Computing Finite and Infinite Free Resolutions with Pommaret-Like Bases

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Abstract

Free resolutions are an important tool in algebraic geometry for the structural analysis of modules over polynomial rings and their quotient rings. Minimal free resolutions are unique up to isomorphism and induce homological invariants in the form of Betti numbers. It is known that Pommaret bases of ideals in the polynomial ring induce finite free resolutions and that the Castelnuovo-Mumford regularity and projective dimension can be read off already from the Pommaret basis. In this article, we generalize this construction to Pommaret-like bases, which are generally smaller. We apply Pommaret-like bases also to infinite resolutions over quotient rings. Over Clements–Lindström rings, we derive bases for the free modules in the resolution using only the Pommaret-like basis. Finally, restricting to monomial ideals in a non-quotient polynomial ring, we derive an explicit formula for the differential of the induced resolution.

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1. Introduction

Involutive Bases have their origin in the works by Janet on the analysis of systems of (linear) partial differential equations (Janet, 1920, 1929). As in Gröbner basis theory, Janet used monomial, and thus combinatorial, structures as a tool with which more complex (differential) algebraic structures can be analysed. Inspired by Janet's—and also Pommaret's (Pommaret, 1978)—works, Zharkov and Blinkov developed involutive bases for polynomial ideals (Zharkov and Blinkov, 1996). Gerdt and Blinkov (1998) studied different types of involutive bases, introducing the framework of involutive divisions in the process. The most well-known involutive divisions—the Janet

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and Pommaret divisions—go back to Janet's works. Further involutive divisions have been studied; see, e.g., (Semenov, 2006; Hashemi et al., 2019).

Like Gröbner bases, involutive bases induce free resolutions of the ideals they generate. For some types of involutive divisions, the syzygy modules in this resolution are generated by involutive bases of the same type (Seiler, 2009b). In the case of the Pommaret division, homological invariants like projective dimension and Castelnuovo-Mumford regularity can be read off directly from the original Pommaret basis. Not every monomial ideal possesses a finite Pommaret basis; those that do are termed *quasi-stable*. For the resolution induced by the Pommaret basis of a quasi-stable monomial ideal, an explicit formula is known (Seiler, 2009b); however, this resolution is not necessarily minimal. This formula generalizes the well-known resolution formula found by Eliahou and Kervaire (1990), which only applies when the Pommaret basis coincides with the minimal generating set of the ideal. A polynomial ideal is said to be in *quasi-stable position* when it possesses a finite Pommaret basis for the given coordinates; moreover, this position is a generic one (Seiler, 2009a). For a comprehensive study and applications of the theory of involutive bases to commutative algebra and the geometric theory of partial differential equations, we refer to (Seiler, 2010).

Our contributions concern the use of relative involutive-like bases for the computation and analysis of free resolutions. For this, we focus on (relative) Pommaret and Pommaret-like bases. While Pommaret bases capture many homological properties of ideals in quasi-stable position (Seiler, 2010), the resolutions induced by them need not be minimal, because already the basis of the ideal might not be a minimal generating system. We show that Pommaret-like bases represent a significant improvement in this respect. Another aspect we investigate is the application to monomial ideals. For these, we are able to identify different classes of (relatively) quasi-stable ideals for which Pommaret-like bases induce the minimal free resolution. Even for other cases, the induced resolution has useful properties like Gröbner-reducedness in all higher syzygy modules. For a subclass of quasi-stable monomial ideals, we obtain closed formulas for the differential of the induced resolution, thereby significantly generalizing the formula by Eliahou and Kervaire (1990) for stable monomial ideals. Moreover, we relate our results to a resolution formula for square-free Borel ideals in zero-dimensional Clements–Lindström rings found by Gasharov et al. (2011).

The article is organized as follows. Section 2 aims to recall well-known facts about involutive bases, syzygies and free resolutions. Section 3 starts by analysing the resolutions induced by relative Pommaret bases. We focus on obtaining minimal Pommaret bases for the syzygy modules in each homological degree and observe phenomena that distinguish the relative situation from the case of resolutions over an ordinary polynomial ring. Pommaret-like bases are generally smaller than their Pommaret counterparts and provide better chances to get minimal resolutions. Section 4 studies resolutions induced by these bases. To carry out an analogous study over quotient rings, we introduce relative involutive-like divisions in Section 5. Section 6 analyses Pommaret-like induced resolutions over Clements–Lindström rings. We get a combinatorial formula for the bigraded Betti numbers of the induced resolutions. In Section 7, we obtain for some classes of monomial ideals explicit formulas for the differential of the Pommaret-like induced resolution, generalizing e.g. constructions by Eliahou and Kervaire (1990) and by Seiler (2010); Albert et al. (2015). Finally, some conclusions are given in Section 8.

2. Preliminaries

Let $\mathcal{P} = \mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[X]$ be the polynomial ring in n variables over a field $\mathbb{K}$, $\mathcal{I} \trianglelefteq \mathcal{P}$ an ideal of $\mathcal{P}$ and $\mathcal{P}/\mathcal{I}$ the quotient ring defined by $\mathcal{I}$.

As a $\mathbb{K}$-vector space, $\mathcal{P}$ has the basis $\mathcal{T} = \{x_1^{\mu_1} \cdots x_n^{\mu_n} \mid \mu_1, \dots, \mu_n \in \mathbb{N}_0\}$ of *terms*, which are products of non-negative integer powers of the variables. To each term $t = x^\mu = x_1^{\mu} \cdots x_n^{\mu_n} \in \mathcal{T}$ we associate its *total degree* $\deg(t) = \sum_{i=1}^n \mu_i$ and its *multidegree* $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}_0^n$. We write $\deg_i(t) = \mu_i$ for the degree of t in the variable x_i. A term ordering on $\mathcal{T}$ is denoted by $\prec$ and throughout we shall assume that $x_1 \prec \cdots \prec x_n$. The leading term of a given polynomial $0 \neq f \in \mathcal{P}$ with respect to $\prec$ is denoted by $\mathrm{lt}(f)$. If $F \subset \mathcal{P}$ is a finite set of polynomials, we denote by $\mathrm{lt}(F)$ the set $\{\mathrm{lt}(f) \mid f \in F\}$. A finite set $G \subset \mathcal{I}$ is called a *Gröbner basis* for $\mathcal{I}$ with respect to $\prec$, if its leading term ideal satisfies $\mathrm{lt}(\mathcal{I}) = \langle \mathrm{lt}(f) \mid f \in \mathcal{I} \rangle = \langle \mathrm{lt}(G) \rangle$. There is a well-known notion of term orderings and Gröbner bases in finitely generated free $\mathcal{P}$-modules; for the leading term of an element $\mathbf{v} \neq \mathbf{0}$ in such a module we write $\mathbf{lt}(\mathbf{v})$. We refer e. g. to (Cox et al., 2015; Adams and Loustaunau, 1994; Mora, 2005, 2016) for more details on Gröbner bases.

For an integer $d \geq 0$, we collect the subset of all terms of degree d in the set $\mathcal{T}_d \subset \mathcal{T}$. $\mathcal{T}_d$ generates the finite dimensional $\mathbb{K}$-vector space $\mathcal{P}_d$ of polynomials *homogeneous of degree d*: $\mathcal{P}_d = \langle \mathcal{T}_d \rangle_{\mathbb{K}}$. This induces the standard grading of $\mathcal{P}$. For a given multidegree $\mu = (\mu_1, \dots, \mu_n)$, we write $\mathcal{P}_\mu = \langle x^\mu \rangle_{\mathbb{K}}$ for the one-dimensional $\mathbb{K}$-vector space of *monomials* supported on the term x^μ. The direct sum $\mathcal{P} = \oplus_{\mu \in \mathbb{N}_0^n} \mathcal{P}_\mu$ induces the *multigrading* of $\mathcal{P}$. We work with ideals that are homogeneous or multihomogeneous; the latter are exactly the monomial ideals.

The main idea of an involutive division is to assign to each generator h in a basis H a subset $\mathrm{M}_L(h, H) \subseteq X$ of multiplicative variables and to consider only $\mathcal{P}$-linear combinations of the generators where each generator $h \in H$ is multiplied by a coefficient depending only on the variables in $\mathrm{M}_L(h, H)$. In contrast to Gröbner bases, not every monomial basis of a monomial ideal is automatically an involutive basis. The rule for the assignment of the multiplicative variables is called an *involutive division*.

Definition 2.1. An *involutive division* L on $\mathcal{T} \subset \mathcal{P}$ associates to any finite set $U \subset \mathcal{T}$ of terms and any term $u \in U$ a set of *L-non-multipliers* $\bar{L}(u, U)$ given by the terms contained in a prime monomial ideal. The variables generating this prime ideal are called the *non-multiplicative variables* $\mathrm{NM}_L(u, U) \subseteq X$ of $u \in U$. The set of *L-multipliers* $L(u, U)$ is given by the order ideal $\mathcal{T} \setminus \bar{L}(u, U)$; it is a subring of $\mathcal{P}$ generated by the set of *multiplicative variables* $\mathrm{M}_L(u, U) = X \setminus \mathrm{NM}_L(u, U)$. For any term $u \in U$, its *involutive cone* is defined as $\mathcal{C}_L(u, U) = u \cdot L(u, U)$. For an involutive division, the involutive cones must satisfy the following conditions:

(i) For two terms $v \neq u \in U$ with $\mathcal{C}_L(u, U) \cap \mathcal{C}_L(v, U) \neq \emptyset$, we have $u \in \mathcal{C}_L(v, U)$ or $v \in \mathcal{C}_L(u, U)$.

(ii) If a term $v \in U$ lies in an involutive cone $\mathcal{C}_L(u, U)$, then $L(v, U) \subset L(u, U)$.

(iii) For any term u in a subset $V \subset U$, we have $L(u, U) \subseteq L(u, V)$.

We write $u \mid_L w$ for a term $u \in U$ and an arbitrary term $w \in \mathcal{T}$, if $w \in \mathcal{C}_L(u, U)$. In this case, u is called an *L-involutive divisor* of w and w an *L-involutive multiple* of u.

Conditions (i) and (ii) ensure that involutive cones can intersect only trivially. Condition (iii) is often called the *filter axiom*. Obviously, it suffices for defining an involutive division to say what are the (non-)multiplicative variables for each term u in a finite set U. Note that involutive divisibility $u \mid_L w$ implies ordinary divisibility, but not vice versa.

As with Gröbner bases, involutive bases are defined via monomial structures. For monomial ideals, we define involutive bases as follows.

Definition 2.2. For a finite set of terms $U \subset \mathcal{T}$ and an involutive division L on $\mathcal{T}$, the *involutive span* of U is the union $C_L(U) = \bigcup_{u \in U} C_L(u, U)$. The set U is an *L-involutive basis* of the ideal generated by it, if $C_L(U) = \mathcal{T} \cdot U$ and the union is disjoint, i. e. every term in $C_L(U)$ has a unique involutive divisor. An involutive division L is *Nœtherian*, if every monomial ideal in $\mathcal{P}$ possesses an L-involutive basis. The L-involutive basis H of a monomial ideal $\mathcal{I}$ is *minimal*, if any other L-involutive basis H' of $\mathcal{I}$ contains H as a subset.

For involutive divisions that are *continuous* (Seiler, 2010, Def. 4.1.3) or even *constructive* (Seiler, 2010, Def. 4.1.7), the following useful properties hold:

Proposition 2.3. *(Seiler, 2010, Prop. 4.1.4) For a continuous involutive division L, a finite set of terms $U \subset \mathcal{T}$ is an L-involutive basis of the monomial ideal $\langle U \rangle$ if and only if, for each $u \in U$ and $x \in \mathrm{NM}_L(u, U)$, we have $xu \in C_L(U)$.*

We call the criterion implied by Proposition 2.3 the criterion of *local involutivity*.

Proposition 2.4. *(Seiler, 2010, Cor. 4.2.4) For a constructive Nœtherian involutive division L, every monomial ideal has a unique minimal L-involutive basis.*

Given a finite set H of polynomials, a term ordering $\prec$ and an involutive division L, we call H an *L-involutive basis*, if $\mathrm{lt}(H)$ is an L-involutive basis of $\mathrm{lt}(\mathcal{I})$ and the generators $h \in H$ have pairwise disjoint leading terms. We assign to each polynomial $h \in H$ the multiplicative variables $\mathrm{M}_L(\mathrm{lt}\,(h), \mathrm{lt}\,(H))$ and define the involutive cone $C_{L,H,\prec}(h) := h\mathbb{K}[\mathrm{M}_L(\mathrm{lt}\,(h), \mathrm{lt}\,(H))]$. An L-involutive basis H of an ideal $\mathcal{I}$ induces then a disjoint decomposition $\mathcal{I} = \bigoplus_{h \in H} C_{L,H,\prec}(h)$ as $\mathbb{K}$-linear spaces. In particular, each ideal element $f \in \mathcal{I}$ has a unique *involutive standard representation* $f = \sum_{h \in H} p_h \cdot h$, in which the coefficients $p_h \in \mathbb{K}[\mathrm{M}_L(\mathrm{lt}\,(h), \mathrm{lt}\,(H))]$ additionally fulfil $\mathrm{lt}(p_h) \cdot \mathrm{lt}(h) \preceq \mathrm{lt}(f)$. H is a *minimal* L-involutive basis of $\mathcal{I}$, if $\mathrm{lt}(H)$ is a minimal L-involutive basis of $\mathrm{lt}(\mathcal{I})$. Note that any involutive basis is also a Gröbner basis.

Two involutive divisions are particularly important in applications: The Janet and Pommaret division, respectively. The *Janet division* was, like the Pommaret division, already introduced by Janet (Janet, 1929, pp. 16-17). Let $U \subset \mathcal{T}$ be a finite set of terms. For each sequence $d_1, \ldots, d_n$ of non-negative integers and for each index $1 \le i \le n$, we introduce the corresponding *Janet class* as the subset

$$U_{[d_i,\ldots,d_n]} = \{u \in U \mid \deg_j(u) = d_j,\ i \le j \le n\} \subseteq U\ . \tag{2.1}$$

The variable x_n is called *Janet multiplicative* (J-multiplicative) for the term $u \in U$, if it holds $\deg_n(u) = \max\{\deg_n(v) \mid v \in U\}$. For $i < n$, x_i is Janet multiplicative for $u \in U_{[d_{i+1},\ldots,d_n]}$, if $\deg_i(u) = \max\{\deg_i(v) \mid v \in U_{[d_{i+1},\ldots,d_n]}\}$. The Janet division is Nœtherian, continuous, and constructive. We sometimes write $\mathrm{MinJB}(\mathcal{I})$ for the minimal

Janet basis of a given monomial ideal $\mathcal{I}$. We write $\mathrm{M}_J(u, U)$ for the set of Janet multiplicative variables of a term $u \in U$, and by $\mathrm{NM}_J(u, U)$ we denote the non-multiplicative variables.

Example 2.5. Consider the ideal $\mathcal{I} = \langle x_1 x_3^2,\ x_2 x_3,\ x_1^2 x_3 \rangle \subset \mathbb{K}[x_1, x_2, x_3]$. The given minimal generating set is not a Janet basis of $\mathcal{I}$, but enlarging the generating set by the term $x_2 x_3^2$, we obtain the Janet basis $\{x_1 x_3^2,\ x_2 x_3,\ x_1^2 x_3,\ x_2 x_3^2\}$ of $\mathcal{I}$.

We now proceed to the Pommaret division. The *class* of a term $1 \neq x^\mu \in \mathcal{T}$ with $\mu = (\mu_1, \dots, \mu_n)$ is defined as the index $\mathrm{cls}\,(x^\mu) = \min\,\{i \mid \mu_i \neq 0\}$. A variable x_i is Pommaret multiplicative for x^μ, if $i \leq \mathrm{cls}\,(x^\mu)$. All variables are Pommaret multiplicative for the trivial term 1. We write $\mathrm{M}_P(u)$ for the set of Pommaret multiplicative variables of a term $u \in \mathcal{T}$, and by $\mathrm{NM}_P(u)$ we denote the non-multiplicative variables. Note that the thus defined *Pommaret division* is global, i. e. the assignment of multiplicative variables is independent of any finite set $U \subset \mathcal{T}$. In contrast to the Janet division, the Pommaret division is not Nœtherian, as e. g. the ideal $\mathcal{I} = \langle x_1 x_2 \rangle$ does not possess a finite Pommaret basis (it does not contain an element of class 2). Nevertheless, the Pommaret division is continuous and constructive. If a monomial ideal $\mathcal{I}$ possesses a Pommaret basis, we sometimes write $\mathrm{MinPB}(\mathcal{I})$ for its minimal Pommaret basis.

For sufficiently large fields $\mathbb{K}$, the non-Nœtherianity of the Pommaret division is only a problem of the used coordinates. After a generic linear change of variables any ideal $\mathcal{I} \subseteq \mathcal{P}$ admits a finite Pommaret basis (Seiler, 2010, Thm. 4.3.15). In this case, $\mathcal{I}$ is said to be in *quasi-stable position*. An in-depth study of this question can be found in (Hashemi et al., 2018) together with a deterministic algorithm for the explicit construction of “good” coordinates for any given ideal $\mathcal{I} \subset \mathcal{P}$. For Pommaret bases, we will always consider the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$, as it is the only class-respecting term ordering (Seiler, 2010, Lem. A.1.8). As generally a monomial ideal does not remain monomial after a linear change of variables, Pommaret bases exist only for a special class of monomial ideals.

Definition 2.6. A monomial ideal $\mathcal{I}$ is called *quasi-stable*, if for any term $x^\mu \in \mathcal{I}$ and for any index i with $\mathrm{cls}(x^\mu) < i \leq n$ an exponent $s \geq 0$ exists such that $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \in \mathcal{I}$. A polynomial ideal $\mathcal{I}$ is in *quasi-stable position*, if $\mathrm{lt}(\mathcal{I})$ is quasi-stable.

