in perfect agreement with the results of Example 6.9, as expected.

7. An explicit formula for the differential

In this section, we will give explicit formulas for the derivatives of resolutions of some monomial ideals induced by Pommaret-like bases over the ordinary polynomial ring $\mathcal{P} = \mathbb{K}[x_1, \dots, x_n]$. These formulas will generalize those described in [21] for resolutions induced by Pommaret bases. While in [21], such a formula was found for all quasi-stable ideals and their minimal Pommaret bases, we will here restrict our attention to a smaller class of ideals.

Our first goal is to establish a subclass of quasi-stable ideals whose minimal Pommaret-like basis satisfies conditions analogous to those found in [21, Lemma 5.4.17] for minimal Pommaret bases of arbitrary quasi-stable ideals. For this subclass, we will then have the technical tools needed to give an explicit formula for the differential of the induced resolution.

Definition 7.1. Let $H = \{h_{\alpha} \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like bases of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$. A is a finite index set. For each $\alpha \in A$, and for each of its Pommaret-like non-multiplicative powers $x_a^{p_a} = x_a^{p(P,h_{\alpha},H,a)}$, there exists exactly one generator $h_{\beta} \in H$ with $x_a^{p_a} \cdot h_{\alpha} \in \mathcal{C}_P(h_{\beta})$. For such a configuration of terms, we write

$$\Delta(\alpha, a) = \beta \tag{7.1}$$

for the index of the Pommaret-like divisor, and

$$t_{\alpha,a} = (x_a^{p_a} \cdot h_\alpha)/h_\beta \tag{7.2}$$

for the Pommaret-like multiplicative cofactor involved.

The following result states some elementary properties satisfied by the objects just defined.

Lemma 7.2. Let $H = \{h_{\alpha} \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$. The associated function Δ and the terms $t_{\alpha,a}$ (as given in Definition 7.1) satisfy the following properties:

- 1. The inequality $\operatorname{cls}(h_{\alpha}) \leq \operatorname{cls}(h_{\Delta(\alpha,a)}) \leq a$ holds for all non-multiplicative indices $a > \operatorname{cls}(h_{\alpha})$.
- 2. Let $b > a > cls(h_{\alpha})$ be two non-multiplicative indices.
 - The variable x_b is non-multiplicative for $h_{\Delta(\alpha,a)}$ and the non-multiplicative power of $h_{\Delta(\alpha,a)}$ at x_b equals that of h_{α} at x_b .
 - If $\operatorname{cls}(h_{\Delta(\alpha,b)}) \ge a$, then $\Delta(\Delta(\alpha,a),b) = \Delta(\alpha,b)$ and $x_a^{p_a} \cdot t_{\alpha,b} = t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b}$.

Proof. Property (1.) follows from the minimality of the Pommaret-like basis H: $h_{\Delta(\alpha,a)}$ is a divisor of $x_a^{p_a} \cdot h_{\alpha}$ and thus its class must be at least as high as that of h_{α} ; it cannot be higher than a, because otherwise $h_{\Delta(\alpha,a)}$ would be a strict Pommaret-like divisor of h_{α} , contradicting minimality.

Property (2.) is split into two items. The first item follows from property (1.) and the definition of the Pommaret-like division, because the terms $h_{\Delta(\alpha,a)}$ and h_{α} must agree in their x_j -degrees for all j > a. Now if, to prove the second item, we additionally assume $\operatorname{cls}(h_{\Delta(\alpha,b)}) \ge a$, then since

 $x_b^{p_b} \cdot h_{\Delta(\alpha,a)}$ and $x_b^{p_b} \cdot h_{\alpha}$ agree in their x_j -degrees for all indices j > a, the same must be true for $h_{\Delta(\Delta(\alpha,a)),b}$ and $h_{\Delta(\alpha,b)}$. We also know that $\deg_a(h_{\Delta(\alpha,b)}) \le \deg_a(h_{\alpha}) < \deg_a(x_b^{p_b} \cdot h_{\Delta(\alpha,a)})$. By the class assumption on $h_{\Delta(\alpha,b)}$, we can now conclude that $h_{\Delta(\alpha,b)}$ is the unique Pommaretlike divisor in H of $x_b^{p_b} \cdot h_{\Delta(\alpha,a)}$. Hence, we have shown $\Delta(\Delta(\alpha,a),b) = \Delta(\alpha,b)$. The remaining statement is a consequence of the following chain of equations:

$$\begin{aligned} x_a^{p_a} \cdot t_{\alpha,b} \cdot h_{\Delta(\alpha,b)} &= x_a^{p_a} \cdot (x_b^{p_b} \cdot h_\alpha) \\ &= x_b^{p_b} \cdot (x_a^{p_a} \cdot h_\alpha) \\ &= x_b^{p_b} \cdot t_{\alpha,a} \cdot h_{\Delta(\alpha,a)} \\ &= t_{\alpha,a} \cdot x_b^{p_b} \cdot h_{\Delta(\alpha,a)} \\ &= t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} \cdot h_{\Delta(\alpha,b)} \\ &= t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} \cdot h_{\Delta(\alpha,b)} \end{aligned}$$

