

Computation of Pommaret Bases Using Syzygies

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Abstract. We investigate the application of syzygies for efficiently computing (finite) Pommaret bases. For this purpose, we first describe a non-trivial variant of Gerdt’s algorithm [10] to construct an involutive basis for the input ideal as well as an involutive basis for the syzygy module of the output basis. Then we apply this new algorithm in the context of Seiler’s method to transform a given ideal into quasi stable position to ensure the existence of a finite Pommaret basis [19]. This new approach allows us to avoid superfluous reductions in the iterative computation of Janet bases required by this method. We conclude the paper by proposing an involutive variant of the signature based algorithm of Gao et al. [8] to compute simultaneously a Gröbner basis for a given ideal and for the syzygy module of the input basis. All the presented algorithms have been implemented in MAPLE and their performance is evaluated via a set of benchmark ideals.

1 Introduction

Gröbner bases provide a powerful computational tool for a wide variety of problems connected to multivariate polynomial ideals. Together with the first algorithm to compute them, they were introduced by Buchberger in his PhD thesis [2]. Later on, he discovered two criteria to improve his algorithm [3] by omitting superfluous reductions. In 1983, Lazard [15] developed a new approach by using linear algebra techniques to compute Gröbner bases. In 1988, Gebauer and Möller [9], by interpreting Buchberger’s criteria in terms of syzygies, presented an efficient way to improve Buchberger’s algorithm. Furthermore, Möller et al. [16] extended this idea and described the first *signature-based* algorithm to compute Gröbner bases. In 1999, Faugère [6], by applying fast linear algebra on sparse matrices, found his F_4 algorithm to compute Gröbner bases. Then, he introduced the well-known F_5 algorithm [7] that uses two new criteria (F_5 and *IsRewritten*) based on the idea of signatures and that performs no useless reduction as long as the input polynomials define a (semi-)regular sequence. Finally,

Gao et al. [8] presented a new approach to compute simultaneously Gröbner bases for an ideal and its syzygy module.

Involutive bases may be considered as a special kind of non-reduced Gröbner bases with additional combinatorial properties. They originate from the works of Janet [14] on the analysis of partial differential equations. By evolving related methods used by Pommaret [17], the notion of *involutive polynomial bases* was introduced by Zharkov and Blinkov [22]. Later, Gerdt and Blinkov [11] generalised these ideas to the concepts of *involutive divisions* and *involutive bases* for polynomial ideals to produce an effective alternative approach to Buchberger's algorithm (for the efficiency analysis of an implementation of Gerdt's algorithm [10], we refer to the web pages <http://invo.jinr.ru>). Recently, Gerdt et al. [12] proposed a signature-based approach to compute involutive bases.

In this article we discuss effective approaches to compute involutive bases and in particular *Pommaret bases*. These bases are a special kind of involutive bases introduced by Zharkov and Blinkov [22]. While finite Pommaret bases do not always exist, every ideal in a sufficiently generic position has one (see [13] for an extensive discussion of this topic). A finite Pommaret basis reflects many (homological) properties of the ideal it generates. For example, many invariants like dimension, depth and Castelnuovo-Mumford regularity can be easily read off from it. We note that all these invariants remain unchanged under coordinate transformations. We refer to [20] for a comprehensive overview of the theory and applications of Pommaret bases.

We first propose a variant of Gerdt's algorithm to compute an involutive basis which simultaneously determines an involutive basis for the syzygy module of the output basis. Based on it, we improve Seiler's method [19] to compute a linear change of coordinates which brings the input ideal into a generic position so that the new ideal has a finite Pommaret basis. Then, as a related work, we describe an involutive version of the approach by Gao et al. [8] to compute simultaneously Gröbner bases of a given ideal and of the syzygy module of the input basis. All the algorithm described in this paper have been implemented in MAPLE and their efficiency is illustrated via a set of benchmark ideals.

This paper is organized as follows. In Section 2, we review basic definitions and notations related to involutive bases. Section 3 is devoted to a variant of Gerdt's algorithm which also computes an involutive basis for the syzygy module of the output basis. In Section 4, we show how to apply it in the computation of Pommaret bases. Finally in Section 5, we conclude by presenting an involutive variant of the algorithm of Gao et al. by combining it with Gerdt's algorithm.