Quasi-stable ideals appear in many places (and are known under many different names like *ideals of Borel type*, *ideals of nested type* or *weakly stable ideals*). Besides the above combinatorial definition, they can be characterised by many algebraic properties. For our purposes, the following characterisation is relevant.

Proposition 2.7. *(Seiler, 2010, Prop. 5.3.4) A monomial ideal $\mathcal{I}$ possesses a finite Pommaret basis, if and only if it is quasi-stable.*

There are two important generalisations of the concept of involutive divisions: Firstly, *relative involutive divisions*, which are defined relative to a given monomial ideal $\mathcal{I} \neq \{0\}$. Given a usual involutive division L, one can derive its relative counterpart $L_{\mathcal{I}}$, which then induces a theory of involutive bases in the quotient ring $\mathcal{P}/\mathcal{I}$. The details are documented in (Hashemi et al., 2021, Sec. 5). Secondly, *involutive-like divisions* are defined by the assignment of non-multiplicative pure variable powers instead of non-multiplicative variables (Hashemi et al., 2023, Sec. 6). The prototype of an involutive-like division is the Janet-like division (Gerdt and Blinkov, 2005b,a):

Definition 2.8. Let $U \subset \mathcal{T}$ be a finite set of terms. For any term $u \in U$ and a Janet non-multiplicative variable $x_i \in \mathrm{NM}_J(u, U)$, the power $x_i^{k_i}$ with

$$k_i = \min\{\deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1},\ldots,d_n]}, \deg_i(v) > \deg_i(u)\}$$

is called a *non-multiplicative* power of u for the *Janet-like division*. The set of all non-multiplicative powers of $u \in U$ is denoted by $\mathrm{NMP}(u, U)$. The elements of the set

$$\mathrm{NM}(u, U) = \{v \in \mathcal{T} \mid \exists w \in \mathrm{NMP}(u, U) : w \mid v\}$$

are called the *J-non-multipliers* for $u \in U$. The terms outside of it are the *J-multipliers* for u. An element $u \in U$ will be called a *Janet-like divisor* of $w \in \mathcal{T}$, if $w = u \cdot v$ with v a J-multiplier for u.

A Janet-like head autoreduced and finite set $U \subset \mathcal{T}$ is called *Janet-like basis* of the monomial ideal $\langle U\rangle$, if every term $t \in \langle U\rangle \cap \mathcal{T}$ has a Janet-like divisor in U. A finite set of polynomials $F \subset \mathcal{P} \setminus \{0\}$ is a *Janet-like basis* of $\mathcal{I} = \langle F\rangle$, if we have $\mathrm{lt}(f) \neq \mathrm{lt}(g)$ for all $f \neq g \in F$ and $\mathrm{lt}(F)$ forms a Janet-like basis for $\mathrm{lt}(\mathcal{I})$.

The *Pommaret-like division* was defined in (Hashemi et al., 2023, Def. 6.11).

Definition 2.9. The *Pommaret-like division P* assigns to each term $t \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T}$ non-multiplicative powers as follows:

1. The Janet non-multiplicative variables x_a with $a > \mathrm{cls}(t)$,
2. The Janet non-multiplicative powers $x_b^{p_b}$ with $b > \mathrm{cls}(t)$.

Note that no non-multiplicative power is assigned to any variable x_b with $b \leq \mathrm{cls}(t)$.

Let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_r\} \subseteq \mathcal{P}^s$ be an enumerated finite subset of a finitely generated free module $\mathcal{P}^s$. The *syzygy module* of F is a submodule of $\mathcal{P}^r$ defined by

$$\mathrm{Syz}(F) = \left\{(g_1, \ldots, g_r)^T \in \mathcal{P}^r \mid \sum_{i=1}^{r} g_i \mathbf{f}_i = 0\right\}.$$

For subsets $F \subseteq (\mathcal{P}/\mathcal{I})^s$, we write $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(F)$ to emphasize that we are working over the quotient ring.

We use syzygies to construct free resolutions of homogeneous ideals $\mathcal{I} \trianglelefteq \mathcal{P}$. A free resolution $\mathbf{F}$ of $\mathcal{I}$ is given by finitely generated free $\mathcal{P}$-modules $F_0, F_1, \ldots$ and homogeneous $\mathcal{P}$-linear maps $\delta_0, \delta_1, \delta_2, \ldots$ as in the following diagram

$$\mathbf{F}: \quad \cdots \xrightarrow{\delta_{m+2}} F_{m+1} \xrightarrow{\delta_{m+1}} F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{I} \to 0,$$

such that $\mathrm{im}(\delta_0) = \mathcal{I}$ and $\mathrm{im}(\delta_{m+1}) = \ker(\delta_m)$ for all $m \in \mathbb{N}_0$. The collection $\{\delta_m\}_{m\geq 0}$ of maps is called the *differential* of the resolution. Leaving aside degree shifts, we can write $F_m = \mathcal{P}^{r_m}$ for $m \geq 0$. Each map δ_m is completely described by the images $\delta(\mathbf{e}_i)$, $i \in \{1, \ldots, r_m\}$; equivalently, δ_m is represented by a matrix $D_m \in \mathcal{P}^{r_{m-1}\times r_m}$, whose i-th column is exactly $\delta_m(\mathbf{e}_i)$. Note that we interpret the module $\mathcal{I}$ as a submodule of $\mathcal{P}^1$, so the matrix D_0 describing δ_0 is of format $(r_0 \times 1)$. Moreover, $D_m \cdot D_{m+1} = 0$ for all m.

The above discussion now implies the next observation: $G = \{\delta_0(\mathbf{e}_1), \ldots, \delta_0(\mathbf{e}_{r_0})\}$ is a homogeneous generating set of $\mathcal{I}$ and the columns of D_1 form a homogeneous generating set G_1 of $\mathrm{Syz}(G)$. Generally, the set G_m of columns of D_m is a homogeneous generating set of the iterated syzygy module $\mathrm{Syz}^m(G)$.

Finally, we note that also for ideals $\mathcal{J} \trianglelefteq \mathcal{P}/\mathcal{I}$ in a quotient ring over a homogeneous ideal $\mathcal{I}$, resolutions by finitely generated free $\mathcal{P}/\mathcal{I}$-modules exist. For these resolutions, $\delta_0(F_0) = \mathcal{J}/\mathcal{I}$ and all modules F_m, $m \geq 0$, are direct sums of copies of $\mathcal{P}/\mathcal{I}$. Otherwise, the terminology is the same.

Since we work with homogeneous ideals $\mathcal{I}$, the matrices in any free resolution of $\mathcal{I}$ have homogeneous polynomials as entries. A free resolution is *minimal* if all entries in the matrices are either 0 or of positive degree. Up to isomorphism, there is exactly one minimal free resolution for each ideal $\mathcal{I}$. Since the ranks of the involved free modules F_m in a minimal free resolution are invariant under isomorphisms, they are a homological invariant of $\mathcal{I}$. They are called *(bigraded) Betti numbers* of $\mathcal{I}$.

Assume that in a minimal free resolution $\mathbf{F}$ of $\mathcal{I}$, $F_m = \oplus_{d\geq 0}\mathcal{P}(-d)^{\beta_{m,d}}$ for $m > 1$; then the numbers $\beta_{m,d} = \beta_{m,d}(\mathcal{I})$ are the Betti numbers of $\mathcal{I}$. By Hilbert's syzygy theorem, the minimal free resolution of $\mathcal{I} \trianglelefteq \mathcal{P}$ is of finite length. Thus, the collection $\{\beta_{m,d}(\mathcal{I})\}_{m,d\geq 0}$ of non-zero Betti numbers of $\mathcal{I}$ is finite. By minimality of $\mathbf{F}$, the sequence $(\min\{d \geq 0 \mid \beta_{m,d}(\mathcal{I}) > 0\})_{m\geq 0}$ is increasing; thus we can present the non-zero Betti numbers in a matrix $(b_{d,m})_{0\leq d\leq r, 0\leq m\leq s} = (\beta_{m,d+m}(\mathcal{I})) \in \mathbb{N}_0^{(r+1)\times(s+1)}$ for some positive integers $s = s(\mathcal{I})$, $r = r(\mathcal{I})$, such that there are neither trailing zero rows nor trailing zero columns.

Consider a homogeneous ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ and a minimal free resolution of it, yielding the numbers $r(\mathcal{I})$ and $s(\mathcal{I})$ of rows and columns in its Betti table. Then $\mathrm{reg}(\mathcal{I}) = r(\mathcal{I})$ is the *Castelnuovo–Mumford regularity*, or simply *regularity*, of $\mathcal{I}$, and $\mathrm{projdim}(\mathcal{I}) = s(\mathcal{I})$ is its *projective dimension*.

The minimal $\mathcal{P}/\mathcal{I}$-free resolutions of homogeneous ideals $\mathcal{J} \trianglelefteq \mathcal{P}/\mathcal{I}$ are in general infinite in the sense that infinitely many non-zero Betti numbers exist. Thus, ideals in $\mathcal{P}/\mathcal{I}$ in general do not have a finite regularity or a finite projective dimension. As a succinct way of writing the infinitely many Betti numbers, we use *Poincaré series*. They are formal power series in two independent variables—say u and s—such that the coefficient of a term $u^m s^d$ is given by the Betti number $\beta_{m,d}(\mathcal{J})$.

Consider an L-involutive basis $H \subset \mathcal{P}$ of a polynomial ideal $\mathcal{I} = \langle H \rangle$ with respect to a continuous involutive division L. The set $\mathrm{lt}(H)$ is an L-involutive basis of the leading ideal $\mathrm{lt}(\mathcal{I})$. One can construct an acyclic directed graph, the *L-graph*, with node set $\mathrm{lt}(H)$ and arrows from $\mathrm{lt}(h_i)$ to $\mathrm{lt}(h_j)$ whenever there is a non-multiplicative variable $x \in \mathrm{NM}_L(\mathrm{lt}(h_i), \mathrm{lt}(H))$ such that $\mathrm{lt}(h_j)$ is an L-divisor of $x\,\mathrm{lt}(h_i)$ (Seiler, 2010, Lem. 5.4.5). Now consider the following method of enumerating $\mathrm{lt}(H)$: As first element $\mathrm{lt}(h_1)$, take any leading term whose node in the L-graph is not the target of any arrow. Deleting $\mathrm{lt}(h_1)$ and its associated arrows from the graph, we obtain another acyclic graph, and as the second element $\mathrm{lt}(h_2)$ in the enumeration we take a leading term whose node is not the target of any arrow in the modified graph. Continuing in this manner, we obtain an *L-ordering* of $\mathrm{lt}(H)$.

Adapting Schreyer's construction (Schreyer, 1980), one can use the L-involutive basis H, ordered according to an L-ordering, to construct a Gröbner basis G_{Syz} of $\mathrm{Syz}(H)$ that has as leading terms the module terms $x\mathbf{e}_i$ with $x \in \mathrm{NM}_L(\mathrm{lt}(h_i), \mathrm{lt}(H))$.

If L is of *Schreyer type* (Seiler, 2010, Def. 5.4.8), G_{Syz} is again an L-involutive basis, and the construction can be iterated to yield a linear, but generally non-minimal, free resolution of $\langle H \rangle$. The Pommaret and Janet divisions are of Schreyer type (Seiler, 2010, Lem. 5.4.9). We use Schreyer-type constructions also for relative involutive bases.

The resolution induced by the Pommaret basis of a homogeneous ideal $\mathcal{I}$ in quasi-stable position can be used to determine the Castelnuovo–Mumford regularity and projective dimension of $\mathcal{I}$ without computing the minimal free resolution of $\mathcal{I}$. The Castelnuovo–Mumord regularity is simply the largest degree of a generator in the Pommaret basis; the projective dimension is the maximal number of non-multiplicative variables that an element of the Pommaret basis can have. For further details, see (Seiler, 2010, Sec. 5.5).

For a quasi-stable monomial ideal $\mathcal{I}$, we refer to (Seiler, 2010, Thm. 5.4.18) for an explicit formula for the differential of the resolution induced by the monomial Pommaret basis. It is immediate from (Seiler, 2010, Eq. (5.53)) that the resolution is minimal if and only if $\mathcal{I}$ is stable. The formula can be read off from the *weighted P-graph* of the basis, which includes for each arrow $h_i \to h_j$ not only the variable $x \in \text{NM}_P(h_i)$ with $xh_i \in C_P(h_j)$, but also the cofactor $t \in \mathbb{K}[\text{M}_P(h_j)]$ such that $xh_i = th_j$.

3. Resolutions induced by relative Pommaret bases

Let us recall the definition from (Hashemi et al., 2021) of the concept of an involutive division $L_\mathcal{I}$ relative to a monomial ideal $\mathcal{I}$ of *Schreyer type*. As our aim is to define a related notion better suited to the computation of free resolutions, we repeat it here for the reader's convenience.

Definition 3.1. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal and $L_\mathcal{I}$ an involutive division relative to $\text{lt}(\mathcal{I})$ induced by a continuous involutive division L on $\mathcal{T}$. We say that $L_\mathcal{I}$ is of *Schreyer type* if, whenever H is an $L_\mathcal{I}$-involutive basis of $\langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$ and G is an L-involutive basis of $\mathcal{I}$, we have that for all $x^\mu \in \text{lt}(H)$ the monomial set

$$B = \left(\left\{ \frac{\text{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \text{lt}(G) \right\} \setminus \text{lt}(\mathcal{I}) \right) \cup \left(\text{NM}_{L_\mathcal{I}}(x^\mu, \text{lt}(H)) \right) \tag{3.1}$$

is an $L_{\text{lt}(\mathcal{I})}$-involutive basis of the ideal $\langle B \rangle + \text{lt}(\mathcal{I})$ relative to $\text{lt}(\mathcal{I})$.

The following example shows that Definition 3.1 is not optimal:

Example 3.2. In $\mathcal{P} = \mathbb{K}[x, y]$, consider the ideals $\mathcal{I} = \langle x^3, y^3 \rangle$ and $\mathcal{J} = \langle x^2, xy, y^2 \rangle$ from (McCullough and Peeva, 2015, Ex. 5.2). Note that the monomial ideals $\mathcal{I}$ and $\mathcal{J}$ are quasi-stable. The minimal Pommaret basis of $\mathcal{I}$ is $G = \{x^3, x^3y, x^3y^2, y^3\}$ and the minimal Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$ is $H = \{x^2, xy, y^2\}$. We can now apply (Hashemi et al., 2021, Prop. 5.14) to obtain a Pommaret basis for $\text{Syz}_{\mathcal{P}/\mathcal{I}}(H)$—note that G and H are already ordered according to a $P_\mathcal{I}$-ordering. Precisely, the enumerations are $g_1 = x^3, g_2 = x^3y, g_3 = x^3y^2, g_4 = y^3$ and $h_1 = x^2, h_2 = xy, h_3 = y^2$.

Using (Hashemi et al., 2021, Thm. 5.13), we compute a Pommaret basis of the first syzygy module of H relative to $\mathcal{I}$, being a subset of the free $\mathcal{P}/\mathcal{I}$-module $(\mathcal{P}/\mathcal{I})^3$ with the canonical basis $\{\mathbf{e}_1^{(1)}, \mathbf{e}_2^{(1)}, \mathbf{e}_3^{(1)}\}$ (the superscript encodes the homological degree.) We underline the leading module terms.

- As A-syzygies, we obtain $\mathbf{A}_1 = \underline{x}\mathbf{e}_1^{(1)}$, $\mathbf{A}_2 = \underline{xy}\mathbf{e}_1^{(1)}$, and $\mathbf{A}_3 = \underline{xy^2}\mathbf{e}_1^{(1)}$ for h_1, $\mathbf{A}_4 = \underline{x^2}\mathbf{e}_2^{(1)}$, $\mathbf{A}_5 = \underline{x^2y}\mathbf{e}_2^{(1)}$, and $\mathbf{A}_6 = \underline{y}^2\mathbf{e}_2^{(1)}$ for h_2, as well as $\mathbf{A}_7 = \underline{y}\mathbf{e}_3^{(1)}$ for h_3.
- As syzygies from non-multiplicative prolongations, we obtain $\mathbf{S}_1 = \underline{y}\mathbf{e}_1^{(1)} - x\mathbf{e}_2^{(1)}$ for h_1 and $\mathbf{S}_2 = \underline{y}\mathbf{e}_2^{(1)} - x\mathbf{e}_3^{(1)}$ for h_2.

We notice immediately that the relative Pommaret basis $\{\mathbf{A}_1, \ldots, \mathbf{A}_7, \mathbf{S}_1, \mathbf{S}_2\}$ is not minimal, as the leading terms of the syzygies $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_5$, $\mathbf{A}_6$ are redundant.

As seen in Example 3.2, for relative Pommaret divisions, Definition 3.1 implies that the relative Pommaret bases computed for the syzygy modules of an $\mathcal{P}/\mathcal{I}$-free resolution are generally non-minimal. Furthermore, relative Janet divisions are in general *not* of Schreyer type if one applies Definition 3.1—see (Hashemi et al., 2021, Ex. 5.15). While it is true that the definition ensures that relative divisions of Schreyer type are suitable for the computation of free resolutions, the construction is not optimal. The reason for this is that the set B in Equality (3.1) is not chosen optimally. Indeed, it is in general not autoreduced with respect to classical (non-restricted) division, as the multipliers of the form $\operatorname{lcm}(x^\nu, x^\mu)/x^\mu$, which are needed for the A-syzygies, may be divisible by non-multiplicative variables. Thus, we propose the following adapted definition.