For arbitrary minimal *Pommaret* bases, the associated Δ functions satisfy a commutativity property of the form

$$\Delta(\Delta(\alpha, a), b) = \Delta(\Delta(\alpha, b), a) \tag{7.3}$$

whenever both of these terms are defined, *i.e.*, when the involved variable indices a, b are always non-multiplicative [21, Lemma 5.4.17]. In general, minimal Pommaret-like bases do not have this property. What is more, for *Pommaret* bases, also the equation $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$ holds in this situation. In contrast to this, there are minimal Pommaret-like bases for which the commutativity property holds, but not the equation just mentioned. This is caused by differences of degrees of nonmultiplicative powers for the same variable.

Example 7.3. Consider the minimal Pommaret-like basis $H = \{h_{\alpha}, h_{\beta}, h_{\gamma}, h_{\delta}, h_{\epsilon}\}$ with $h_{\alpha} = xy$, $h_{\beta} = y^4$, $h_{\gamma} = xz$, $h_{\delta} = y^2z$, and $h_{\epsilon} = z^3$. Its associated Δ function satisfies the commutativity property of Equation (7.3). For this only one condition needs to be checked:

$$\Delta(\Delta(\alpha, y), z) = \delta = \Delta(\Delta(\alpha, z), y).$$

However, we have $t_{\alpha,y} = x$, $t_{\Delta(\alpha,y),z} = y^2$, $t_{\alpha,z} = y$, and $t_{\Delta(\alpha,z),y} = x$, so that $t_{\alpha,y} \cdot t_{\Delta(\alpha,y),z} = xy^2 \neq xy = t_{\alpha,z} \cdot t_{\Delta(\alpha,z),y}$. This is caused by a difference in the degrees of the non-multiplicative powers at the variable y between h_{α} (degree 3) and h_{γ} (degree 2).

We now define a subclass of quasi-stable ideals having Δ -functions with properties useful for the analysis of their Pommaret-like resolutions:

Definition 7.4. Let $H = \{h_{\alpha} \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasistable ideal $\mathcal{I} = \langle H \rangle$. The ideal \mathcal{I} together with the basis H is called Δ -commuting if the associated function Δ and the terms $t_{\alpha,a}$ (as given in Definition 7.1) satisfy the following properties:

- 1. If $b > a > \operatorname{cls}(h_{\alpha})$ are two non-multiplicative indices and $\operatorname{cls}(h_{\Delta(\alpha,b)}) < a$, then the exponent of the non-multiplicative power of $h_{\Delta(\alpha,b)}$ at the variable x_a equals that of the non-multiplicative power of h_{α} at the variable x_a .
- 2. We have $\Delta(\Delta(\alpha, a), b) = \Delta(\Delta(\alpha, b), a)$.
- 3. We have $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$.

For Δ -commuting quasi-stable ideals, we are able to give an explicit formula for the differential of the resolution induced by the minimal Pommaret-like basis. As is usual for such formulas, the summands obey a certain sign rule, and for this we need the following definition:

Definition 7.5. Let $x_i \in A \subseteq \{x_1, \dots, x_n\}$ be a variable contained in a subset A of variables. Then we write

$$\operatorname{sgn}(x_i, A) \coloneqq (-1)^{|\{x_j \in A \mid j > i\}|}.$$

Theorem 7.6. Let $H = \{h_{\alpha} \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the Δ -commuting quasi-stable ideal $\mathcal{I} = \langle H \rangle$. We write $\text{NMP}(h_{\alpha}, H) = \{x_j^{p_j} \mid j > \text{cls}(h_{\alpha})\}$. The Pommaret-like induced resolution of \mathcal{I} is supported on free generators of the form $\mathbf{e}_{h_{\alpha},x^{\mu}}$, where the x^{μ} are square-free terms supported on $\{x_j \mid j > \text{cls}(h_{\alpha})\}$. The differential δ of the resolution is given by $\delta(\mathbf{e}_{\alpha,1}) = h_{\alpha}$, and, for $\deg(x^{\mu}) > 0$,

$$\delta(\mathbf{e}_{\alpha,x^{\mu}}) = \sum_{x_j \in \mathrm{supp}(x^{\mu})} \mathrm{sgn}(x_j, \mathrm{supp}(x^{\mu})) \cdot \left(x_j^{p_j} \mathbf{e}_{\alpha,x^{\mu}/x_j} - t_{\alpha,j} \mathbf{e}_{\Delta(\alpha,j),x^{\mu}/x_j}\right).$$
(7.4)

In this formula, we interpret all summands to be zero which involve a non-existent free generator $\mathbf{e}_{\beta,x^{\nu}}$, i.e., an expression of this form for which $\operatorname{supp}(x^{\nu}) \notin \{x_j \mid j > \operatorname{cls}(h_{\beta})\}$.