2 Preliminaries

In this section, we review basic notations and preliminaries needed in the subsequent sections. Throughout this paper, we assume that $\mathcal{P} = \mathbb{k}[x_1, \dots, x_n]$ is the polynomial ring over an infinite field \mathbb{k} . We consider polynomials $f_1, \dots, f_k \in \mathcal{P}$ and the ideal $\mathcal{I} = \langle f_1, \dots, f_k \rangle$ generated by them. The total degree and the degree w.r.t. a variable x_i of a polynomial in $f \in \mathcal{P}$ are denoted by $\deg(f)$

and $\deg_i(f)$, respectively. In addition, $\mathcal{M} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\}$ stands for the monoid of all monomials in \mathcal{P} . We use throughout the reverse degree lexicographic ordering with $x_n \prec \cdots \prec x_1$. The leading monomial of a given polynomial $f \in \mathcal{P}$ w.r.t. \prec is denoted by $\text{LM}(f)$. If $F \subset \mathcal{P}$ is a finite set of polynomials, $\text{LM}(F)$ denotes the set $\{\text{LM}(f) \mid f \in F\}$. The leading coefficient of f , denoted by $\text{LC}(f)$, is the coefficient of $\text{LM}(f)$. The leading term of f is defined to be $\text{LT}(f) = \text{LM}(f)\text{LC}(f)$. A finite set $G = \{g_1, \dots, g_t\} \subset \mathcal{P}$ is called a *Gröbner basis* of \mathcal{I} w.r.t. \prec if $\text{LM}(\mathcal{I}) = \langle \text{LM}(g_1), \dots, \text{LM}(g_t) \rangle$ where $\text{LM}(\mathcal{I}) = \langle \text{LM}(f) \mid f \in \mathcal{I} \rangle$. We refer e.g. to the book of Cox et al. [4] for further details on Gröbner bases.

An analogous notion of Gröbner bases may be defined for sub-modules of \mathcal{P}^t for some t , see [5]. In this direction, let us recall some basic notations and results. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_t\}$ be the standard basis of \mathcal{P}^t . A module monomial in \mathcal{P}^t is an element of the form $x^\alpha \mathbf{e}_i$ for some i , where x^α is a monomial in \mathcal{P} . So, each $f \in \mathcal{P}^t$ can be written as a \mathbb{k} -linear combination of module monomials in \mathcal{P}^t . A total ordering $<$ on the set of monomials of \mathcal{P}^t is called a *module monomial ordering* if the following conditions are satisfied:

- if \mathbf{m} and \mathbf{n} are two module monomials such that $\mathbf{n} < \mathbf{m}$ and $x^\alpha \in \mathcal{P}$ is a monomial then $x^\alpha \mathbf{n} < x^\beta \mathbf{m}$,
- $<$ is well-ordering.

In addition, we say that $x^\alpha \mathbf{e}_i$ divides $x^\beta \mathbf{e}_j$ if $i = j$ and x^α divides x^β . Based on these definitions, one is able to extend the theory of Gröbner bases to sub-modules of the \mathcal{P} -modules of finite rank. Some well-known examples of module monomial ordering are term over position (TOP), position over term (POT) and the Schreyer ordering.

Definition 1. Let $\{g_1, \dots, g_t\} \subset \mathcal{P}$ and \prec a monomial ordering on \mathcal{P} . We define the Schreyer module ordering on \mathcal{P}^t as follows: We write $x^\alpha \mathbf{e}_i \prec_s x^\beta \mathbf{e}_j$ if either $\text{LM}(x^\alpha g_i) \prec \text{LM}(x^\beta g_j)$, or $\text{LM}(x^\alpha g_i) = \text{LM}(x^\beta g_j)$ and $j < i$.

Schreyer proposed in his master thesis [18] a slight modification of Buchberger's algorithm to compute a Gröbner basis for the syzygy module of a Gröbner basis.

Definition 2. Let us consider $G = (g_1, \dots, g_t) \in \mathcal{P}^t$. The (first) syzygy module of G is defined to be $\text{Syz}(G) = \{(h_1, \dots, h_t) \mid h_i \in \mathcal{P}, \sum_{i=1}^t h_i g_i = 0\}$.