Definition 3.3. Let $\mathcal{I}$ and $L_\mathcal{I}$ be as in Definition 3.1. Then $L_\mathcal{I}$ is of *strong Schreyer type* if, whenever H is an $L_\mathcal{I}$-involutive basis of $\langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$ and G is an L-involutive basis for $\mathcal{I}$, then for all $x^\mu \in \operatorname{lt}(H)$, the set

$$M(x^\mu, \operatorname{lt}(H), \operatorname{lt}(G)) \cup \mathrm{NM}_{L_\mathcal{I}}(x^\mu, \operatorname{lt}(H))$$

is an $L_\mathcal{I}$-involutive basis for the monomial ideal it generates relative to $\operatorname{lt}(\mathcal{I})$ where the set of *multiplicative A-multipliers* $M(x^\mu, \operatorname{lt}(H), \operatorname{lt}(G))$ is defined by

$$M(x^\mu, \operatorname{lt}(H), \operatorname{lt}(G)) = \left\{ \frac{\operatorname{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \operatorname{lt}(G) \right\} \setminus \Big(\operatorname{lm}(\mathcal{I}) + \langle \mathrm{NM}_{L_\mathcal{I}}(x^\mu, \operatorname{lt}(H))\rangle\Big). \quad (3.2)$$

Note that the set $M(x^\mu, \operatorname{lt}(H), \operatorname{lt}(G)) \cup \mathrm{NM}_{L_\mathcal{I}}(x^\mu, \operatorname{lt}(H))$ from Definition 3.3 is a subset of the set B defined in Equality (3.1).

Proposition 3.4. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal in quasi-stable position and P the Pommaret division on $\mathcal{T}$. Then the relative involutive division $P_{\operatorname{lt}(\mathcal{I})}$ induced by P is of strong Schreyer type.*

Proof. Let G be the Pommaret basis of $\mathcal{I}$ and let H be a Pommaret basis of the ideal $\langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$. For each $x^\mu \in \operatorname{lt}(H)$, we have to show that the set $B' = M(x^\mu, \operatorname{lt}(H), \operatorname{lt}(G)) \cup \mathrm{NM}_{P_{\operatorname{lt}(\mathcal{I})}}(x^\mu)$ is a Pommaret basis of the ideal it generates relative to $\operatorname{lt}(\mathcal{I})$. We know that $B' \subseteq B$, where B is defined as in Equality (3.1). Moreover, from the definitions, it is easy to see that $\langle B, \operatorname{lt}(\mathcal{I})\rangle = \langle B', \operatorname{lt}(\mathcal{I})\rangle$. We still have to show that B' is a relative Pommaret basis. Note that B contains a Pommaret basis and each term in $t \in B \setminus B'$ is divisible by a variable x_j with $j > \operatorname{cls}(x^\mu)$, i.e., by a non-multiplicative variable for x^μ. Assume that j is the maximal index having this property. Then, $t \in \mathcal{C}_P(x_j)$. We can deduce that B' also contains a Pommaret basis. Also, it is clear that the Pommaret cones $\mathcal{C}_P(x_j)$ and $\mathcal{C}_P(t)$, where $x_j \in \mathrm{NM}_P(x^\mu)$ and $t \in \mathbb{K}[\mathrm{M}_P(x^\mu)] \cap B'$, have empty intersection (look at the x_j-degrees). Finally, we need to show that all Pommaret cones $C_P(t)$, $C_P(s)$, where $s \neq t \in \mathbb{K}[\mathrm{M}_P(x^\mu)] \cap B'$, have empty intersection. For this, first note that $\mathrm{cls}(s) = \mathrm{cls}(\overline{s})$ and $\mathrm{cls}(t) = \mathrm{cls}(\overline{t})$, where $\overline{s}, \overline{t} \in \mathrm{lt}(G)$ are the terms inducing the multipliers s, t for x^μ. Indeed, s and t are Pommaret multiplicative for x^μ, and so the only indices i for which $\deg_i(\overline{s}) > \deg_i(x^\mu)$ is possible are indices $i \leq \mathrm{cls}(x^\mu)$. And there must be an index with this property, as otherwise $\overline{s}|x^\mu$, which is impossible. The minimal such index then obviously is just $\mathrm{cls}(\overline{s})$, and similarly for $\overline{t}$. Hence, a non-empty intersection of the Pommaret cones of s and t would imply a non-empty intersection of the Pommaret cones $\overline{s}$ and $\overline{t}$. This is impossible, because $\overline{s}, \overline{t}$ are elements of the Pommaret basis of the ideal $\mathrm{lt}(\mathcal{I})$. □

Remark 3.5. The Janet division $J_{\mathcal{I}}$ relative to a monomial ideal $\mathcal{I}$ is not of strong Schreyer type, even when $\mathcal{I}$ is quasi-stable: Consider $\mathcal{I} = \langle x_1^2 x_3^2, x_3^3, x_2 x_3^2 \rangle = \langle G \rangle$ and $H = \{x_1, x_2\}$. $\mathcal{I}$ is a stable ideal, and H is a Janet basis relative to $\mathcal{I}$. Consider the term $x^\mu = x_1$. The set $M(x^\mu, H, G) \cup \mathrm{NM}_{J_{\mathcal{I}}}(x^\mu, H)$ is given by $\{x_1 x_3^2\} \cup \{x_2\}$. This is not a Janet basis relative to $\mathcal{I}$, because the variable x_3 is non-multiplicative for x_2 in this set, but $x_2 x_3$ does not possess a Janet divisor in the same set.

Proposition 3.4 ensures that we get minimal Pommaret bases in each step of the resolution computation. We use Schreyer orderings for these Pommaret bases, which depend on P-orderings. There is an easy procedure by which P-orderings can be obtained automatically for the next syzygy module. Indeed, for any given generator of the current module, we need to take first the multiplicative A-syzygies in the order that is induced by the ordering on G. Then we take the non-multiplicative variables in ascending order. We do this for each generator sequentially, and we obtain a minimal Pommaret basis, already P-ordered, for the next syzygy module.

Example 3.6. Consider $\mathcal{I} = \langle x^2, xy, y^2 \rangle$ and $\mathcal{J} = \langle x^3, y^3 \rangle = \langle x^3, x^3y, x^3y^2, y^3 \rangle$ as in Example 3.2. Applying Proposition 3.4 repeatedly, we obtain relative Pommaret bases for the next iterated syzygy modules as follows. Note that that the columns of D_k represent the minimal relative Pommaret basis for the k-th iterated syzygy module.

$$D_2 = \begin{pmatrix} x^2 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & y^2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & x & y & 0 & 0 \\ 0 & 0 & xy & 0 & -x^2 & y^2 & 0 \\ 0 & 0 & x^2 & 0 & 0 & xy & y^2 \end{pmatrix}, D_3 = \begin{pmatrix} x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & x^2 & y & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & -x & y^2 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & x^2 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & y \end{pmatrix},$$

$$D_4 = \begin{pmatrix} x^2 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & -x^2 & y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & 0 & 0 & xy & 0 & 0 & 0 & -x^2 & y^2 & 0 \\ 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & xy & y^2 \end{pmatrix}, \ldots$$

Remark 3.7. Example 3.6 shows that constants can appear in some homological degree k of the Pommaret-induced resolution even if there are no constants in the differential at the previous homological degree $k-1$. This is for instance the case for the homological degree 4 in Example 3.6: the matrix D_4 contains constants even though D_3 does not. This behaviour of the induced resolution is new, compared to the Pommaret-induced resolutions for $\mathcal{P}$-modules: in (Seiler, 2010, Lem. 5.5.1), it is shown that a Pommaret-induced resolution over $\mathcal{P}$ is minimal if and only if the first differential does not contain any constant terms.

Example 3.8. In $\mathcal{P} = \mathbb{K}[x, y, z]$, consider the ideals $\mathcal{I} = \langle z^3 \rangle$, $\mathcal{J} = \langle xyz, y^2z, yz^2, \mathcal{I} \rangle$. With the usual notations, we verify by computation that the Pommaret-induced resolution is the minimal $\mathcal{P}/\mathcal{I}$-free resolution of the $\mathcal{P}/\mathcal{I}$-module $\mathcal{J}/\mathcal{I}$:

$$D_0 = \begin{pmatrix} xyz & y^2z & yz^2 \end{pmatrix},\ D_1 = \begin{pmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},\ D_2 = \begin{pmatrix} z & 0 & 0 & 0 \\ -y & z^2 & 0 & 0 \\ x & 0 & z^2 & 0 \\ 0 & xz & yz & z^2 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} z^2 & 0 & 0 & 0 \\ y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},\ D_4 = D_2.$$

4. Resolutions induced by Pommaret-like bases

For analysing free resolutions induced by Pommaret-like bases, a necessary first step is to understand resolutions over the polynomial ring $\mathcal{P}$ of quasi-stable monomial ideals $\mathcal{I}$ generated by a Pommaret-like basis H. If H is also a Pommaret basis, then the structure of the induced resolution is known (Seiler, 2010). In the case that $\mathcal{I}$ is a monomial ideal, the resolution is minimal if and only if H is the minimal generating set of $\mathcal{I}$, see (Seiler, 2010). Note that if $\mathcal{I}$ is not monomial, this result does not hold in general, see e.g. (Seiler, 2010, Ex. 5.5.9). As a first step to a similar result for Pommaret-like bases, some combinatorial characterization of monomial ideals whose minimal generating set is also a Pommaret-like basis are helpful.

Remark 4.1. If a Pommaret-like basis H of a monomial ideal $\mathcal{I}$ is given, then ordering the elements ascendingly with respect to the lexicographic ordering with $x_1 \prec \cdots \prec x_n$ gives a P-ordering, because for each $h \in H$ and $x_j^{p_j} \in \mathrm{NMP}_P(h, H)$, the Pommaret-like divisor $u \in H$ of $x_j^{p_j} \cdot h$ fulfils $\deg_j(u) = \deg_j(h) + p_j$ and $\deg_\ell(u) = \deg_\ell(h)$ for $\ell > j$. Thus, $h \prec_{\text{lex}} u$. From this P-ordering, one can derive a Schreyer ordering in the syzygy module which has non-multiplicative powers as leading terms.

A Pommaret-like basis H of an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ in quasi-stable position induces a free resolution of $\mathcal{I}$ over $\mathcal{P}$, and at each homological degree, the corresponding syzygy module is generated by a Pommaret-like basis (Hashemi et al., 2023). There are special classes of ideals for which this induced resolution is in fact the minimal free resolution. One class of ideals for which this is true is the class of *componentwise linear ideals* (provided that the ideal is in componentwise quasi-stable position (Hashemi et al., 2018, Thm. 19)). We can apply (Seiler, 2010, Thm. 5.5.2) to see this, even though that result is concerned with Pommaret bases, because Pommaret bases are a special kind of Pommaret-like bases. Moreover, for *stable* monomial ideals the induced Pommaret-like resolution is also minimal because the Pommaret resolution is (Seiler, 2010, Prop. 5.5.6). The following result shows that the class of monomial ideals for which the Pommaret-like resolution is minimal is larger than the class of ideals for which the Pommaret resolution is minimal i.e., stable monomial ideals:

Theorem 4.2. *Let $\{0\} \neq \mathcal{I} \triangleleft \mathcal{P}$ be a quasi-stable monomial ideal such that its minimal basis is simultaneously its minimal Pommaret-like basis $H \subset \mathcal{I} \cap \mathcal{T}$. Moreover, assume that for each Pommaret-like non-multiplicative power $x_j^{p_j}$ of t with respect to the set H, it holds $(t/x_{\mathrm{cls}(t)})x_j^{p_j} \in \mathcal{I}$. Then the free resolution of $\mathcal{I}$ over $\mathcal{P}$ induced by the basis H is the minimal free resolution of $\mathcal{I}$ over $\mathcal{P}$.*

Before we prove Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let $H \subset \mathcal{T}$ be a minimal Pommaret-like basis generating the ideal $\mathcal{I} = \langle H \rangle$. The condition $(x_j^{p_j} \cdot t)/x_{\mathrm{cls}(t)} \in \mathcal{I}$ in Theorem 4.2 is equivalent to the statement that the unique Pommaret-like divisor $s \in H$ of $x_j^{p_j} \cdot t$ fulfils $\mathrm{cls}((x_j^{p_j} \cdot t)/s) \leq \mathrm{cls}(s)$.*

Proof. Let $k = \mathrm{cls}(t)$ and let s be the unique Pommaret-like divisor in H of $(x_j^{p_j} \cdot t)$. Note that $j > k$. Moreover, since H is minimal, $f = (x_j^{p_j} \cdot t)/s \neq 1$. By the definition of Pommaret-like non-multiplicative powers, it is clear that $(x_j^{p_j} \cdot t)|_{x_1=\cdots=x_{j-1}=1} = s|_{x_1=\cdots=x_{j-1}=1}$. Hence we have $\mathrm{cls}(f) < j$. Now, if $\mathrm{cls}(f) \leq \mathrm{cls}(s)$, then $k = \mathrm{cls}(f)$, and x_k is multiplicative for s. From this we see $(x_j^{p_j} \cdot t)/x_k \in \mathcal{I}$. Conversely, if if $\mathrm{cls}(f) > \mathrm{cls}(s)$, then $s \cdot f$ is an element of the minimal Pommaret basis of $\mathcal{I}$, and $k = \mathrm{cls}(s)$. Thus $(s \cdot f)/x_k \notin \mathcal{I}$ and consequently $(x_j^{p_j} \cdot t)/x_k \notin \mathcal{I}$. □

Proof of Theorem 4.2. We show that no non-zero constant terms appear in the matrices describing the differentials of the induced resolution. Write the resolution as

$$\mathbf{F} : \cdots \xrightarrow{d_3} \mathcal{P}^{b_2} \xrightarrow{d_2} \mathcal{P}^{b_1} \xrightarrow{d_1} \mathcal{P}^{|H|} \xrightarrow{d_0} \mathcal{I} \to 0.$$

The matrix D_0 describing d_0 consists of one row containing the elements of H as entries. Hence, no constant terms appear there. As the next step, we show that in the matrix D_1 describing d_1 there are no constant terms. By construction and using the fact that $\mathcal{I}$ is a monomial ideal, each column of D_1 contains only two non-zero entries: $x_a^{p_a}$ (a non-multiplicative power of a term $t \in H$) and a cofactor $f \in \mathcal{T}$ such that

$$t \cdot x_a^{p_a} = s \cdot f, \tag{4.1}$$

where $s \in H$ is the unique term such that $t \cdot x_a^{p_a} \in C_P(s, H)$. Since the set H is by assumption the minimal monomial generating set of $\mathcal{I}$, we have $f \neq 1$. Hence, no column of D_1 contains any constant term and the whole D_1 is free of constant terms.

The columns of D_1 represent a minimal Pommaret-like basis of the first syzygy module $\mathrm{Syz}(H) \subset \mathcal{P}^{|H|}$ of H. The leading module terms $x^\mu \mathbf{e}_i^{(1)}$ of this syzygy module are exactly of the form $x_a^{p_a}\mathbf{e}_i^{(1)}$ where $x_a^{p_a}$ is a non-multiplicative power of the i-th element of H. They are found in the i-th row of D_1. There may be other non-zero entries in the said row, but they are cofactors f as given in Equality (4.1). Moreover, in the situation of Equality (4.1), it is clear that $\text{cls}(t) \leq \text{cls}(s)$ and hence, by Lemma 4.3,

$$\text{cls}(f) \leq \text{cls}(s). \tag{4.2}$$

From this it follows that $\text{cls}(f) < \text{cls}(x_b^{p_b})$ for all non-multiplicative powers $x_b^{p_b}$ of s. We will use this property in the next step.

The matrix D_2 has as many rows as D_1 has columns. Each column of D_2 contains at least the non-zero entry $x_c^{p_c}$, a non-multiplicative power of a generator of the leading module of $\text{Syz}(H)$. Since this leading term module is generated by module terms whose monomial parts are the non-multiplicative powers of the set H, also the non-multiplicative powers of this leading term module will have monomial parts of the same form. These non-multiplicative powers are obviously not constants. The further non-zero entries of a column of D_2 result from the involutive-like standard representation of the vector $x_c^{p_c} \cdot \mathbf{c}$, where $\mathbf{c}$ is a column of D_1, with respect to the set of all columns of D_1. We focus on the possible non-zero entries that can be generated by the cancellations which happen in the i-th row. During the involutive-like reduction process, it can happen that an intermediate result has a non-zero entry there, but this entry will be of the form $f \cdot p$, where p is some polynomial and f is a term with the properties given in (4.2). In the column of D_2 encoding the involutive-like reduction we are studying at present, a non-zero entry (other than the one already analysed) can be created in row j only if the j-th column $\mathbf{c}_j$ of D_1 has as its leading module term $x_b^{p_b}\mathbf{e}_i^{(1)}$, where $x_b^{p_b}$ is as studied in Equality (4.2). The class condition given in Equality (4.2) now guarantees that the non-zero entry generated in the j-th row of the column of D_2 will be free of constant terms. What is more, all terms in the support of this entry will have class less or equal to $\text{cls}(f)$. Now, since the indices i and j in the discussion above were arbitrary, we have proved that also the matrix D_2 does not contain any constant terms.

The last thing we need to prove is that, also in D_2, we have a condition on the classes of terms analogous to that given in Equality (4.2). If we can show this, then an iteration of the arguments used for the analysis of D_2 can be applied to all successive matrices in the resolution.