Proof. The proof is a straightforward adaptation of the proof of [21, Thm. 5.4.18], replacing non-multiplicative variables by their associated non-multiplicative powers where appropriate. \Box

Corollary 7.7. Let $\mathcal{I} = \langle H \rangle$ be a Δ -commuting quasi-stable ideal minimally generated by the set $H \subset \mathcal{T}$, for which H is also a Pommaret-like basis. Then the Pommaret-like resolution of \mathcal{I} induced by H is minimal.

Proof. By minimality of H, we have $t_{\alpha,a} \neq 1$ for the terms defined in Definition 7.1. Now, the minimality of the induced resolution is a trivial consequence of the explicit differential formula 7.4, which applies because all assumptions of Theorem 7.6 are fulfilled for \mathcal{I} and H.

Example 7.8. Let us continue Example 4.5. We have the minimal Pommaret-like basis

$$\begin{split} H &= \{h_{\alpha} = w^9 x^3 y^2 z^2, h_{\beta} = x^5 y^2 z^2, h_{\gamma} = w^7 y^4 z^2, h_{\delta} = x^3 y^4 z^2, \\ h_{\epsilon} &= y^6 z^2, h_{\zeta} = x^3 y^2 z^4, h_{\eta} = y^4 z^4, h_{\theta} = z^8 \}. \end{split}$$

Using Formula 7.4, we obtain the following values of the differential δ of the induced resolution for basis elements of homological degrees 2 and 3:

$\delta(\mathbf{e}_{\alpha,xy})$ = $y^2 \mathbf{e}_{\alpha,x}$	$-x^2 \mathbf{e}_{\alpha,y}$	$+w^9\mathbf{e}_{\beta,y}$
$\delta(\mathbf{e}_{\alpha,xz})$ = $z^2 \mathbf{e}_{\alpha,x}$	$-x^2 \mathbf{e}_{\alpha,z}$	$+w^9\mathbf{e}_{\beta,z}$
$\delta(\mathbf{e}_{\alpha,yz})$ = $z^2 \mathbf{e}_{\alpha,y}$	$-w^9 \mathbf{e}_{\zeta,y} - y^2 \mathbf{e}_{\alpha,z}$	$+x^2 \mathbf{e}_{\delta,z}$
$\delta(\mathbf{e}_{\beta,yz})$ = $z^2 \mathbf{e}_{\beta,y}$	$-x^2 \mathbf{e}_{\zeta,y} - y^2 \mathbf{e}_{\beta,z}$	$+x^2 \mathbf{e}_{\delta,z}$
$\delta(\mathbf{e}_{\gamma,xy})$ = $y^2 \mathbf{e}_{\gamma,x}$	$-x^3 \mathbf{e}_{\gamma,y}$	$+w^7 \mathbf{e}_{\delta,y}$
$\delta(\mathbf{e}_{\gamma,xz})$ = $z^2 \mathbf{e}_{\gamma,x}$	$-x^3 \mathbf{e}_{\gamma,z}$	$+w^7 \mathbf{e}_{\delta,z}$
$\delta(\mathbf{e}_{\gamma,yz})$ = $z^2 \mathbf{e}_{\gamma,y}$	$-y^2 \mathbf{e}_{\gamma,z}$	$+w^7 \mathbf{e}_{\epsilon,z}$
$\delta(\mathbf{e}_{\delta,yz})$ = $z^2 \mathbf{e}_{\delta,y}$	$-y^2 \mathbf{e}_{\delta,z}$	$+x^3 \mathbf{e}_{\epsilon,z}$
$\delta(\mathbf{e}_{\zeta,yz}) = z^4 \mathbf{e}_{\zeta,y}$	$-y^2 \mathbf{e}_{\zeta,z}$	$+x^3 \mathbf{e}_{\eta,z}$

$$\delta(\mathbf{e}_{\alpha,xyz}) = z^2 \mathbf{e}_{\alpha,xy} \qquad -y^2 \mathbf{e}_{\alpha,xz} + x^2 \mathbf{e}_{\alpha,yz} \qquad -w^9 \mathbf{e}_{\beta,yz}$$

$$\delta(\mathbf{e}_{\gamma,xyz}) = z^2 \mathbf{e}_{\gamma,xy} \qquad -y^2 \mathbf{e}_{\gamma,xz} + x^3 \mathbf{e}_{\gamma,yz} \qquad -w^7 \mathbf{e}_{\delta,yz}$$

The *P*-graph of this Pommaret-like basis is given in Figure 1.