To prove this class condition, again consider the j-th row of D_2, where a non-zero entry q with $\text{cls}(q) \leq \text{cls}(f)$ is located resulting from a step in an involutive-like reduction which uses the leading module term $x_b^{p_b}\mathbf{e}_i^{(1)}$ of the j-th column of D_1. We need to compare this class with the classes of all leading module terms of $\text{Syz}^2(H)$ of the form $u \cdot \mathbf{e}_j^{(2)}$. But these leading module terms arise from non-multiplicative powers of the leading module term $x_b^{p_b}\mathbf{e}_i^{(1)}$ in the leading module of $\text{Syz}(H)$, and hence $u = x_d^{p_d}$ for some index $b < d \leq n$. Using Equality (4.2), it is now clear that $\text{cls}(q) \leq \text{cls}(f) < \text{cls}(x_b^{p_b}) < \text{cls}(x_d^{p_d})$, i.e., the class condition we need is fulfilled. □

Below, we continue by giving two examples for minimal free resolutions induced by Pommaret-like bases.

Example 4.4. Here, we show that the class of monomials satisfying the conditions of Theorem 4.2 is larger than the class of monomial ideals for which we can construct minimal free resolutions as proved in (Seiler, 2010). Let $a, b, c \geq 1$ be any three positive integers and let $\mathcal{I} = \langle x^a, y^b, z^c \rangle$ be an irreducible monomial ideal given by its minimal generating system $H = \{x^a, y^b, z^c\}$, which is easily seen to be also a Pommaret-like basis. Moreover, H satisfies the additional assumptions of Theorem 4.2. Hence, it induces a minimal Pommaret-like free resolution of $\mathcal{I}$. The matrices of the differentials are given as follows:

$$D_0 = \begin{pmatrix} x^a & y^b & z^c \end{pmatrix},\ D_1 = \begin{pmatrix} y^b & z^c & 0 \\ -x^a & 0 & z^c \\ 0 & -x^a & -y^b \end{pmatrix},\ D_2 = \begin{pmatrix} z^c \\ -y^b \\ x^a \end{pmatrix}.$$

Example 4.5. In the polynomial ring $\mathbb{K}[w, x, y, z]$ with $w \prec x \prec y \prec z$, consider the monomial ideal $\mathcal{I} = \langle H \rangle$ with

$$H = \{w^9x^3y^2z^2, x^5y^2z^2, w^7y^4z^2, x^3y^4z^2, y^6z^2, x^3y^2z^4, y^4z^4, z^8\}.$$

(The elements have been ordered lexicographically from lowest to highest.) One can verify that H is simultaneously the minimal generating system of $\mathcal{I}$ and a Pommaret-like basis satisfying the additional assumptions of Theorem 4.2. Hence, it induces a minimal Pommaret-like free resolution of $\mathcal{I}$. The matrices of the differentials are given as follows:

$$D_0 = \begin{pmatrix} w^9x^3y^2z^2 & x^5y^2z^2 & w^7y^4z^2 & x^3y^4z^2 & y^6z^2 & x^3y^2z^4 & y^4z^4 & z^8 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} x^2 & y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -w^9 & 0 & 0 & y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^3 & y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w^9 & 0 & -x^2 & 0 & -w^7 & 0 & 0 & y^2 & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -w^7 & 0 & -x^3 & 0 & z^2 & 0 & 0 & 0 \\ 0 & 0 & -w^9 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & z^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w^7 & 0 & -x^3 & -y^2 & -x^3 & 0 & z^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^3y^2 & -y^4 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x^2 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & -y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ w^9 & 0 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & w^9 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y^2 & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x^3 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x^3 & -y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & w^7 & 0 & 0 & z^2 & 0 \\ 0 & 0 & w^9 & x^2 & 0 & w^7 & 0 & -y^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w^7 & x^3 & 0 \\ 0 & 0 & -w^9 & -x^2 & 0 & 0 & 0 & 0 & z^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \end{pmatrix},\ D_3 = \begin{pmatrix} z^2 & 0 \\ -y^2 & 0 \\ x^2 & 0 \\ -w^9 & 0 \\ 0 & z^2 \\ 0 & -y^2 \\ 0 & x^3 \\ 0 & -w^7 \\ 0 & 0 \end{pmatrix}.$$

We shall notice that Theorem 4.2 does not completely cover the class of quasi-stable monomial ideals whose Pommaret-like bases induce minimal free resolutions. In other words, there exist quasi-stable monomial ideals that do not satisfy the theorem's assumptions but whose Pommaret-like bases nevertheless induce the minimal free resolution:

Example 4.6. In the polynomial ring $\mathbb{K}[x, y, z]$, consider the monomial ideal $\mathcal{I}$ with minimal generating set $G = \{xy, y^3, xz, y^2z, z^2\}$. As one can check, G is a Pommaret-like basis. The generator $t = xy$ has the non-multiplicative powers y^2 and z. While it is true that $(t/x) \cdot y^2 = y^3 \in \mathcal{I}$, we have $(t/x) \cdot z = yz \notin \mathcal{I}$. Only if we increase the exponent of the variable z to 2, i.e., higher than the non-multiplicative power, we reach the term $yz^2 \in \mathcal{I}$. The Pommaret-like basis G induces a minimal free resolution with differential represented by the following matrices:

$$D_0 = \begin{pmatrix} xy & y^3 & xz & y^2z & z^2 \end{pmatrix},\; D_1 = \begin{pmatrix} y^2 & z & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ 0 & -y & 0 & y^2 & z & 0 \\ 0 & 0 & -y & -x & 0 & z \\ 0 & 0 & 0 & 0 & -x & -y^2 \end{pmatrix},\; D_2 = \begin{pmatrix} z & 0 \\ -y^2 & 0 \\ x & 0 \\ -y & z \\ 0 & -y^2 \\ 0 & x \end{pmatrix}.$$

We finish this section with a result that is useful for relating resolutions induced by Pommaret-like bases to other free resolutions.

Proposition 4.7. *Let $\{0\} \neq \mathcal{I} \neq \mathcal{P}$ be a polynomial ideal in quasi-stable position and H its minimal Pommaret-like basis. Then the free resolution induced by H consists of* reduced *Gröbner bases for all syzygy modules* $\mathrm{Syz}^m(H)$*, $m \geq 1$. In other words, in each homological degree $m \geq 1$, the set of columns of the matrix describing the differential is the unique reduced Gröbner basis of* $\mathrm{Syz}^m(H)$ *for the chosen module term order.*

Proof. For $m \geq 1$ let D_m be the matrix in the induced free resolution that represents the differential map δ_m. By (Hashemi et al., 2023, Thm. 7.7), the set of columns of D_1, say $C(D_1)$, is a Pommaret-like basis of $\mathrm{Syz}(H)$. The leading terms of this basis are given by $Z = \{x_a^{p_a} \cdot \mathbf{e}_i \mid h_i \in H \wedge x_a^{p_a} \in \mathrm{NMP}_P(h_i, H)\}$, and $|C(D_1)| = |Z|$. Since Z is the minimal generating set for the module it generates, $C(D_1)$ is a minimal Gröbner basis. It is reduced because all non-leading module terms arise as coefficients in an involutive-like reduction computation. More precisely, if the first syzygy generators induced by H do not form a reduced set, then there exists a Pommaret-like multiplicative term t for $x^{\mu_i} \in H$ which is divisible by a Pommaret-like non-multiplicative term s for x^{μ_i}, leading to a contradiction. Thus we have shown the claim for $m = 1$. Note that $C(D_1)$ is in particular again a Pommaret-like basis. The claim now follows by induction on m and using (Hashemi et al., 2023, Thm. 7.7). □

5. Relative involutive-like divisions

In this section, we introduce a combination of the concepts of relative involutive divisions and involutive-like divisions. For a detailed explanation of relative involutive divisions, see (Hashemi et al., 2021); for the definition and properties of involutive-like divisions, see (Hashemi et al., 2023, Sec. 6).

Definition 5.1. Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{P}$ be a nonzero monomial ideal. An *involutive-like division $L_{\mathcal{I}}$ relative to $\mathcal{I}$* associates to any finite set $U \subset \mathcal{T} \setminus \mathcal{I}$ of terms and any term $u \in U$ a set of *$L_{\mathcal{I}}$-non-multipliers* $\overline{L_{\mathcal{I}}}(u, U)$ given by the terms contained in an irreducible monomial ideal. The powers generating this irreducible ideal are called the *non-multiplicative powers* and the set of these powers is denoted by $\mathrm{NMP}_{L_{\mathcal{I}}}(u, U)$. The set of *$L_{\mathcal{I}}$-multipliers* $L_{\mathcal{I}}(u, U)$ is given by the order ideal $\mathcal{T} \setminus \overline{L_{\mathcal{I}}}(u, U)$. For any term $u \in U$, its *relative involutive-like cone* is defined as $C_{L_{\mathcal{I}}}(u, U) = u \cdot L_{\mathcal{I}}(u, U) \setminus \mathcal{I}$. For a relative involutive-like division, the relative involutive-like cones must satisfy the following conditions:

1. For two terms $v \neq u \in U$ with $C_{L_{\mathcal{I}}}(u, U) \cap C_{L_{\mathcal{I}}}(v, U) \neq \emptyset$, we have $u \in C_{L_{\mathcal{I}}}(v, U)$ or $v \in C_{L_{\mathcal{I}}}(u, U)$.
2. If a term $v \in U$ lies in an involutive cone $C_{L_{\mathcal{I}}}(u, U)$, then $L_{\mathcal{I}}(v, U) \subset L_{\mathcal{I}}(u, U)$.

Remark 5.2. As discussed in (Hashemi et al., 2023, Def. 6.1), it is not possible to include in Definition 5.1 an analogon to the filter axiom.

It is straightforward to prove that from an involutive-like division L on $\mathcal{T}$, one can derive a relative involutive-like division $L_{\mathcal{I}}$ by using the same rule for the assignment of non-multiplicative powers as for L and merely adapting the cones to make them subsets of $\mathcal{T} \setminus \mathcal{I}$. One can do this, in particular, for the important special case of the Janet-like division J.

Definition 5.3. Let $\mathcal{I} \lhd \mathcal{P}$ be a nonzero monomial ideal and let $U \subset \mathcal{T} \setminus \mathcal{I}$ be a finite set of terms. Let $u \in U$ be a term. Then the non-multiplicative powers of u with respect to U, $\mathcal{I}$ and the relative Janet-like division $J_{\mathcal{I}}$ are defined as follows:

$$\mathrm{NMP}_{J_{\mathcal{I}}}(u, U) = \mathrm{NMP}_{J}(u, U) \setminus (\mathcal{I} : u). \tag{5.1}$$

Therefore and in other words, the relative Janet-like division uses the same rule for the assignment of non-multiplicative powers as the Janet-like division J, but it excludes variable powers that form part of the ideal quotient associated to the term u in question.

If x_a is a variable for which a relative Janet-like non-multiplicative power for u exists, then we write the exponent of this power as $p(J_{\mathcal{I}}, u, U, a)$.

Remark 5.4. The relative Janet-like division $J_{\mathcal{I}}$ is an involutive-like division relative to $\mathcal{I}$. This fact can be easily proven by using the properties of the Janet-like division J. Also other properties like the continuity of the Janet-like division J are inherited by $J_{\mathcal{I}}$.

The definition of the Pommaret-like division $P_{\mathcal{I}}$ relative to a monomial ideal $\mathcal{I}$ should aim to guarantee that the following properties are fulfilled:

1. Cones should be disjoint, if they are not contained in each other.
2. For $u \in U$, no non-multiplicative powers should be assigned for $x_1, , \ldots, x_{\mathrm{cls}(u)}$.
3. For $\mathcal{J} \supset \mathcal{I}$, a relative Pommaret-like basis should exist if and only if $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ (see Definition 5.8 below).
4. A unique minimal relative Pommaret-like basis should exist for any monomial ideal that is quasi-stable relative to $\mathcal{I}$.
5. The minimal relative Pommaret-like basis should be as small as possible.

These considerations lead to the following definition.

Definition 5.5. Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{P}$ be a nonzero monomial ideal. The *Pommaret-like division $P_{\mathcal{I}}$ relative to $\mathcal{I}$* assigns to each term $u \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T} \setminus \mathcal{I}$ non-multiplicative powers as follows: For each x_a with $a > \mathrm{cls}(u)$, if $x_a \in \mathrm{NM}_J(u, U)$, then set $p(P_{\mathcal{I}}, u, U, a) = p(J_{\mathcal{I}}, u, U, a)$. If $x_a \in \mathrm{M}_J(u, U)$ and there does not exist any exponent $s \in \mathbb{N}$ with $u \cdot x_a^s \in \mathcal{I}$, set $p(P_{\mathcal{I}}, u, U, a) = 1$. No other variable gets assigned a non-multiplicative power with respect to the relative Pommaret-like division $P_{\mathcal{I}}$. In particular, no variable x_b with $b \le \operatorname{cls}(u)$ is assigned a relative Pommaret-like non-multiplicative power for the term u.

Proposition 5.6. *The relative Pommaret-like division $P_{\mathcal{I}}$ is a relative involutive-like division.*

Proof. Let $u \neq v \in U$ be two terms in the finite subset $U \subset \mathcal{T} \setminus \mathcal{I}$. Let $k = \max\{\operatorname{cls}(u), \operatorname{cls}(v)\}$. If $k = n = \operatorname{cls}(u) = \operatorname{cls}(v)$, disjointness of the relative Pommaret-like cones is easily seen, as also in the case where $k = n$ but one of $\operatorname{cls}(u), \operatorname{cls}(v)$ is less than n. If $k < n$ and the projections $u|_{x_1=\cdots=x_k=1}$, $v|_{x_1=\cdots=x_k=1}$ are equal, then either disjointness or containment of the relative Pommaret-like cones is also easily seen. It remains the case when $k < n$ but the projections on the subring $\mathbb{K}[x_{k+1}, \ldots, x_n]$ are not equal. There, note that from any two elements $u' = su$ and $v' = tv$, where s and t are $P_{\mathcal{I}}$-multiplicative terms, we get in the subring that the projections of s and t are Janet-like multipliers of the projections of u and v. Hence the projections of the relative Pommaret-like cones of u and v on the same subring are either contained one in the other or they are disjoint. If they are disjoint, the same also holds true for the full cones in the whole ring $\mathcal{P}$. If they are contained one in the other, then checking the k-degrees of u and v will yield that the full cones are either disjoint or contained. A containment will hold if and only if the term with larger class, without loss of generality v, has a smaller or equal x_k-degree compared to that of the other term and the projection of the cone of v in the subring is a superset of the other cone projection. □

Definition 5.7. Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{P}$ be a nonzero monomial ideal and let $\mathcal{J} \supset \mathcal{I}$ be a further monomial ideal in $\mathcal{P}$. Let $L_{\mathcal{I}}$ be an involutive-like division relative to $\mathcal{I}$. A finite set of terms $H \subset \mathcal{T} \cap (\mathcal{J} \setminus \mathcal{I})$ is called an *$L_{\mathcal{I}}$-involutive like basis of $\mathcal{J}$ relative to $\mathcal{I}$* if every term $t \in \mathcal{T} \cap (\mathcal{J} \setminus \mathcal{I})$ has a unique $L_{\mathcal{I}}$ involutive-like divisor in the set H.

We recall the following definition from (Hashemi et al., 2021, Def. 7.1).

Definition 5.8. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. We say that $\mathcal{J}$ is *quasi-stable relative to $\mathcal{I}$*, if for all terms $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ and for all indices i with $\operatorname{cls}(x^\mu) < i \le n$ there exists an exponent $s \ge 0$ such that either $x_i^s x^\mu \in \mathcal{I}$ or $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$.

Theorem 5.9. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. Then there exists a Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$ if and only if $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$.*

Proof. First assume that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. By (Hashemi et al., 2021, Prop. 7.4), we know that there exists a relative Pommaret basis H of $\mathcal{J}$. Since relative Pommaret-like cones are always supersets of relative Pommaret cones, the set H is also a relative Pommaret-like basis of $\mathcal{J}$, but not necessarily a minimal one.

Now assume that there exists a Pommaret-like basis $H \subset \mathcal{T} \cap (\mathcal{J} \setminus \mathcal{I})$ of $\mathcal{J}$ relative to $\mathcal{I}$. Arguing by *reductio ad absurdum*, suppose that $\mathcal{J}$ is not quasi-stable relative to $\mathcal{I}$. In particular, H is a generating set of $\mathcal{J}$ relative to $\mathcal{I}$. Since $\mathcal{J}$ is not quasi-stable relative to $\mathcal{I}$, there is a term $1 \neq h \in H$ and an index $j > k = \operatorname{cls}(h)$ such that for every exponent $s \in \mathbb{N}$ we have $(h/x_k)x_j^s \notin \mathcal{J}$ and $hx_j^s \notin \mathcal{I}$. Consider the Janet class

$C = H_{[\deg_{j+1}(h),\dots,\deg_n(h)]}$. Among the terms in C, there is one with maximal x_j-degree. Let this degree be denoted by d.

Now, since H is a relative Pommaret-like basis, the term $h \cdot x_j^{d-\deg_j(h)}$ has a $P_{\mathcal{I}}$-divisor u in H. By definition of the $P_{\mathcal{I}}$-like division, u must be an element of the Janet class $H_{[d,\deg_{j+1}(h),\dots,\deg_n(h)]}$. Moreover, it must be a divisor (in the non-involutive sense) of $h \cdot x_j^{d-\deg_j(h)}$, so there is a term $t \in \mathcal{T}$ with $h \cdot x_j^{d-\deg_j(h)} = u \cdot t$. Now, if there were an exponent $e \in \mathbb{N}$ with $u \cdot x_j^e \in \mathcal{I}$, then also $u \cdot t \cdot x_j^e \in \mathcal{I}$ and hence $h \cdot x_j^{e+d-\deg_j(h)} \in \mathcal{I}$, in contradiction to the assumptions made for h. Hence, such an exponent e does not exist. Moreover, by construction, $x_j \in \mathrm{M}_J(u, H)$. Additionally, it is not possible that $\mathrm{cls}(u) \geq j$, because otherwise u would be a divisor of $(h/x_k)x_j^{d-\deg_j(h)}$, again in contradiction to the assumptions made for h.

By the statements just shown and by Definition 5.5, one $P_{\mathcal{I}}$-non-multiplicative power of u with respect to H is x_j^1. Now, $u \cdot x_j \in \mathcal{J} \setminus \mathcal{I}$, and it cannot have any $P_{\mathcal{I}}$-divisor in H, since such a divisor would be an element of a Janet class $H_{[d+1,\deg_j(h),\dots,\deg_n(j)]}$. But this Janet class is empty by the maximality property of d. All in all, we have shown that there is a term in $\mathcal{J} \setminus \mathcal{I}$ which has no $P_{\mathcal{I}}$-like divisor in the set H. This contradicts the assumption that H is a relative Pommaret-like basis of $\mathcal{J}$. □

Example 5.10. Consider the ideals $\mathcal{I} = \langle x^6, y^6, z^6 \rangle$ and $\mathcal{J} = \langle \mathcal{I}, xy, yz \rangle$ taken from (Gasharov et al., 2011, Ex. 4.12). The set $H = \{xz, yz\}$ is a Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$. Indeed, for the Janet division $\mathrm{M}_J(yz, H) = \{x, y, z\}$ and $\mathrm{M}_J(xz, H) = \{x, z\}$. The set of Janet-like non-multiplicative powers of xz is $\mathrm{NMP}_J(xz, H) = \{y\}$. Note that for each term $h \in H$ and for each variable, by multiplying h with a high enough power of the variable, we get a term in $\mathcal{I}$. Hence, the relative Pommaret-like non-multiplicative powers of the terms in H are $\mathrm{NMP}_{P_{\mathcal{I}}}(yz, H) = \emptyset$, $\mathrm{NMP}_{P_{\mathcal{I}}}(xz, H) = \{y\}$. Now, it is clear that all non-multiplicative multiples of xz are in the relative Pommaret-like cone of yz and H is a relative Pommaret-like basis as claimed.

Example 5.10 can be generalized. Gasharov et al. (2011) studied the (infinite) free resolution of a special type of squarefree ideal in a particular quotient ring, a so-called Clements–Lindström ring. In the rest of this paper, we give an alternative approach to construct such a resolution via relative Pommaret-like bases. As the first step, we show in Proposition 5.13 that the minimal generating set of a square-free Borel monomial ideal is a relative minimal Pommaret-like basis. In this direction, we need two definitions. The first is taken from (Gasharov et al., 2011). The second goes back essentially to Clements and Lindström (1969). We adapt both to our conventions on variable orderings. Below, for a term s, we refer to $\mathrm{supp}(s)$ as the set of all variables appearing in s. Furthermore, for a given monomial ideal $\mathcal{I}$, $\mathrm{Min}(\mathcal{I})$ stands for its minimal generating set of terms.

Definition 5.11. We call a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ generated by squarefree terms *squarefree Borel*, if for any (necessarily squarefree) term $s \in \mathrm{Min}(\mathcal{I})$ the following holds: For any variable $x_i \in \mathrm{supp}(s)$ and any index j with $i < j \leq n$ such that $x_j \notin \mathrm{supp}(s)$, we have $(s/x_i) \cdot x_j \in \mathcal{I}$.

Definition 5.12. We call an irreducible, non-zero monomial ideal $\mathcal{I} \lhd \mathcal{P}$ is *Clements–Lindström*, if $\mathrm{Min}(\mathcal{I})$ is of the form $\{x_i^{a_i}, x_{i+1}^{a_{i+1}}, \dots, x_n^{a_n}\}$ with $2 \le a_n \le a_{n-1} \le \cdots \le a_{i+1} \le a_i$. We call $\mathcal{P}/\mathcal{I}$ a *Clements–Lindström ring*.

Proposition 5.13. *Let $\mathcal{I}$ be a zero-dimensional Clements–Lindström ideal and let H be the minimal generating set of a square-free Borel monomial ideal. Then the ideal $\mathcal{J} = \langle \mathcal{I}, H\rangle$ is quasi-stable relative to $\mathcal{I}$ and the set H is the minimal Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. As a zero-dimensional Clements–Lindström ideal, $\mathcal{I} = \langle x_1^{a_1}, \dots, x_n^{a_n}\rangle$ with $a_1 \ge \cdots \ge a_n \ge 2$. The square-free minimal generating set H of $\mathcal{J}$ is disjoint from $\mathcal{I}$. Hence, it is the minimal generating set of $\mathcal{J}$ relative to $\mathcal{I}$. Exclude in the following the trivial special case $H = \{1\}$. H fulfils the square-free Borel property. Hence, for any term $1 \ne h \in H$ and any index $j > k = \mathrm{cls}(h)$ such that $x_j \notin \mathrm{supp}(h)$, there is another term $u_1 \in H$ dividing $(h/x_k)x_j$. Since H is a minimal generating set, we have $\mathrm{cls}(u_1) \le j$. If either $\mathrm{cls}(u_1) = j$ or

$$\mathrm{supp}(u_1) \supseteq \{x_j\} \cup \{x_a \in \mathrm{supp}(h) \mid a > j\}$$

then we have are done and obtained the desired term u_1. Otherwise, we repeat this process, by finding $u_2 \in H$ such that u_2 divides $(h/x_{\mathrm{cls}(u_1)})x_\ell$ with $\ell > j$ and $x_\ell \in \mathrm{supp}(h)$. We know that for i, the constructed term u_i fulfils the condition $\mathrm{cls}(u_i) \le j$. It is clear to see that the sequence of $u_1, u_2, \dots$ is finite. Assume that u_m is the last constructed term. If $\mathrm{supp}(u_m) \supseteq \{x_j\} \cup \{x_a \in \mathrm{supp}(h) \mid a > j\}$ then we are done. Otherwise, we have $\mathrm{cls}(u_m) = j$ and $\mathrm{supp}(u_m) \not\supseteq \{x_j\} \cup \{x_a \in \mathrm{supp}(h) \mid a > j\}$. Repeating the mentioned process on u_m leads to a contradiction to the minimality of H. Thus, at the end, for each index j with $j > \mathrm{cls}(h)$, we arrive at a term v such that $\mathrm{supp}(v) \cap \{x_{\mathrm{cls}(v)}, \dots, x_n\} = \{x_j\} \cup \{x_a \in \mathrm{supp}(h) \mid a \ge \mathrm{cls}(v)\}$.

Now, for a given index $j > \mathrm{cls}(h)$, assume that the term v satisfies the above equality. Both h and v are in the Janet class $H_{[\deg_{j+1}(h), \dots, \deg_n(h)]}$ and this shows that $x_j \in \mathrm{NM}_J(h, H)$ and x_j is a $P_{\mathcal{I}}$-non-multiplicative power of h with respect to H. On the other hand, we know that each variable $x_a \in \mathrm{supp}(h)$ with $a > \mathrm{cls}(h)$ lies in $\mathrm{M}_J(h, H)$, because H is square-free and $\deg_a(h) = 1$. Additionally, for each such variable x_a, of course there is an exponent $e \in \mathbb{N}$ such that $hx_a^e \in \mathcal{I}$, because $\mathcal{I}$ is zero-dimensional. Hence, for such variable x_a, no $P_{\mathcal{I}}$-non-multiplicative power exists for h. These arguments imply that $x_j \cdot h \in \mathcal{C}_{P_{\mathcal{I}}}(v)$. Applying a local involution argument, we see that H is a relative Pommaret-like basis of $\mathcal{J}$. □

Remark 5.14. A close inspection of the proof of Proposition 5.13 shows that the proposition holds also under slightly weaker conditions. Let $k \in \{1, \dots, n\}$ be defined as $\min\{\mathrm{cls}(h) \mid h \in H\}$. Then the proposition also holds if $\mathcal{I}$ is an irreducible quasi-stable ideal for which some power of x_{k+1} is a minimal generator.

6. Pommaret-like free resolutions over Clements–Lindström rings

Since relative Pommaret-like bases are a special kind of relative Gröbner bases, they induce free resolutions via the relative Schreyer Theorem (Hashemi et al., 2021).

If we assume that the ambient quotient ring is $\mathcal{P}/\mathcal{I}$, where $\mathcal{I}$ is a quasi-stable monomial ideal, and if we complete the relative Pommaret-like basis to a relative Pommaret basis, then the induced resolution will consist of Pommaret bases for the syzygy modules in each homological degree. In this section, we will show that if we restrict to the class of *irreducible* quasi-stable monomial ideals, then we can skip the completion step from Pommaret-like basis to Pommaret basis: The relative Pommaret-like basis will then induce a free resolution which consists of Pommaret-like bases for each syzygy module. This resolution is in general closer to a minimal free resolution than that induced by relative Pommaret bases. Up to a permutation of coordinates, the class of irreducible quasi-stable monomial ideals is equivalent to the class of Clements–Lindström ideals. We will formulate our results in the most general form possible, but for simplicity one can think of the ring in which computations take place as a Clements–Lindström ring $\mathcal{P}/\langle x_k^{a_k}, \ldots, x_n^{a_n}\rangle$ with $a_k \geq \cdots \geq a_n \geq 2$. We shall note that a variant of the next proposition in the non-relative setting was given in (Hashemi et al., 2023, Thm. 7.7).

Proposition 6.1. *Let $\mathcal{I}$ be an irreducible quasi-stable monomial ideal and consider the ring $\mathcal{P}/\mathcal{I}$. Let H be a Pommaret-like basis relative to $\mathcal{I}$ of the (polynomial) ideal $\mathcal{J} \supset \mathcal{I}$. If H is ordered according to a P-ordering, then a Pommaret-like basis of* $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$ *is given by the S-polynomials of H induced by non-multiplicative multiples of the leading terms and the A-polynomials induced by multiplicatively annihilating leading terms of H modulo $\mathcal{I}$. Iteration of this result implies that a free resolution is induced consisting of relative Pommaret-like bases in each homological degree.*

Proof. We only sketch a proof, as it is similar to that of the corresponding results in (Hashemi et al., 2021, Thm. 5.13 and Prop. 5.14) where relative Pommaret bases are treated. As in the proof of (Hashemi et al., 2021, Prop. 5.14), quasi-stability of $\mathcal{I}$ is needed to ensure the relative quasi-stability of the leading module of the first syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$; the irreduciblity of $\mathcal{I}$ implies that the leading module terms have pure powers as their polynomial parts, from which it is easily seen that the set of these leading terms form a relative Pommaret-like basis. Moreover, the proof uses the continuity of the Pommaret-like division (Hashemi et al., 2023, Prop. 6.15) (for the P-ordering) and the relative Schreyer Theorem (Hashemi et al., 2021, Thm. 3.12). Note that "gaps" can appear in the lists of leading module terms in some syzygy module components. These gaps appear for the variables where one can reach $\mathcal{I}$ by multiplying by a power of that variable. That one can reach $\mathcal{I}$ implies that relative quasi-stability is not destroyed by these gaps. □

Remark 6.2. Only those A-polynomials whose annihilating factor is not identical to a generator of $\mathcal{I}$ contribute non-zero syzygies (Hashemi et al., 2021, Cor. 4.9).

As in the non-relative case, we are interested in a description of at least a part of the class of monomial ideals $\mathcal{J} \supset \mathcal{I}$ quasi-stable relative to $\mathcal{I}$ whose relative Pommaret-like bases induce *minimal* free resolutions by the process of Proposition 6.1. Recall that an estimate for the classes of "tail" terms compared to the classes of leading module terms was central to the proof of Theorem 4.2. In order to be able to use a similar argument, we need to impose even stricter assumptions on the relative Pommaret-like basis generating $\mathcal{J}$ than we had to impose in Theorem 4.2. The reason for this is that in the relative case, the contributions of A-polynomials have an effect which amounts to a "non-increasing" property for the classes of leading module terms in the resolution.

Example 6.3. Continuing with Example 4.5, in $\mathbb{K}[w, x, y, z]/\langle x^{10}, y^{10}, z^{10}\rangle$, we consider the monomial ideal $\mathcal{J} = \langle H\rangle$ minimally generated by the relative Pommaret-like basis

$$H = \left\{ w^9x^3y^2z^2,\ x^5y^2z^2,\ w^7y^4z^2,\ x^3y^4z^2,\ y^6z^2,\ x^3y^2z^4,\ y^4z^4,\ z^8 \right\}.$$

Note that $y^2 \cdot x^5y^2z^2 - x^2 \cdot x^3y^4z^2 = 0$ (compare the fourth column of the matrix D_1 in Example 4.5); moreover, $x^5 \cdot x^5y^2z^2 = 0 = x^7 \cdot x^3y^4z^2$. We obtain the three elements $\mathbf{S}_1 = y^2\mathbf{e}_2 - x^2\mathbf{e}_4$, $\mathbf{S}_2 = x^5\mathbf{e}_2$, and $\mathbf{S}_3 = x^7\mathbf{e}_4$ of the first syzygy module of H. Observe that y^2 is a Pommaret-like non-multiplicative power of $\mathbf{S}_2$. Multiplying and reducing, we see that $y^2\mathbf{S}_2 - x^5\mathbf{S}_1 - \mathbf{S}_3 = 0$. Thus, a constant appears in the Pommaret-like induced resolution, which is consequently not minimal.

Theorem 6.4. *Let $\{0\} \neq \mathcal{I} \lhd \mathcal{P}$ be an irreducible quasi-stable monomial ideal and $\mathcal{J} \supset \mathcal{I}$ a larger monomial ideal generated by a minimal Pommaret-like basis $H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$ relative to $\mathcal{I}$. Assume that H is simultaneously the minimal monomial generating set of $\mathcal{J}$ relative to $\mathcal{I}$. Moreover, let H be such that for each $t \in H$ and $x_a^{p_a} \in \mathrm{NMP}_{P_\mathcal{I}}(t, H)$, the unique $P_\mathcal{I}$-divisor $s \in H$ of $t \cdot x_a^{p_a}$ is of greater class than t i. e.* $\mathrm{cls}(s) > \mathrm{cls}(t)$. *Then the free resolution of $\mathcal{J}$ over $\mathcal{P}/\mathcal{I}$ induced by the basis H is minimal.*

Moreover, for each $m \geq 1$, the set of columns of the matrix D_m describing the differential consists of the unique relative reduced Gröbner basis of $\mathrm{Syz}^m_{\mathcal{P}/\mathcal{I}}(H)$ *for the chosen module monomial ordering.*

Proof. We need to show that the matrices describing the differentials do not contain any constant terms. By assumption, $H \neq \{1\}$ and hence it does not contain any constant. We now analyse the matrices $D_1, D_2, \ldots$ iteratively. For every $h \in H$, the matrix D_1 contains as leading module terms the non-multiplicative powers of h as well as, for $k = \mathrm{cls}(h)$, a factor $x_k^{d_k - \deg_k(h)}$ if $x_k^{d_k}$ is a minimal generator of $\mathcal{I}$. The tail terms in D_1 arise by division of terms $h \cdot x_j^{p_j}$, where $x_j^{p_j}$ is a non-multiplicative power, by their unique Pommaret-like divisor s in H:

$$h \cdot x_j^{p_j} = s \cdot u. \tag{6.1}$$

Since H is the minimal relative generating set of $\mathcal{J}$, these tail terms are not constant. Moreover, by assumption, we have $\mathrm{cls}(s) > \mathrm{cls}(h)$ and in turn $\mathrm{cls}(u) = \mathrm{cls}(h)$. A tail term u will be found in the row corresponding to the generator $s \in H$, and the leading terms in that row will be of class $\geq \mathrm{cls}(s)$, and so u has strictly smaller class than the leading terms in the same row. Note that columns of D_1 belonging to annihilating factors do not have any tail term. Summarizing, D_1 does not contain any constant terms and all tail terms have a strictly smaller class than the leading terms in the same row.

It is now straightforward to proceed analogously as in the proof of Theorem 4.2, showing by induction on the homological degree that no constant terms appear in the resolution, and thus to show its minimality.

Central to this induction proof is the fact that tail terms always have strictly smaller class than leading terms in the same row. The reducedness of the Gröbner bases in each degree is an obvious consequence. □

As a consequence, we see below that using this theorem and Proposition 5.13, we are able to describe minimal free resolutions for the class of monomial ideals considered in Proposition 5.13.

Corollary 6.5. *Let $\mathcal{I}$ be a zero-dimensional Clements–Lindström ideal and let H be the minimal generating set of a square-free Borel monomial ideal. Then the free resolution induced by H is the minimal.*

Proof. A close inspection of the proof of Proposition 5.13 shows that the elements of H fulfil the class condition imposed in Theorem 6.4 and this completes the proof. □

Example 6.6. Let us continue Example 5.10 by considering the ideals $\mathcal{I} = \langle x^6, y^6, z^6\rangle$ and $\mathcal{J} = \langle \mathcal{I}, xz, yz\rangle$. The set $H = \{xz, yz\}$ is the minimal generating system of the ideal $\mathcal{J}$ relative to $\mathcal{I}$, and it is simultaneously a relative Pommaret-like basis, as proven in Example 5.10. Since $\mathrm{cls}(xz) < \mathrm{cls}(yz)$, the additional conditions of Theorem 6.4 are also fulfilled. Hence, H induces an infinite minimal free resolution of $\mathcal{J}$ over $\mathcal{P}/\mathcal{I}$, with the first differential matrices given by:

$$D_0 = \begin{pmatrix} xz & yz \end{pmatrix}, \quad D_1 = \begin{pmatrix} x^5 & y & z^5 & 0 & 0 \\ 0 & -x & 0 & y^5 & z^5 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} x & y & z^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^5 & 0 & y^5 & z^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x^5 & 0 & -y & z & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & y & z^5 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & -y^5 & z \end{pmatrix}.$$

Remark 6.7. The minimal free resolution given in (Gasharov et al., 2011, Cons. 4.4) for square-free borel ideals relative to a zero-dimensional Clements–Lindström ring is necessarily isomorphic to the Pommaret-like resolution of the same ideal, since both resolutions are minimal.