Example 7.9. Let us continue Example 4.6. Thus, we consider the minimal Pommaret-like basis $H = \{xy, y^3, xz, y^2z, z^2\}$ and we write $h_{\alpha} = xy$, $h_{\beta} = y^3$, $h_{\gamma} = xz$, $h_{\delta} = y^2z$, and $h_{\epsilon} = z^2$. Using



FIGURE 1. *P*-graph of Pommaret-like basis of Example 7.8. Each arrow is labelled with a Pommaret-like non-multiplicative power of the basis element belonging to the source. This non-multiplicative power is printed bold. Moreover, the label contains the associated cofactor, which is Pommaret-like multiplicative for the basis element belonging to the target of the arrow.

Formula 7.4, we obtain the following values of the differential δ of the induced resolution for basis elements of homological degree 2:

$\delta(\mathbf{e}_{lpha,yz})$ = $z\mathbf{e}_{lpha,y}$	$-y\mathbf{e}_{\gamma,y}-y^2\mathbf{e}_{lpha,z}$	$+x\mathbf{e}_{\beta,z}$
$\delta(\mathbf{e}_{\gamma,yz})$ = $z\mathbf{e}_{\gamma,y}$	$-y^2 \mathbf{e}_{\gamma,z}$	$+x\mathbf{e}_{\delta,z}$

The P-graph of this Pommaret-like basis is given in Figure 2.

In the remainder of this section, we aim to find a family of quasi-stable monomial ideals as large as possible such that for each ideal in the family, the resolution induced by its minimal Pommaret-like basis admits an explicit differential formula akin to Equation (7.4). For a given minimal Pommaret-like basis $H = \{h_{\alpha} \mid \alpha \in A\}$, we would like to find terms $u_{\alpha,j,\mu}$ such that the Pommaret-like induced resolution of the ideal $\langle H \rangle$ is described by the formula

$$\delta(\mathbf{e}_{\alpha,x^{\mu}}) = \sum_{x_j \in \mathrm{supp}(x^{\mu})} \mathrm{sgn}(x_j, \mathrm{supp}(x^{\mu})) \cdot \left(x_j^{p_j} \mathbf{e}_{\alpha,x^{\mu}/x_j} - u_{\alpha,j,\mu} \mathbf{e}_{\Delta(\alpha,j),x^{\mu}/x_j}\right).$$
(7.5)

In particular, we still work with resolutions supported on module basis elements $\mathbf{e}_{\alpha,x^{\mu}}$ where x^{μ} is a square-free term supported on $\{x_{\operatorname{cls}(h_{\alpha})}, \ldots, x_n\}$, and $\operatorname{deg}(x^{\mu})$ is the homological degree of the basis element. Moreover, the leading terms of the involved syzygies have polynomial parts $x_a^{p(a,h_{\alpha},H)}$, i.e., they are Pommaret-like non-multiplicative powers of some original basis element. Thus we can associate the multidegree $h_{\alpha} \cdot \prod_{x_a \in \operatorname{supp}(x^{\mu})} x_a^{p(a,h_{\alpha},H)}$ to the basis element $\mathbf{e}_{\alpha,x^{\mu}}$.

The original ideal is monomial or, in other words, multihomogeneous; thus, so is the induced resolution. The terms $u_{\alpha,j,\mu}$ need to ensure the multihomogeneity. The resolution is assumed to be



FIGURE 2. *P*-graph of Pommaret-like basis of Example 7.9. See Figure 1 for instructions on how to read this graph.