In fact, one can always find an isomorphism that consists only of permutations of basis elements. One can prove this by assigning leading terms to the syzygies defined in (Gasharov et al., 2011, Eq. 4.10). This assignment can be done in such a way that the leading terms for each homological degree will coincide with the leading terms in the Pommaret-like resolution. The sets of leading terms being equal, we can conclude that the syzygies of (Gasharov et al., 2011, Eq. 4.10) form Gröbner bases in each homological degree; the reducedness can then be shown in a straightforward manner using a basic result on Borel monomial ideals.

The uniqueness of the reduced Gröbner basis then shows that the resolution of (Gasharov et al., 2011, Cons. 4.4) and the Pommaret-like resolution coincide. This also gives an explicit formula for the differential, depending only on the data contained in the first two matrices D_0 and D_1.

The next example shows that our construction covers many elementary cases:

Example 6.8. Let $a_1, \ldots, a_n$ be positive integers. By fixing $i \in \{1, \ldots, n\}$, let $1 \leq b_i < a_i$ be another integer. Then, relative to the irreducible monomial ideal $\mathcal{I} = \langle x_1^{a_1}, \ldots, x_n^{a_n}\rangle$, the set $H = \{x_i^{b_i}\}$ is a relative Pommaret-like basis of $\mathcal{J} = \langle H, \mathcal{I}\rangle$ and the induced resolution over $\mathcal{P}/\mathcal{I}$ is the obvious 2-periodic minimal free resolution with differentials described by the following matrices:

$$D_0 = \begin{pmatrix} x_i^{b_i} \end{pmatrix}, \quad D_1 = \begin{pmatrix} x_i^{a_i - b_i} \end{pmatrix}, \quad D_2 = \begin{pmatrix} x_i^{b_i} \end{pmatrix} = D_0.$$

A final, more or less "generic", example, shows the general behavior of the construction:

Example 6.9. If $\mathcal{I} = \langle y^4, z^5 \rangle$ and $\mathcal{J} = \langle \mathcal{I}, x^2y^3, xy^2z^2, y^3z^2, z^3 \rangle$, then we conclude that $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$ is the minimal relative generating set of $\mathcal{J}$, and it is simultaneously a relative Pommaret-like basis satisfying the additional conditions of Theorem 6.4. Hence, it induces a minimal free resolution of $\mathcal{J}$ over $\mathcal{P}/\mathcal{I}$, with the first maps of the differential represented by the following matrices:

$$D_0 = \begin{pmatrix} x^2y^3 & xy^2z^2 & y^3z^2 & z^3 \end{pmatrix}, \quad D_1 = \begin{pmatrix} y & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 & 0 \\ 0 & -x^2 & -x & 0 & y & z & 0 \\ 0 & 0 & 0 & -xy^2 & 0 & -y^3 & z^2 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} y^3 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^3 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & z^4 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & 0 & xy^2 & 0 & 0 & y^3 & z & 0 & 0 \\ 0 & 0 & x^2z^2 & 0 & x & 0 & 0 & -y & z^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy^2z^2 & 0 & 0 & y^3z^2 & z^3 \end{pmatrix}.$$

In the remainder of this section, we will derive formulae for the Betti numbers of ideals generated by relative Pommaret-like bases over factor rings of the form $\mathcal{P}/\mathcal{I}$, where $\mathcal{I}$ is an irreducible quasi-stable monomial ideal. We understand this to include also the case $\mathcal{I} = \{0\}$, and hence all resolutions induced by Pommaret-like bases over the ordinary polynomial ring $\mathcal{P} = \mathcal{P}/\{0\} = \mathbb{K}[x_1, \ldots, x_n]$. The results can also be applied to non-minimal free resolutions induced by Pommaret-like bases, but then one gets only formulae for the ranks of the free modules in these non-minimal resolutions, which yield, degree by degree, upper bounds for the Betti numbers of the resolved ideals.

Definition 6.10. Let $\mathcal{I} = \langle x_k^{a_k}, x_{k+1}^{a_{k+1}}, \ldots, x_n^{a_n} \rangle$ be an irreducible quasi-stable monomial ideal. Then we write $\operatorname{cls}(\mathcal{I}) = k$ and $\operatorname{supp}(\mathcal{I}) = \{x_k, x_{k+1}, \ldots, x_n\}$.

Let now $\mathcal{J} \supseteq \mathcal{I}$ be any homogeneous polynomial ideal in quasi-stable position relative to $\mathcal{I}$ with respect to the degrevlex term ordering, and let H be its minimal relative Pommaret-like basis. We construct a basis for the bigraded free $\mathcal{P}/\mathcal{I}$-module supporting the induced resolution, only using the Pommaret-like basis of the leading ideal of $\mathcal{J}$ relative to $\mathcal{I}$.

The resolution induced by H is supported on free $\mathcal{P}/\mathcal{I}$-modules. The first free $\mathcal{P}/\mathcal{I}$-module M_0 has a basis that we enumerate as $\{\mathbf{e}_\alpha \mid h_\alpha \in H\}$. Write $t_\alpha = \operatorname{lt}(h_\alpha)$ for each $h_\alpha \in H$. As always, we order H according to a P-ordering. The next free module M_1 has a basis whose cardinality equals that of the minimal Pommaret-like basis of $\operatorname{Syz}(H)$

with respect to the Schreyer module term order—note that this is a reduced Gröbner basis. Hence, the free basis of M_1 is in bijection with the elements of this Gröbner basis; in other words, it is in bijection with the leading module terms of the Gröbner basis. These leading module terms are given as follows (cf. Proposition 6.1):

- $x_a^{p_a} \cdot \mathbf{e}_\alpha$, where $x_a^{p_a} \in \mathrm{NMP}_{P_\mathcal{I}}(t_\alpha, \mathrm{lt}(H))$,
- $x_i^{a_i - \deg_i(t_\alpha)} \cdot \mathbf{e}_\alpha$, where $x_i \in \mathrm{supp}(t_\alpha) \cap \mathrm{supp}(\mathcal{I})$ and there is no $P_\mathcal{I}$-nonmultiplicative power for t_α at x_i. (If $\ell = \mathrm{cls}(t_\alpha) \geq \mathrm{cls}(\mathcal{I})$, then this case will always include $x_\ell^{a_\ell - \deg_\ell(t_\alpha)} \cdot \mathbf{e}_\alpha$.)

Since the two cases are mutually exclusive, and each concerns leading module terms whose polynomial parts are pure variable powers, we can identify each leading module term by its position and the variable involved. Thus, a free basis of M_1 can be enumerated as

$$\{\mathbf{e}_{\alpha,x_i} \mid x_i \geq \mathrm{cls}(t_\alpha) \wedge (x_i \in \mathrm{NM}_{P_\mathcal{I}}(t_\alpha, \mathrm{lt}(H)) \vee x_i \in \mathrm{supp}(t_\alpha) \cap \mathrm{supp}(\mathcal{I}))\}.$$

We keep the condition "$x_i \geq \mathrm{cls}(t_\alpha)$" for clarity, even though it could be omitted, being implicit in the other conditions. At this stage, it is useful to introduce notations for the leading ideals in each module component of M_1, because we can use them to describe, by an iteration, all further leading terms in the resolution. Set

$$\mathcal{J}_\alpha = \langle x_i^{d_i} \mid x_i^{d_i} \cdot \mathbf{e}_\alpha \in \mathrm{lt}(\mathrm{Syz}(H))\rangle.$$

These ideals are irreducible and we will use the notation $\mathrm{supp}(\mathcal{J}_\alpha)$ for the set of variables appearing in their respective generating sets.

Consider now the leading terms of the Pommaret-like basis of $\mathrm{Syz}^2(H)$, which are in bijection to a free basis of the next module in the resolution, M_2. Each of them is induced by a leading term of the basis of $\mathrm{Syz}(H)$. Such a leading term, $x_i^{d_i} \cdot \mathbf{e}_\alpha$, say, induces exactly the following leading terms in $\mathrm{Syz}^2(H)$:

- $x_j^{d_j} \cdot \mathbf{e}_{\alpha,x_i}$, where $x_j \in \mathrm{supp}(\mathcal{J}_\alpha)$ and $j > i$,
- $x_i^{a_i - d_i} \cdot \mathbf{e}_{\alpha,x_i}$, if $x_i \in \mathrm{supp}(\mathcal{I})$.

Note that the polynomial part of the new leading term will be supported on a variable whose index is not less than that of the polynomial part of the term which induces it. We can now list the free basis of M_2: Leading terms induced as in the first case correspond to basis elements $\mathbf{e}_{\alpha,x_i x_j}$, whereas leading terms induced as in the second case correspond to basis elements $\mathbf{e}_{\alpha,x_i^2}$.

We can iterate this construction. For the r-th module in the resolution, M_r, it yields a basis consisting of elements of the form $\mathbf{e}_{\alpha,x^\mu}$, where x^μ is a term of degree r with $\mathrm{cls}(x^\mu) \geq \mathrm{cls}(t_\alpha)$. Moreover, x^μ is supported on $\mathrm{supp}(\mathcal{J}_\alpha)$, and if for each $x_i \in \mathrm{supp}(\mathcal{I})$ we substitute 1 into x^μ, we get a squarefree term supported on $\mathrm{supp}(\mathcal{J}_\alpha) \setminus \mathrm{supp}(\mathcal{I})$.

From this description of the free bases, we obtain the following formula for the total Betti numbers of the resolution, where we write $\mathcal{S}_\alpha$ for $\mathrm{supp}(\mathcal{J}_\alpha)$ and $\mathcal{S}$ for $\mathrm{supp}(\mathcal{I})$:

For $r = 0$, $\operatorname{rank}(M_0) = |H|$; for $r \geq 1$,

$$\operatorname{rank}(M_r) = \sum_{\substack{h_\alpha \in H \\ \mathcal{S}_\alpha \cap \mathcal{S} \neq \emptyset}} \sum_{j=0}^{\min\{r,\, |\mathcal{S}_\alpha \setminus \mathcal{S}|\}} \binom{|\mathcal{S}_\alpha \setminus \mathcal{S}|}{j} \cdot \binom{|\mathcal{S}_\alpha \cap \mathcal{S}| + r - j - 1}{|\mathcal{S}_\alpha \cap \mathcal{S}| - 1} + \sum_{\substack{h_\alpha \in H \\ \mathcal{S}_\alpha \cap \mathcal{S} = \emptyset}} \big[r \leq |\mathcal{S}_\alpha| \big] \binom{|\mathcal{S}_\alpha|}{r}, \tag{6.2}$$

where the product of binomial coefficients counts the number of terms x^μ of degree r supported on $\mathcal{J}_\alpha$ with the additional restriction of being squarefree outside $\operatorname{supp}(\mathcal{I})$. Moreover, the Kronecker–Iverson symbol $[r \leq |\mathcal{S}_\alpha|]$ yields 1 when the statement enclosed in the square brackets is true and 0 otherwise.

We turn to the bigraded Betti numbers, which we will compute in the form of a Poincaré series, i. e. a formal power series in two variables, which we name s and u. The first variable encodes homological degrees and the second encodes degrees as given by the ordinary grading of the polynomial ring $\mathcal{P}$. Recall that each basis element $\mathbf{e}_{\alpha,x^\mu}$ has homological degree $\deg(x^\mu)$. Its polynomial degree is the sum of $\deg(t_\alpha)$ (recall $t_\alpha = \operatorname{lt}(h_\alpha)$) and the degrees of the polynomial parts of all leading module terms involved in the building up of the syzygy $\mathbf{S}_{\alpha,x^\mu} \in \operatorname{Syz}^{\deg(x^\mu)}(H)$. These polynomial parts are pure powers of variables from $\operatorname{supp}(\mathcal{J}_\alpha)$. Moreover, their indices form a non-decreasing sequence. There can be repeated indices in this sequence, and if an index j is repeated, it means that the next syzygy is formed from the annihilation of the current leading term. So if a module term with polynomial part $x_j^{c_j}$ is to annihilate, the next leading term will have polynomial part $x_j^{a_j - c_j}$ (recall that $\mathcal{I}$ is generated by the terms $x_j^{a_j}$). More repetitions of the same index will cause the involved leading terms to have polynomial parts oscillating between $x_j^{c_j}$ and $x_j^{a_j - c_j}$. This means that the contribution of x_j-terms to the overall polynomial degree of S_{α,x^μ} depends, on one hand, on the parity of μ_j, and the remaining part is just $h_j \cdot \lfloor \mu_j/2 \rfloor$. Since $\mathcal{J}_\alpha$ is generated by terms $x_j^{d_j}$, we get the following formula for the Poincaré series of our resolution, where we write $\mathcal{S}_\alpha$ for $\operatorname{supp}(\mathcal{J}_\alpha)$ and $\mathcal{S}$ for $\operatorname{supp}(\mathcal{I})$:

$$\sum_{h_\alpha \in H} \left[u^{\deg(t_\alpha)} \cdot \left(1 + \sum_{B \subseteq \mathcal{S}_\alpha} \binom{|\mathcal{S}_\alpha|}{|B|} s^{|B|} \prod_{x_b \in B} u^{d_b} \prod_{x_j \in \mathcal{S}_\alpha \cap \mathcal{S}} \frac{1}{1 - s^2 u^{a_j}} \right) \right]. \tag{6.3}$$

Example 6.11. Let us continue with Example 6.9 where we had $\mathcal{I} = \langle y^4, z^5 \rangle$ and $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$. We will use Equality (6.2) to compute the Betti numbers of the ideal generated by H relative to $\mathcal{I}$ and then compare it with the results of Example 6.9.

We write $h_\alpha = x^2y^3$, $h_\beta = xy^2z^2$, $h_\gamma = y^3z^2$, and $h_\delta = z^3$. An analysis of the Pommaret-like non-multiplicative powers of these generators shows that $\mathcal{J}_\alpha = \{y, z^2\}$, $\mathcal{J}_\beta = \{y, z\}$, $\mathcal{J}_\gamma = \{y, z\}$, and $\mathcal{J}_\delta = \{z^2\}$. Since $\operatorname{supp}(\mathcal{I}) = \{x, y\}$, we have $\operatorname{supp}(\mathcal{J}_\alpha) = \operatorname{supp}(\mathcal{J}_\alpha) \cap \operatorname{supp}(\mathcal{I})$, and the same equality holds also for the other indices. Thus, Equality (6.2) reduces to

$$\operatorname{rank}(M_r) = \sum_{h_\alpha \in H} \binom{|\operatorname{supp}(\mathcal{J}_\alpha)| + r - 1}{|\operatorname{supp}(\mathcal{J}_\alpha)| - 1},$$

and this gives, since we have three generators with $|\operatorname{supp}(\mathcal{J}_\bullet)| = 2$ and one generator with $|\operatorname{supp}(\mathcal{J}_\bullet)| = 1$, the formula

$$\operatorname{rank}(M_r) = 3\binom{1+r}{1} + \binom{r}{0} = 4 + 3r,$$

which is for $r \in \{1, 2, 3\}$ in perfect agreement with the results of Example 6.9.

7. An explicit formula for the differential

In this section, we will give explicit formulae for the differentials of resolutions of some monomial ideals induced by Pommaret-like bases over the ordinary polynomial ring $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_n]$. These formulas will generalize those given in (Seiler, 2010) for resolutions induced by Pommaret bases. While in (Seiler, 2010), such a formula was found for all quasi-stable ideals and their minimal Pommaret bases, we will here restrict our attention to a smaller class of ideals. Our first goal is to establish a subclass of quasi-stable ideals whose minimal Pommaret-like basis satisfy conditions analogous to those found in (Seiler, 2010, Lem. 5.4.17) for minimal Pommaret bases of arbitrary quasi-stable ideals. For this subclass, we will then have the technical tools needed to give an explicit formula for the differential of the induced resolution.

Definition 7.1. Let $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$ where A is a finite index set. For each $\alpha \in A$, and for each of its Pommaret-like non-multiplicative powers $x_a^{p_a} = x_a^{p(P,h_\alpha,H,a)}$, there exists exactly one generator $h_\beta \in H$ with $x_a^{p_a} \cdot h_\alpha \in C_P(h_\beta)$. For such a configuration of terms, we write

$$\Delta(\alpha, a) = \beta \tag{7.1}$$

for the index of the Pommaret-like divisor, and

$$t_{\alpha,a} = (x_a^{p_a} \cdot h_\alpha)/h_\beta \tag{7.2}$$

for the Pommaret-like multiplicative cofactor involved.

Lemma 7.2. *Let $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$. The associated function Δ and the terms $t_{\alpha,a}$ (as given in Definition 7.1) satisfy the following properties:*

1. *The inequality $\operatorname{cls}(h_\alpha) \leq \operatorname{cls}(h_{\Delta(\alpha,a)}) \leq a$ holds for all non-multiplicative indices $a > \operatorname{cls}(h_\alpha)$.*
2. *Let $b > a > \operatorname{cls}(h_\alpha)$ be two non-multiplicative indices.*
   - *The variable x_b is non-multiplicative for $h_{\Delta(\alpha,a)}$ and the non-multiplicative power of $h_{\Delta(\alpha,a)}$ at x_b equals that of h_α at x_b.*
   - *If $\operatorname{cls}(h_{\Delta(\alpha,b)}) \geq a$, then $\Delta(\Delta(\alpha, a), b) = \Delta(\alpha, b)$ and $x_a^{p_a} \cdot t_{\alpha,b} = t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b}$.*

Proof. Property (1.) follows from the minimality of the Pommaret-like basis H: $h_{\Delta(\alpha,a)}$ is a divisor of $x_a^{p_a} \cdot h_\alpha$ and thus its class must be at least as high as that of h_α; it cannot be higher than a, because otherwise $h_{\Delta(\alpha,a)}$ would be a strict Pommaret-like divisor of h_α, contradicting minimality.

Property (2.) splits into two items. The first item follows from Property (1.) and the definition of the Pommaret-like division, as the terms $h_{\Delta(\alpha,a)}$ and h_α must agree in their x_j-degrees for all $j > a$. Now if, to prove the second item, we additionally assume $\mathrm{cls}(h_{\Delta(\alpha,b)}) \geq a$, then – since $x_b^{p_b} \cdot h_{\Delta(\alpha,a)}$ and $x_b^{p_b} \cdot h_\alpha$ agree in their x_j-degrees for all indices $j > a$ – the same must be true for $h_{\Delta(\Delta(\alpha,a)),b}$ and $h_{\Delta(\alpha,b)}$. We also know that $\deg_a(h_{\Delta(\alpha,b)}) \leq \deg_a(h_\alpha) < \deg_a(x_b^{p_b} \cdot h_{\Delta(\alpha,a)})$. By the class assumption on $h_{\Delta(\alpha,b)}$, we can now conclude that $h_{\Delta(\alpha,b)}$ is the unique Pommaret-like divisor in H of $x_b^{p_b} \cdot h_{\Delta(\alpha,a)}$. Hence, we have shown $\Delta(\Delta(\alpha,a),b) = \Delta(\alpha,b)$. The remaining statement is a consequence of the following chain of equalities:

$$\begin{aligned} x_a^{p_a} \cdot t_{\alpha,b} \cdot h_{\Delta(\alpha,b)} &= x_a^{p_a} \cdot (x_b^{p_b} \cdot h_\alpha) = x_b^{p_b} \cdot (x_a^{p_a} \cdot h_\alpha) \\ &= x_b^{p_b} \cdot t_{\alpha,a} \cdot h_{\Delta(\alpha,a)} = t_{\alpha,a} \cdot x_b^{p_b} \cdot h_{\Delta(\alpha,a)} \\ &= t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} \cdot h_{\Delta(\Delta(\alpha,a),b)} = t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} \cdot h_{\Delta(\alpha,b)}. \end{aligned}$$

□

For arbitrary minimal *Pommaret* bases, the associated Δ functions satisfy a commutativity property of the form

$$\Delta(\Delta(\alpha,a),b) = \Delta(\Delta(\alpha,b),a) \tag{7.3}$$

whenever both of these terms are defined, i. e. when the involved variable indices a, b are always non-multiplicative (Seiler, 2010, Lem. 5.4.17). In general, minimal Pommaret-like bases do not have this property. What is more, for *Pommaret* bases, also the equality $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$ holds in this situation. By contrast, there are minimal Pommaret-like bases for which the commutativity property holds, but not the equality just mentioned. This is caused by differences of degrees of non-multiplicative powers for the same variable.

Example 7.3. Consider the minimal Pommaret-like basis $H = \{h_\alpha, h_\beta, h_\gamma, h_\delta, h_\epsilon\}$ with $h_\alpha = xy$, $h_\beta = y^4$, $h_\gamma = xz$, $h_\delta = y^2z$, and $h_\epsilon = z^3$. Its associated Δ function satisfies the commutativity property of Equality (7.3). For this only one condition needs to be checked:

$$\Delta(\Delta(\alpha,y),z) = \delta = \Delta(\Delta(\alpha,z),y).$$

However, we have $t_{\alpha,y} = x$, $t_{\Delta(\alpha,y),z} = y^2$, $t_{\alpha,z} = y$, and $t_{\Delta(\alpha,z),y} = x$, so that $t_{\alpha,y} \cdot t_{\Delta(\alpha,y),z} = xy^2 \neq xy = t_{\alpha,z} \cdot t_{\Delta(\alpha,z),y}$. This is caused by a difference in the degrees of the non-multiplicative powers at the variable y between h_α (degree 3) and h_γ (degree 2).

We now define a subclass of quasi-stable ideals having Δ-functions with properties useful for the analysis of their Pommaret-like resolutions:

Definition 7.4. Let $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$. The ideal $\mathcal{I}$ together with the basis H is called Δ-*commuting* if the associated function Δ and the terms $t_{\alpha,a}$ (as given in Definition 7.1) satisfy the following properties:

1. If $b > a > \mathrm{cls}(h_\alpha)$ are two non-multiplicative indices and $\mathrm{cls}(h_{\Delta(\alpha,b)}) < a$, then the exponent of the non-multiplicative power of $h_{\Delta(\alpha,b)}$ at the variable x_a equals that of the non-multiplicative power of h_α at the variable x_a.

2. We have $\Delta(\Delta(\alpha, a), b) = \Delta(\Delta(\alpha, b), a)$.
3. We have $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$.

For Δ-commuting quasi-stable ideals, we are able to give an explicit formula for the differential of the resolution induced by the minimal Pommaret-like basis. As is usual for such formulas, the summands obey a certain sign rule, and for this we need the following definition:

Definition 7.5. Let $x_i \in Y \subseteq \{x_1, \dots, x_n\}$. Then we write

$$\mathrm{sgn}(x_i, Y) = (-1)^{|\{x_j \in Y \mid j > i\}|}.$$

Theorem 7.6. *Let $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the Δ-commuting quasi-stable ideal $\mathcal{I} = \langle H \rangle$. We write $\mathrm{NMP}(h_\alpha, H) = \{x_j^{p_j} \mid j > \mathrm{cls}(h_\alpha)\}$. The Pommaret-like induced resolution of $\mathcal{I}$ is supported on free generators of the form $\mathbf{e}_{h_\alpha, x^\mu}$, where the x^μ are square-free terms supported on $\{x_j \mid j > \mathrm{cls}(h_\alpha)\}$. The differential δ of the resolution is given by $\delta(\mathbf{e}_{\alpha,1}) = h_\alpha$, and, for $\deg(x^\mu) > 0$,*

$$\delta(\mathbf{e}_{\alpha,x^\mu}) = \sum_{x_j \in \mathrm{supp}(x^\mu)} \mathrm{sgn}(x_j, \mathrm{supp}(x^\mu)) \cdot \left(x_j^{p_j} \mathbf{e}_{\alpha, x^\mu/x_j} - t_{\alpha,j} \mathbf{e}_{\Delta(\alpha,j), x^\mu/x_j} \right). \tag{7.4}$$

In this formula, we interpret all summands to be zero which involve a non-existent free generator $\mathbf{e}_{\beta,x^\nu}$, i. e. an expression of this form for which $\mathrm{supp}(x^\nu) \nsubseteq \{x_j \mid j > \mathrm{cls}(h_\beta)\}$.

The proof is a straightforward adaptation of the one of (Seiler, 2010, Thm. 5.4.18), replacing non-multiplicative variables by their associated non-multiplicative powers where appropriate and therefore omitted.

Corollary 7.7. *Let $\mathcal{I} = \langle H \rangle$ be a Δ-commuting quasi-stable ideal minimally generated by the set $H \subset \mathcal{T}$, for which H is also a Pommaret-like basis. Then the Pommaret-like resolution of $\mathcal{I}$ induced by H is minimal.*

Proof. By minimality of H, we have $t_{\alpha,a} \neq 1$ for the terms defined in Definition 7.1. Now, the minimality of the induced resolution is a trivial consequence of the explicit differential formula 7.4, which applies because all assumptions of Theorem 7.6 are fulfilled for $\mathcal{I}$ and H. □

Example 7.8. Let us continue Example 4.5. We have the minimal Pommaret-like basis

$$H = \{h_\alpha = w^9x^3y^2z^2, h_\beta = x^5y^2z^2, h_\gamma = w^7y^4z^2, h_\delta = x^3y^4z^2,$$
$$h_\epsilon = y^6z^2, h_\zeta = x^3y^2z^4, h_\eta = y^4z^4, h_\theta = z^8\}.$$

Using Formula 7.4, we obtain the following values of the differential δ of the induced resolution for basis elements of homological degrees 2 and 3:

$$\begin{aligned}
\delta(\mathbf{e}_{\alpha,xy}) &= y^2\mathbf{e}_{\alpha,x} && - x^2\mathbf{e}_{\alpha,y} && +w^9\mathbf{e}_{\beta,y}\\
\delta(\mathbf{e}_{\alpha,xz}) &= z^2\mathbf{e}_{\alpha,x} && - x^2\mathbf{e}_{\alpha,z} && +w^9\mathbf{e}_{\beta,z}\\
\delta(\mathbf{e}_{\alpha,yz}) &= z^2\mathbf{e}_{\alpha,y} && -w^9\mathbf{e}_{\zeta,y} - y^2\mathbf{e}_{\alpha,z} && +w^9\mathbf{e}_{\delta,z}\\
\delta(\mathbf{e}_{\beta,yz}) &= z^2\mathbf{e}_{\beta,y} && -x^2\mathbf{e}_{\zeta,y} - y^2\mathbf{e}_{\beta,z} && +x^2\mathbf{e}_{\delta,z}\\
\delta(\mathbf{e}_{\gamma,xy}) &= y^2\mathbf{e}_{\gamma,x} && - x^3\mathbf{e}_{\gamma,y} && +w^7\mathbf{e}_{\delta,y}\\
\delta(\mathbf{e}_{\gamma,xz}) &= z^2\mathbf{e}_{\gamma,x} && - x^3\mathbf{e}_{\gamma,z} && +w^7\mathbf{e}_{\delta,z}\\
\delta(\mathbf{e}_{\gamma,yz}) &= z^2\mathbf{e}_{\gamma,y} && - y^2\mathbf{e}_{\gamma,z} && +w^7\mathbf{e}_{\epsilon,z}\\
\delta(\mathbf{e}_{\delta,yz}) &= z^2\mathbf{e}_{\delta,y} && - y^2\mathbf{e}_{\delta,z} && +x^3\mathbf{e}_{\epsilon,z}\\
\delta(\mathbf{e}_{\zeta,yz}) &= z^4\mathbf{e}_{\zeta,y} && - y^2\mathbf{e}_{\zeta,z} && +x^3\mathbf{e}_{\eta,z}
\end{aligned}$$

$$\begin{aligned}
\delta(\mathbf{e}_{\alpha,xyz}) &= z^2\mathbf{e}_{\alpha,xy} && -y^2\mathbf{e}_{\alpha,xz} + x^2\mathbf{e}_{\alpha,yz} && -w^9\mathbf{e}_{\beta,yz}\\
\delta(\mathbf{e}_{\gamma,xyz}) &= z^2\mathbf{e}_{\gamma,xy} && -y^2\mathbf{e}_{\gamma,xz} + x^3\mathbf{e}_{\gamma,yz} && -w^7\mathbf{e}_{\delta,yz}.
\end{aligned}$$

The P-graph of this Pommaret-like basis is given in Figure 1.

Example 7.9. Let us continue Example 4.6. Thus, we consider the minimal Pommaret-like basis $H = \{xy, y^3, xz, y^2z, z^2\}$ and we write $h_\alpha = xy$, $h_\beta = y^3$, $h_\gamma = xz$, $h_\delta = y^2z$, and $h_\epsilon = z^2$. Using Formula 7.4, we obtain the following values of the differential δ of the induced resolution for basis elements of homological degree 2:

$$\begin{aligned}
\delta(\mathbf{e}_{\alpha,yz}) &= z\mathbf{e}_{\alpha,y} && -y\mathbf{e}_{\gamma,y} - y^2\mathbf{e}_{\alpha,z} && +x\mathbf{e}_{\beta,z}\\
\delta(\mathbf{e}_{\gamma,yz}) &= z\mathbf{e}_{\gamma,y} && - y^2\mathbf{e}_{\gamma,z} && +x\mathbf{e}_{\delta,z}.
\end{aligned}$$

The P-graph of this Pommaret-like basis is given in Figure 2.

In the remainder of this section, we aim to find a family of quasi-stable monomial ideals as large as possible such that for each ideal in the family, the resolution induced by its minimal Pommaret-like basis admits an explicit differential formula akin to Equality (7.4). For a given minimal Pommaret-like basis $H = \{h_\alpha \mid \alpha \in A\}$, we would like to find terms $u_{\alpha,j,\mu}$ such that the Pommaret-like induced resolution of the ideal $\langle H\rangle$ is described by the formula

$$\delta(\mathbf{e}_{\alpha,x^\mu}) = \sum_{x_j\in\operatorname{supp}(x^\mu)} \operatorname{sgn}(x_j, \operatorname{supp}(x^\mu)) \cdot \left(x_j^{p_j}\mathbf{e}_{\alpha,x^\mu/x_j} - u_{\alpha,j,\mu}\mathbf{e}_{\Delta(\alpha,j),x^\mu/x_j}\right). \tag{7.5}$$

In particular, we still work with resolutions supported on module basis elements $\mathbf{e}_{\alpha,x^\mu}$ where x^μ is a square-free term supported on $\{x_{\operatorname{cls}(h_\alpha)}, \dots, x_n\}$, and $\deg(x^\mu)$ is the homological degree of the basis element. Moreover, the leading terms of the involved syzygies have polynomial parts $x_a^{p(a,h_\alpha,H)}$, i.e., they are Pommaret-like non-multiplicative powers of some original basis element. Thus we can associate the multidegree $h_\alpha \cdot \prod_{x_a\in\operatorname{supp}(x^\mu)} x_a^{p(a,h_\alpha,H)}$ to the basis element $\mathbf{e}_{\alpha,x^\mu}$.

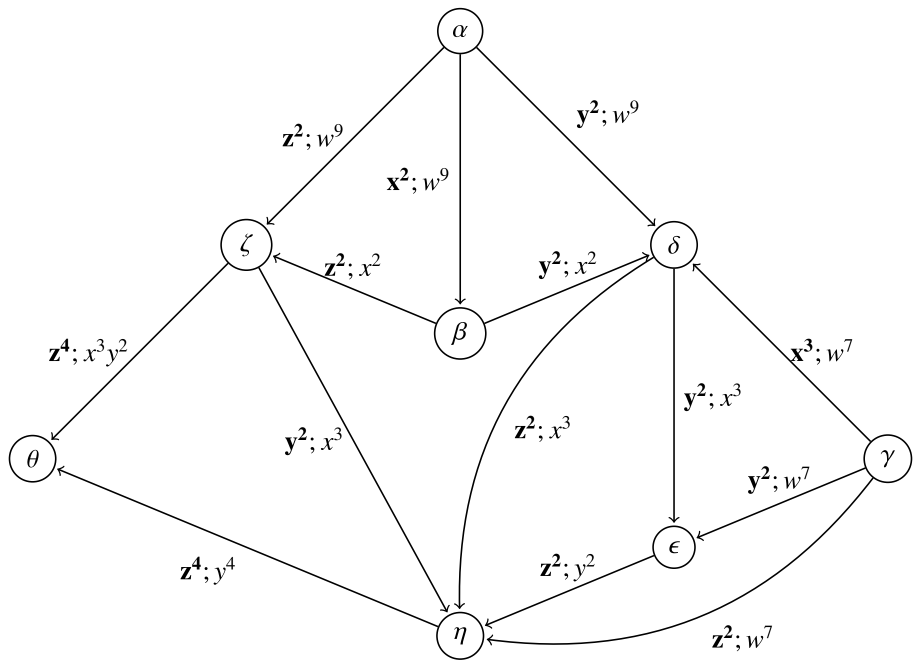

Figure 1: *P*-graph of Pommaret-like basis of Example 7.8. Each arrow is labelled with a Pommaret-like non-multiplicative power of the basis element belonging to the source. This non-multiplicative power is printed bold. Moreover, the label contains the associated cofactor, which is Pommaret-like multiplicative for the basis element belonging to the target of the arrow.

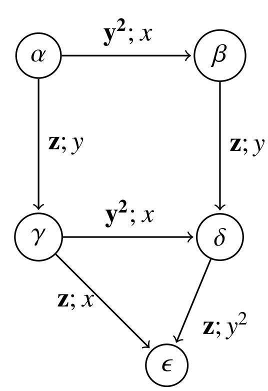

Figure 2: *P*-graph of Pommaret-like basis of Example 7.9. See Figure 1 for instructions on how to read it.

The original ideal is monomial or, in other words, multihomogeneous; thus, so is the induced resolution. The terms $u_{\alpha,j,\mu}$ need to ensure the multihomogeneity. The resolution is assumed to be Pommaret-like induced; hence, the leading terms, which are given by non-multiplicative powers, determine the multidegrees of all involved syzygies. The differential δ is 0-multihomogeneous. Hence all terms in $\delta(\mathbf{e}_{\alpha,x^\mu})$ must be exactly of multidegree $h_\alpha \prod_{x_a\in\operatorname{supp}(x^\mu)} x_a^{p(a,h_\alpha,H)}$. This is true for the terms $x_j^{p_j}\mathbf{e}_{\alpha,x^\mu/x_j}$ since the multidegree of $\mathbf{e}_{\alpha,x^\mu/x_j}$ is $h_\alpha \prod_{\substack{x_a\in\operatorname{supp}(x^\mu)\\ a\neq j}} x_a^{p(a,h_\alpha,H)}$. For each index j, a natural candidate for $u_{\alpha,j,\mu}$ is $t_{\alpha,j}$ since $x_j^{p_j}\cdot h_\alpha = t_{\alpha,j}\cdot h_{\Delta(\alpha,j)}$ and thus, the multidegree of $t_{\alpha,j}\mathbf{e}_{\Delta(\alpha,j),x^\mu/x_j}$ is $h_\alpha\cdot x_j^{p_j}\cdot\prod_{\substack{x_a\in\operatorname{supp}(x^\mu)\\ a\neq j}} x_a^{p(a,h_{\Delta(\alpha,j)},H)}$. Consider now the term

$$v_{\alpha,j,\mu} = \prod_{\substack{x_a\in\operatorname{supp}(x^\mu)\\ a\neq j}} x_a^{p(a,h_\alpha,H)-p(a,h_{\Delta(\alpha,j)},H)} \in \mathbb{K}[x_1^{\pm1},\dots,x_n^{\pm1}].$$

If $t_{\alpha,j}\cdot v_{\alpha,j,\mu}\in\mathcal{T}$, then the terms $u_{\alpha,j,\mu} = t_{\alpha,j}\cdot v_{\alpha,j,\mu}$ make (7.5) multihomogeneous. It is not hard to see that Δ-commuting quasi-stable ideals satisfy these conditions.