Pommaret-like induced; hence, the leading terms, which are given by non-multiplicative powers, determine the multidegrees of all involved syzygies. The differential δ is 0-multihomogeneous. Hence all terms in $\delta(\mathbf{e}_{\alpha,x^{\mu}})$ must be exactly of multidegree $h_{\alpha} \prod_{x_a \in \mathrm{supp}(x^{\mu})} x_a^{p(a,h_{\alpha},H)}$. This is true for the terms $x_j^{p_j} \mathbf{e}_{\alpha,x^{\mu}/x_j}$ since the multidegree of $\mathbf{e}_{\alpha,x^{\mu}/x_j}$ is $h_{\alpha} \prod_{x_a \in \mathrm{supp}(x^{\mu})} x_a^{p(a,h_{\alpha},H)}$. For each index j, a natural candidate for $u_{\alpha,j,\mu}$ is $t_{\alpha,j}$ since $x_j^{p_j} \cdot h_{\alpha} = t_{\alpha,j} \cdot h_{\Delta(\alpha,j)}$ and thus, the multidegree of $t_{\alpha,j} \mathbf{e}_{\Delta(\alpha,j),x^{\mu}/x_j}$ is $h_{\alpha} \cdot x_j^{p_j} \cdot \prod_{x_a \in \mathrm{supp}(x^{\mu})} x_a^{p(a,h_{\alpha,j}),H)}$. Consider now the term $v_{\alpha,j,\mu} = \prod_{\substack{x_a \in \mathrm{supp}(x^{\mu}) \\ a \neq j}} x_a^{p(a,h_{\alpha},H) - p(a,h_{\Delta(\alpha,j)},H)} \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$

If $t_{\alpha,j} \cdot v_{\alpha,j,\mu} \in \mathcal{T}$, then the terms $u_{\alpha,j,\mu} \coloneqq t_{\alpha,j} \cdot v_{\alpha,j,\mu}$ make (7.5) multihomogeneous. It is not hard to see that Δ -commuting quasi-stable ideals satisfy these conditions.

Before proving that Equation (7.5) indeed describes the Pommaret-like induced resolution, we first need to make explicit the P-orderings that we use in each homological degree of the Pommaret-like induced resolution.

Remark 7.10. Let $H = \{h_{\alpha} \mid \alpha \in A\} \subset \mathcal{T}$ be a minimal Pommaret-like basis and let $\mathbf{e}_{\alpha,x^{\mu}}, \mathbf{e}_{\beta,x^{\nu}}$ with $\deg(x^{\mu}) = \deg(x^{\nu}) = d$ be two basis elements of the free module F_d in the induced resolution. We work with the following *P*-ordering: $\mathbf{e}_{\alpha,x^{\mu}}$ precedes $\mathbf{e}_{\beta,x^{\nu}}$ if and only if either h_{α} precedes h_{β} in the *P*-ordering of *H*, or $\alpha = \beta$ and $x^{\mu} <_{\text{revlex}} x^{\nu}$.

Definition 7.11. Let $H = \{h_{\alpha} \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$. The ideal \mathcal{I} together with the basis H is called *weakly* Δ -*commuting* if the associated function Δ and the terms $t_{\alpha,a}$ (as given in Definition 7.1) satisfy the following property:

If $b > a > \operatorname{cls}(h_{\alpha})$ are two non-multiplicative indices and $\operatorname{cls}(h_{\Delta(\alpha,b)}) < a$, then we have $\Delta(\Delta(\alpha,a),b) = \Delta(\Delta(\alpha,b),a)$.

Before proceeding to the main theorem, we need the following elementary lemma:

Lemma 7.12. Let x^{μ} be a squarefree monomial and let $x_j, x_j \in \text{supp}(x^{\mu}) =: A$ with i < j be two variables that divide x^{μ} . Then $\text{sgn}(x_i, A) \cdot \text{sgn}(x_j, A \setminus \{x_i\}) = -\text{sgn}(x_j, A) \cdot \text{sgn}(x_i, A \setminus \{x_j\})$

Proof. Without loss of generality, we may assume that x_j has the highest index in $\operatorname{supp}(x^{\mu})$. Then, $\operatorname{sgn}(x_j, A \setminus \{x_i\}) = \operatorname{sgn}(x_j, A) = 1$. Thus, we need to show that $\operatorname{sgn}(x_i, A) = -\operatorname{sgn}(x_i, A \setminus \{x_j\})$. But this is clear, as $x_j \in A$ and j > i.

Theorem 7.13. Let $H = \{h_{\alpha} \mid \alpha \in A\}$ be the minimal Pommaret-like basis of the weakly Δ commuting quasi-stable monomial ideal $\langle H \rangle$. Let the terms $u_{\alpha,j,\mu}$ be given such that Equation (7.5)
together with $\delta(\mathbf{e}_{\alpha,1}) = h_{\alpha}$ defines a 0-multihomogeneous map δ of homological degree -1. Then δ is the differential of the Pommaret-like induced resolution of $\langle H \rangle$.

Proof. Let ∂ denote the differential of the Pommaret-like induced resolution. We need to show that $\delta = \partial$. First, note that the two maps are defined on the same free module(s). It is clear by definition that $\delta_0 = \partial_0$. Assume that we can prove that $\delta^2 = 0$. Then, elements of the form $\delta(\mathbf{e}_{\alpha,x_j})$ are in $\operatorname{Syz}^1(H)$, and, more generally, if $\deg(x^{\mu}) = d$, we have $\delta(\mathbf{e}_{\alpha,x^{\mu}}) \in \operatorname{Syz}^d(H)$. We work with the term ordering induced by the *P*-ordering on the basis *H*. For each $d \in \mathbb{N}_0$, we consider the set $G_d := \{\delta(\mathbf{e}_{\alpha,x^{\mu}}) \mid \deg(x^{\mu}) = d + 1\}$; we denote by F_d the analogous set for the differential ∂ . If we can show that $\operatorname{It}(G_d) = \operatorname{It}(F_d)$ for all *d*, and that G_d is autoreduced with respect to these leading terms, then since F_d is a reduced Gröbner basis and by the uniqueness of reduced Gröbner bases, it must hold that $G_d = F_d$.

We first show that $\delta^2 = 0$. So let $\mathbf{e}_{\alpha,x^{\mu}}$ be a module generator with $\deg(x^{\mu}) \ge 2$. We claim that $S := \delta^2(\mathbf{e}_{\alpha,x^{\mu}}) = 0$. S is a sum of module monomials supported on module basis elements of the form $\mathbf{e}_{\beta,x^{\mu}/x_ix_j}$, where $\{x_i, x_j\} \in \operatorname{supp}(x^{\mu})$ (assume i < j), and β lies in the index set $I(i,j) := \{\alpha, \Delta(\alpha, i), \Delta(\alpha, j), \Delta(\Delta(\alpha, i), j)\}$. (Note that, if $\Delta(\Delta(\alpha, j), i)$ exists, then it is equal to $\Delta(\Delta(\alpha, i), j)$.) As module monomials supported on basis elements of the form $\mathbf{e}_{\beta,x^{\mu}/x_ix_j}$ and $\mathbf{e}_{\beta,x^{\mu}/x_kx_\ell}$, where $\{i, j\} \neq \{k, \ell\}$, cannot cancel, it remains only to be seen that the summands supported on basis elements $\mathbf{e}_{\beta,x^{\mu}/x_ix_j}$, with i and j now fixed, cancel. As δ is multihomogeneous, we only need to determine the existing summands and show that their signs sum to zero.

The index set I(i, j) has at least 3 distinct elements, as $\alpha < \Delta(\alpha, i) < \Delta(\Delta(\alpha, i), j)$ in the *P*-order of the Pommaret-like basis *H*. We now distinguish two main cases:

If |I(i,j)| = 4, then $\Delta(\alpha,j) \neq \Delta(\Delta(\alpha,i),j)$ and hence $\Delta(\Delta(\alpha,j),i) = \Delta(\Delta(\alpha,i),j)$. These four nodes form a square in the *P*-graph. For each node, *S* contains exactly two summands. The sum of the the signs of the monomials supported on $\mathbf{e}_{\beta,x^{\mu}/x_ix_j}$ is as follows, where $A = \sup(x^{\mu})$:

• For $\beta = \alpha$:

 $\operatorname{sgn}(x_i, A)\operatorname{sgn}(x_j, A \setminus \{x_i\}) + \operatorname{sgn}(x_j, A)\operatorname{sgn}(x_i, A \setminus \{x_j\}),$

• For $\beta = \Delta(\alpha, i)$:

 $\operatorname{sgn}(x_i, A)(-1)\operatorname{sgn}(x_i, A \setminus \{x_i\}) + (-1)\operatorname{sgn}(x_i, A)\operatorname{sgn}(x_i, A \setminus \{x_i\}),$

• For $\beta = \Delta(\alpha, j)$:

$$\operatorname{gn}(x_i, A)(-1)\operatorname{sgn}(x_j, A \setminus \{x_i\}) + (-1)\operatorname{sgn}(x_j, A)\operatorname{sgn}(x_i, A \setminus \{x_j\}),$$

• For $\beta = \Delta(\Delta(\alpha, i), j)$:

$$(-1)$$
sgn $(x_i, A)(-1)$ sgn $(x_j, A \setminus \{x_i\}) + (-1)$ sgn $(x_j, A)(-1)$ sgn $(x_i, A \setminus \{x_j\})$.

The monomials in each of the four cases sum to zero by Lemma 7.12 and multihomogeneity of δ .