Before proving that Equality (7.5) indeed describes the Pommaret-like induced resolution, we first need to make explicit the P-orderings that we use in each homological degree of the Pommaret-like induced resolution.

Remark 7.10. Let $H = \{h_\alpha \mid \alpha\in A\}\subset\mathcal{T}$ be a minimal Pommaret-like basis and let $\mathbf{e}_{\alpha,x^\mu}$, $\mathbf{e}_{\beta,x^\nu}$ with $\deg(x^\mu)=\deg(x^\nu)=d$ be two basis elements of the free module F_d in the induced resolution. We work with the following P-ordering: $\mathbf{e}_{\alpha,x^\mu}$ precedes $\mathbf{e}_{\beta,x^\nu}$ if and only if either h_α precedes h_β in the P-ordering of H, or $\alpha=\beta$ and $x^\mu \prec_{\text{revlex}} x^\nu$.

Definition 7.11. Let $H = \{h_\alpha \mid \alpha\in A\}\subset\mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H\rangle$. The ideal $\mathcal{I}$ together with the basis H is called *weakly Δ-commuting* if the associated function Δ and the terms $t_{\alpha,a}$ (as given in Definition 7.1) satisfies the following property: If $b>a>\operatorname{cls}(h_\alpha)$ are two non-multiplicative indices and $\operatorname{cls}(h_{\Delta(\alpha,b)})<a$, then we have $\Delta(\Delta(\alpha,a),b)=\Delta(\Delta(\alpha,b),a)$.

Lemma 7.12. *Let x^μ be a squarefree monomial and let $x_j, x_j\in\operatorname{supp}(x^\mu)=Y$ with $i<j$ be two variables that divide x^μ. Then* $\operatorname{sgn}(x_i,Y)\cdot\operatorname{sgn}(x_j,Y\setminus\{x_i\}) = -\operatorname{sgn}(x_j,Y)\cdot\operatorname{sgn}(x_i,Y\setminus\{x_j\})$

Proof. Without loss of generality, we may assume that x_j has the highest index in $\operatorname{supp}(x^\mu)$. Then, $\operatorname{sgn}(x_j,Y\setminus\{x_i\}) = \operatorname{sgn}(x_j,Y) = 1$. Thus, we need to show that $\operatorname{sgn}(x_i,Y) = -\operatorname{sgn}(x_i,Y\setminus\{x_j\})$. But this is clear, as $x_j\in Y$ and $j>i$. □

Theorem 7.13. *Let $H=\{h_\alpha\mid\alpha\in A\}$ be the minimal Pommaret-like basis of the weakly Δ-commuting quasi-stable monomial ideal $\langle H\rangle$. Let the terms $u_{\alpha,j,\mu}$ be given such that Equality* (7.5) *together with $\delta(\mathbf{e}_{\alpha,1})=h_\alpha$ defines a* 0*-multihomogeneous map δ of homological degree* -1. *Then δ is the differential of the Pommaret-like induced resolution of $\langle H\rangle$.*

Proof. Let ∂ denote the differential of the Pommaret-like induced resolution. We need to show that $\delta=\partial$. First, note that the two maps are defined on the same free module(s).

It is clear by definition that $\delta_0 = \partial_0$. Assume that we can prove that $\delta^2 = 0$. Then, elements of the form $\delta(\mathbf{e}_{\alpha,x_j})$ are in $\mathrm{Syz}^1(H)$, and, more generally, if $\deg(x^\mu) = d$, we have $\delta(\mathbf{e}_{\alpha,x^\mu}) \in \mathrm{Syz}^d(H)$. We work with the term ordering induced by the P-ordering on the basis H. For each $d \in \mathbb{N}_0$, we consider the set $G_d = \{\delta(\mathbf{e}_{\alpha,x^\mu}) \mid \deg(x^\mu) = d+1\}$; we denote by F_d the analogous set for the differential ∂. If we can show that $\mathbf{lt}(G_d) = \mathbf{lt}(F_d)$ for all d, and that G_d is autoreduced with respect to these leading terms, then since F_d is a reduced Gröbner basis and by the uniqueness of reduced Gröbner bases, it must hold that $G_d = F_d$.

We first show that $\delta^2 = 0$. So let $\mathbf{e}_{\alpha,x^\mu}$ be a module generator with $\deg(x^\mu) \geq 2$. We claim that $S = \delta^2(\mathbf{e}_{\alpha,x^\mu}) = 0$. S is a sum of module monomials supported on module basis elements of the form $\mathbf{e}_{\beta,x^\mu/x_ix_j}$, where $\{x_i, x_j\} \in \mathrm{supp}(x^\mu)$ (assume $i < j$), and β lies in the index set $I(i,j) = \{\alpha, \Delta(\alpha,i), \Delta(\alpha,j), \Delta(\Delta(\alpha,i),j)\}$. (Note that, if $\Delta(\Delta(\alpha,j),i)$ exists, then it is equal to $\Delta(\Delta(\alpha,i),j)$.) As module monomials supported on basis elements of the form $\mathbf{e}_{\beta,x^\mu/x_ix_j}$ and $\mathbf{e}_{\beta,x^\mu/x_kx_\ell}$, where $\{i,j\} \neq \{k,\ell\}$, cannot cancel, it remains only to be seen that the summands supported on basis elements $\mathbf{e}_{\beta,x^\mu/x_ix_j}$, with i and j now fixed, cancel. As δ is multihomogeneous, we only need to determine the existing summands and show that their signs sum to zero.

The index set $I(i,j)$ has at least 3 distinct elements, as $\alpha < \Delta(\alpha,i) < \Delta(\Delta(\alpha,i),j)$ in the P-order of the Pommaret-like basis H. We now distinguish two main cases:

If $|I(i,j)| = 4$, then $\Delta(\alpha,j) \neq \Delta(\Delta(\alpha,i),j)$ and hence $\Delta(\Delta(\alpha,j),i) = \Delta(\Delta(\alpha,i),j)$. These four nodes form a square in the P-graph. For each node, S contains exactly two summands. The sum of the signs of the monomials supported on $\mathbf{e}_{\beta,x^\mu/x_ix_j}$ is as follows, where $Y = \mathrm{supp}(x^\mu)$:

- For $\beta = \alpha$:

$$\mathrm{sgn}(x_i, Y)\mathrm{sgn}(x_j, Y \setminus \{x_i\}) + \mathrm{sgn}(x_j, Y)\mathrm{sgn}(x_i, Y \setminus \{x_j\}),$$

- For $\beta = \Delta(\alpha,i)$:

$$\mathrm{sgn}(x_j, Y)(-1)\mathrm{sgn}(x_i, Y \setminus \{x_j\}) + (-1)\mathrm{sgn}(x_i, Y)\mathrm{sgn}(x_j, Y \setminus \{x_i\}),$$

- For $\beta = \Delta(\alpha,j)$:

$$\mathrm{sgn}(x_i, Y)(-1)\mathrm{sgn}(x_j, Y \setminus \{x_i\}) + (-1)\mathrm{sgn}(x_j, Y)\mathrm{sgn}(x_i, Y \setminus \{x_j\}),$$

- For $\beta = \Delta(\Delta(\alpha,i),j)$:

$$(-1)\mathrm{sgn}(x_i, Y)(-1)\mathrm{sgn}(x_j, Y \setminus \{x_i\}) + (-1)\mathrm{sgn}(x_j, Y)(-1)\mathrm{sgn}(x_i, Y \setminus \{x_j\}).$$

The monomials in each of the four cases sum to zero by Lemma 7.12 and multihomogeneity of δ.

If $|I(i,j)| = 3$, then, as $\alpha < \Delta(\alpha,j)$ and $\Delta(\alpha,j) \neq \Delta(\alpha,i)$, we must have $\Delta(\alpha,j) = \Delta(\Delta(\alpha,i),j)$. The three nodes in $I(i,j)$ form a triangle in the P-graph. For each node, S contains exactly two summands, but no others, as $\mathbf{e}_{\Delta(\alpha,j),x^\mu/x_j} = 0$ by convention, because $\mathrm{cls}(h_{\Delta(\alpha,j)}) \geq i$. The sum of the signs of the monomials supported on $\mathbf{e}_{\beta,x^\mu/x_ix_j}$ is as follows, where $Y = \mathrm{supp}(x^\mu)$:

- For $\beta = \alpha$: As in the case $|I(i,j)| = 4$.

- For $\beta = \Delta(\alpha, i)$: As in the case $|I(i, j)| = 4$.
- For $\beta = \Delta(\alpha, j) = \Delta(\Delta(\alpha, i), j)$:

$$(-1)\mathrm{sgn}(x_i, Y)(-1)\mathrm{sgn}(x_j, Y \setminus \{x_i\}) + \mathrm{sgn}(x_i, Y)(-1)\mathrm{sgn}(x_j, Y \setminus \{x_i\}).$$

The monomials in each of the three cases sum to zero by Lemma 7.12 and multihomogeneity of δ. Thus we have shown $\delta^2 = 0$.

Now we proceed by analysing leading terms and showing autoreducedness. For $\mathbf{S} = \delta(\mathbf{e}_{\alpha,x^\mu}) \in G_d$, define $\hat{x} = x_{\hat{j}}$ as the variable with maximal index dividing x^μ. Then $\mathbf{lt}(\mathbf{S}) = x_{\hat{j}}^{p_{\hat{j}}}\mathbf{e}_{\alpha,x^\mu/x_{\hat{j}}}$, because terms of the form $x^\nu\mathbf{e}_{\alpha,\bullet}$ precede terms of the form $x^\rho\mathbf{e}_{\Delta(\alpha,\bullet),\bullet}$ in the P-ordering and $x^\mu/x_{\hat{j}}$ is the revlex-smallest term among terms of the form x^μ/x_i. So $\mathbf{lt}(\mathbf{S})$ equals the leading term of the element $\partial(\mathbf{e}_{\alpha,x^\mu})$ as desired. It remains to show autoreducedness. Assume that $\mathbf{lt}(\mathbf{S})$ divides a term in the support of $\mathbf{T} = \delta(\mathbf{e}_{\beta.x^\nu})$. Necessarily, $\deg(x^\nu) = \deg(x^\mu)$. If $\beta = \alpha$, then it is clear that $\mathbf{lt}(\mathbf{S})$ does not divide any term of the form $x^\zeta\mathbf{e}_{\Delta(\beta,\bullet),\ldots}$ in the support of $\mathbf{T}$, as $\Delta(\beta, \bullet) \neq \beta = \alpha$. Assume it divides a term in the support of $\mathbf{T}$ of the form $x_a^{p_a}\mathbf{e}_{\alpha,x^\nu/x_a}$. Then we must have $a = \hat{j}$ and $x^\nu = x^\mu$, i.e., $\mathbf{S} = \mathbf{T}$. If $\beta \neq \alpha$, then it is clear that $\mathbf{lt}(\mathbf{S})$ does not divide any term of the form $x_a^{p_a}\mathbf{e}_{\beta,x^\nu/x_a}$ in the support of $\mathbf{T}$, as $\beta \neq \alpha$. Assume now that $\mathbf{lt}(\mathbf{S})$ divides a term of the form $u_{\beta,r}\mathbf{e}_{\Delta(\beta,r),x^\nu/x_r}$ in the support of $\mathbf{T}$. Then we must have $\Delta(\beta, r) = \alpha$ and $x^\nu/x_r = x^\mu/x_{\hat{j}}$. The latter relation implies $x_{\hat{j}} \notin \mathrm{supp}(x^\nu)$. Thus, the definition of $u_{\beta,r}$ gives $\deg_{\hat{j}}(u_{\beta,r}) = \deg_{\hat{j}}(t_{\beta,r})$. But, as $\Delta(\beta, r) = \alpha$, $t_{\beta,r}$ is Pommaret-like multiplicative for h_α, and so $\deg_{\hat{j}}(u_{\beta,r}) < p(\hat{j}, h_\alpha, H) = \deg_{\hat{j}}(\mathbf{lt}(\mathbf{S}))$, contradicting the assumed divisibility. Thus the collection of all the $\delta(\mathbf{e}_{\alpha,x^\mu})$ is indeed autoreduced as claimed. ☐

Example 7.14. Consider the minimal Pommaret-like basis $H = \{h_\alpha = xy, h_\beta = y^4, h_\gamma = xyz, h_\delta = y^2z, h_\epsilon = z^3\} \subset \mathbb{K}[x, y, z]$. It induces a minimal free resolution with differential represented by the following matrices:

$$D_0 = \begin{pmatrix} xy & y^4 & xyz & y^2z & z^2 \end{pmatrix},\ D_1 = \begin{pmatrix} y^3 & z & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ 0 & -1 & 0 & y & z^2 & 0 \\ 0 & 0 & -y^2 & -x & 0 & z^2 \\ 0 & 0 & 0 & 0 & -xy & -y^2 \end{pmatrix},\ D_2 = \begin{pmatrix} z & 0 \\ -y^3 & 0 \\ x & 0 \\ -y^2 & z^2 \\ 0 & -y \\ 0 & x \end{pmatrix}.$$

One checks easily that the P-graph of H is, up to labels, the same as in Figure 2. Thus, H is weakly Δ-commuting. However, it is not Δ-commuting, because, even though $\Delta(\alpha, z) = \gamma$ and $\mathrm{cls}(h_\gamma) = x < z$, the non-multiplicative powers of h_α and h_γ differ, being y^3 and y, respectively. Thus we cannot use Theorem 7.6 for finding the differential, but must resort to using Theorem 7.13. Assume we wanted to use Theorem 7.6. Then we would obtain the expression

$$z\mathbf{e}_{\alpha,y} - \mathbf{e}_{\gamma,y} - y^3\mathbf{e}_{\alpha,z} + x\mathbf{e}_{\beta,z}.$$

This does not correspond to the first column of D_2. Using Theorem 7.13, the coefficient of $\mathbf{e}_{\gamma,y}$ is multiplied by the correction term y^2, which is exactly the quotient of the non-multiplicative powers at y of h_α and h_γ, respectively. This way, we obtain the correct differential value

$$\delta(\mathbf{e}_{\alpha,yz}) = z\mathbf{e}_{\alpha,y} - y^2\mathbf{e}_{\gamma,y} - y^3\mathbf{e}_{\alpha,z} + x\mathbf{e}_{\beta,z}.$$

Example 7.15. Consider the minimal Pommaret-like basis $H = \{h_\alpha = x^3y, h_\beta = y^5, h_\gamma = x^3z, h_\delta = xy^3z, h_\epsilon = y^4z, h_\zeta = z^4\} \subset \mathbb{K}[x, y, z]$. It is not weakly Δ-commuting, because $\Delta(\Delta(\alpha, y), z) = \epsilon \neq \delta = \Delta(\Delta(\alpha, z), y)$. In fact, $\Delta(\Delta(\Delta(\alpha, z), y), y) = \epsilon$ and the P-graph of H contains a pentagon.

8. Conclusion

The Pommaret involutive division is known to be well suited to the analysis of free resolutions and homological invariants of polynomial ideals in quasi-stable position. In this article, we studied the resolutions obtained by using several generalizations of this division: The relative Pommaret division and the Pommaret-like division, respectively. These divisions allowed us to study on the one hand ideals in quotient rings and on the other hand to obtain smaller resolutions which are closer to the minimal one. At the same time, we showed that the combinatorial properties of the Pommaret division are preserved. Our results not only enlarge the scope of the study of involutive bases, but also open new opportunities for enhancing the efficiency of involutive algorithms for the computation of free resolutions.

We showed that the induced free resolutions have good algorithmic properties like Gröbner-reducedness in all higher syzygy modules. Moreover, for special types of quasi-stable monomial ideals, we showed that the induced resolutions are even minimal. For some ideal types we obtained explicit formulas for the differential which only depend on the data needed to compute the original Pommaret-like bases.

One may expect that the resolutions induced by Pommaret-like bases can be proven to be induced by mapping cones (see (Herzog and Takayama, 2002) for a definition), as it is the case for the resolutions induced by Pommaret bases (Albert et al., 2015). Moreover, it may be worthwhile to investigate whether the Pommaret-like induced resolution is cellular, for instance using the techniques described by Iglesias and Sáenz de Cabezón (2021).

In this work, we focused on resolutions over the ordinary polynomial ring and over Clements–Lindström rings, as the latter ones have found much attention in recent years. However, the presented techniques are valid for any quotient ring defined by an ideal in quasi-stable position. A natural direction for further research is to find more general classes of such quotient rings for which the induced resolution has special properties, in particular for which one can show that it is minimal.

We finally mention that the here developed tools are also of interest for the analysis of Hilbert and Quot schemes. For the classical case, the use of the Pommaret division in this context was pioneered by Bertone et al. (2013) and further developed by Albert et al. (2020). The extension to the relative case was recently exploited for the first time in (Bertone et al., 2023).

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