If |I(i,j)| = 3, then, as $\alpha < \Delta(\alpha,j)$ and $\Delta(\alpha,j) \neq \Delta(\alpha,i)$, we must have $\Delta(\alpha,j) = \Delta(\Delta(\alpha,i),j)$. The three nodes in I(i,j) form a triangle in the *P*-graph. For each node, *S* contains exactly two summands, but no others, as $\mathbf{e}_{\Delta(\alpha,j),x^{\mu}/x_{j}} = 0$ by convention, because $\operatorname{cls}(h_{\Delta(\alpha,j)}) \geq i$. The sum of the signs of the monomials supported on $\mathbf{e}_{\beta,x^{\mu}/x_{i}x_{j}}$ is as follows, where $A = \operatorname{supp}(x^{\mu})$:

- For $\beta = \alpha$: As in the case |I(i, j)| = 4.
- For $\beta = \Delta(\alpha, i)$: As in the case |I(i, j)| = 4.
- For $\beta = \Delta(\alpha, j) = \Delta(\Delta(\alpha, i), j)$:

 $(-1)\operatorname{sgn}(x_i, A)(-1)\operatorname{sgn}(x_j, A \setminus \{x_i\}) + \operatorname{sgn}(x_i, A)(-1)\operatorname{sgn}(x_j, A \setminus \{x_i\}).$

The monomials in each of the three cases sum to zero by Lemma 7.12 and multihomogeneity of δ . Thus we have shown $\delta^2 = 0$.

Now we proceed by analysing leading terms and showing autoreducedness. For $\mathbf{S} := \delta(\mathbf{e}_{\alpha,x^{\mu}}) \in G_d$, define $\hat{x} = x_{\hat{j}}$ as the variable with maximal index dividing x^{μ} . Then $\mathbf{lt}(\mathbf{S}) = x_{\hat{j}}^{p_j} \mathbf{e}_{\alpha,x^{\mu}/x_j}$, because terms of the form $x^{\nu} \mathbf{e}_{\alpha,\bullet}$, precede terms of the form $x^{\rho} \mathbf{e}_{\Delta(\alpha,\bullet),\bullet}$ in the *P*-ordering and $x^{\mu}/x_{\hat{j}}$ is the revlex-smallest term among terms of the form x^{μ}/x_i . So $\mathbf{lt}(\mathbf{S})$ equals the leading term of the element $\partial(\mathbf{e}_{\alpha,x^{\mu}})$ as desired. It remains to show autoreducedness. Assume that $\mathbf{lt}(\mathbf{S})$ divides a term in the support of $\mathbf{T} := \delta(\mathbf{e}_{\beta,x^{\nu}})$. Necessarily, $\deg(x^{\nu}) = \deg(x^{\mu})$. If $\beta = \alpha$, then it is clear that $\mathbf{lt}(\mathbf{S})$ does not divide any term of the form $x^{\zeta} \mathbf{e}_{\Delta(\beta,\bullet),\ldots}$ in the support of \mathbf{T} , as $\Delta(\beta, \bullet) \neq \beta = \alpha$. Assume it divides a term in the support of \mathbf{T} of the form $x_a^{p_a} \mathbf{e}_{\alpha,x^{\nu}/x_a}$. Then we must have $a = \hat{j}$ and $x^{\nu} = x^{\mu}$, i.e., $\mathbf{S} = \mathbf{T}$. If $\beta \neq \alpha$, then it is clear that $\mathbf{lt}(\mathbf{S})$ does not divide any term of the form $x_{a}^{p_a} \mathbf{e}_{\alpha,x^{\nu}/x_a}$. Then we must have $a = \hat{j}$ and $x^{\nu} = x^{\mu}$, i.e., $\mathbf{S} = \mathbf{T}$. If $\beta \neq \alpha$, then it is clear that $\mathbf{lt}(\mathbf{S})$ does not divide any term of the form $x_a^{p_a} \mathbf{e}_{\beta,x^{\nu}/x_a}$ in the support of \mathbf{T} , as $\beta \neq \alpha$. Assume now that $\mathbf{lt}(\mathbf{S})$ divides a term of the form $u_{\beta,r} \mathbf{e}_{\Delta(\beta,r),x^{\nu}/x_r}$ in the support of \mathbf{T} . Then we must have $\Delta(\beta, r) = \alpha$ and $x^{\nu}/x_r = x^{\mu}/x_{\hat{j}}$. The latter relation implies $x_{\hat{j}} \notin \mathrm{supp}(x^{\nu})$. Thus, the definition of $u_{\beta,r}$ gives $\deg_{\hat{j}}(u_{\beta,r}) < \gcd_{\hat{j}}(t_{\beta,r})$. But, as $\Delta(\beta, r) = \alpha$, $t_{\beta,r}$ is Pommaret-like multiplicative for h_{α} , and so $\deg_{\hat{j}}(u_{\beta,r}) < p(\hat{j},h_{\alpha},H) = \deg_{\hat{j}}(\mathbf{lt}(\mathbf{S}))$, contradicting the assumed divisibility. Thus the collection of all the $\delta(\mathbf{e}_{\alpha,x^{\mu}})$ is indeed autoreduced as claimed.

Example 7.14. Consider the minimal Pommaret-like basis $H = \{h_{\alpha} = xy, h_{\beta} = y^4, h_{\gamma} = xyz, h_{\delta} = y^2z, h_{\epsilon} = z^3\} \subset \mathbb{K}[x, y, z].$

It induces a minimal free resolution with differential represented by the following matrices:

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$$D_{0} = \begin{pmatrix} xy & y^{4} & xyz & y^{2}z & z^{2} \end{pmatrix}, D_{1} = \begin{pmatrix} y^{3} & z & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ 0 & -1 & 0 & y & z^{2} & 0 \\ 0 & 0 & -y^{2} & -x & 0 & z^{2} \\ 0 & 0 & 0 & 0 & -xy & -y^{2} \end{pmatrix}, D_{2} = \begin{pmatrix} z & 0 \\ -y^{3} & 0 \\ x & 0 \\ -y^{2} & z^{2} \\ 0 & -y \\ 0 & x \end{pmatrix}$$

One checks easily that the *P*-graph of *H* is, up to labels, the same as in Figure 2. Thus, *H* is weakly Δ -commuting. However, it is not Δ -commuting, because, even though $\Delta(\alpha, z) = \gamma$ and $\operatorname{cls}(h_{\gamma}) = x < z$, the non-multiplicative powers of h_{α} and h_{γ} differ, being y^3 and y, respectively.

Thus we cannot use Theorem 7.6 for finding the differential, but must resort to using Theorem 7.13. Assume we wanted to use Theorem 7.6. Then we would obtain the expression

$$z\mathbf{e}_{\alpha,y} - \mathbf{e}_{\gamma,y} - y^3\mathbf{e}_{\alpha,z} + x\mathbf{e}_{\beta,z}.$$

This does not correspond to the first column of D_2 . Using Theorem 7.13, the coefficient of $\mathbf{e}_{\gamma,y}$ is multiplied by the correction term y^2 , which is exactly the quotient of the non-multiplicative powers at y of h_{α} and h_{γ} , respectively. This way, we obtain the correct differential value

$$\delta(\mathbf{e}_{\alpha,yz}) = z\mathbf{e}_{\alpha,y} - y^2\mathbf{e}_{\gamma,y} - y^3\mathbf{e}_{\alpha,z} + x\mathbf{e}_{\beta,z}.$$

Example 7.15. Consider the minimal Pommaret-like basis $H = \{h_{\alpha} = x^3y, h_{\beta} = y^5, h_{\gamma} = x^3z, h_{\delta} = xy^3z, h_{\epsilon} = y^4z, h_{\zeta} = z^4\} \subset \mathbb{K}[x, y, z]$. It is not weakly Δ -commuting, because $\Delta(\Delta(\alpha, y), z) = \epsilon \neq \delta = \Delta(\Delta(\alpha, z), y)$. In fact, $\Delta(\Delta(\Delta(\alpha, z), y), y) = \epsilon$ and the *P*-graph of *H* contains a pentagon.

8. Conclusion

The Pommaret involutive division is known to be well suited to the analysis of free resolutions and homological invariants of polynomial ideals in quasi-stable position and of quasi-stable monomial ideals. In this article, we studied the resolutions obtained by using several generalizations of this division: The relative Pommaret and Pommaret-like divisions. These divisions allowed us to study also ideals in polynomial quotient rings as well as to obtain smaller bases. At the same time, we showed that the combinatorial properties of the Pommaret division are preserved. Our results not only enlarge the scope of the study of involutive bases, but also open new opportunities for enhancing the efficiency of involutive algorithms for the computation of free resolutions.

It is to be expected that the resolutions induced by Pommaret bases can be proven to be induced by mapping cones (see [12] for a definition), as is true for the resolutions induced by Pommaret bases [1]. Moreover, it may be worthwhile to investigate whether the Pommaret-like induced resolution is cellular, for instance using the techniques described by Iglesias and Sáenz de Cabezón [13].

In this work, we focused on resolutions over the ordinary polynomial ring and over quotient rings defined by Clements–Lindström ideals. However, the presented techniques are valid for any quotient ring defined by an ideal in quasi-stable position. A natural direction for further research is to find more general classes of such quotient rings for which the induced resolution has special properties.

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