Let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis. By Buchberger's criterion, each S -polynomial has a standard representation: $\text{SPoly}(g_i, g_j) = a_{ji} m_{ji} g_i - a_{ij} m_{ij} g_j = h_{ij1} g_1 + \cdots + h_{ijt} g_t$ where $a_{ji}, a_{ij} \in \mathbb{k}$, $h_{ijl} \in \mathcal{P}$ and m_{ji}, m_{ij} are monomials. Let $\mathbf{S}_{ij} = a_{ji} m_{ji} \mathbf{e}_i - a_{ij} m_{ij} \mathbf{e}_j - h_{ij1} \mathbf{e}_1 - \cdots - h_{ijt} \mathbf{e}_t$ be the corresponding syzygy.

Theorem 3 (Schreyer's Theorem). With the above introduced notations, the set $\{\mathbf{S}_{ij} \mid 1 \leq i < j \leq t\}$ is a Gröbner basis for $\text{Syz}(g_1, \dots, g_t)$ w.r.t. \prec_s .

Example 4. Let $F = \{xy - x, x^2 - y\} \subset \mathbb{k}[x, y]$. The Gröbner basis of F w.r.t. $x \prec_{dex} y$ is $G = \{g_1 = xy - x, g_2 = x^2 - y, g_3 = y^2 - y\}$ and the Gröbner basis of $\text{Syz}(g_1, g_2, g_3)$ is $\{(x, -y + 1, -1), (-x, y^2 - 1, -x^2 + y + 1), (y, 0, -x)\}$.

If $F = \{f_1, \dots, f_k\}$ is *not* a Gröbner basis, Wall [21] proposed an effective method to compute $\text{Syz}(F)$. If the extended set $G = f_1, \dots, f_k, f_{k+1}, \dots, f_t$ is a Gröbner basis of $\langle F \rangle$, then $\text{Syz}(F) = \{As \mid s \in \text{Syz}(G)\}$ where A is a matrix such that $G = FA$.

We conclude this section by recalling some definitions and results from the theory of involutive bases (see [10, 20] for more details). Given a set of polynomials, an involutive division partitions the variables into two disjoint subsets of *multiplicative* and *non-multiplicative* variables.

Definition 5. An involutive division \mathcal{L} is given on \mathcal{M} if for any finite set $U \subset \mathcal{M}$ and any $u \in U$, the set of variables is partitioned into the subsets of multiplicative variables $M_{\mathcal{L}}(u, U)$ and non-multiplicative variables $NM_{\mathcal{L}}(u, U)$ such that the following conditions hold where $\mathcal{L}(u, U)$ denotes the monoid generated by $M_{\mathcal{L}}(u, U)$:

1. $v, u \in U, u\mathcal{L}(u, U) \cap v\mathcal{L}(v, U) \neq \emptyset \Rightarrow u \in v\mathcal{L}(v, U)$ or $v \in u\mathcal{L}(u, U)$,
2. $v \in U, v \in u\mathcal{L}(u, U) \Rightarrow \mathcal{L}(v, U) \subset \mathcal{L}(u, U)$,
3. $V \subset U$ and $u \in V \Rightarrow \mathcal{L}(u, U) \subset \mathcal{L}(u, V)$.

We shall write $u \mid_{\mathcal{L}} w$ if $w \in u\mathcal{L}(u, U)$. In this case, u is called an \mathcal{L} -involutive divisor of w and w an \mathcal{L} -involutive multiple of u .

We recall the definitions of the Janet and Pommaret division, respectively.

Example 6. Let $U \subset \mathcal{P}$ be a finite set of monomials. For each sequence d_1, \dots, d_n of non-negative integers and for each $1 \leq i \leq n$ we define

$$[d_1, \dots, d_i] = \{u \in U \mid d_j = \deg_j(u), 1 \leq j \leq i\}.$$

The variable x_1 is Janet multiplicative (denoted by \mathcal{J} -multiplicative) for $u \in U$ if $\deg_1(u) = \max\{\deg_1(v) \mid v \in U\}$. For $i > 1$ the variable x_i is Janet multiplicative for $u \in [d_1, \dots, d_{i-1}]$ if $\deg_i(u) = \max\{\deg_i(v) \mid v \in [d_1, \dots, d_{i-1}]\}$.

Example 7. For $u = x_1^{d_1} \cdots x_k^{d_k}$ with $d_k > 0$ the variables $\{x_k, \dots, x_n\}$ are considered as Pommaret multiplicative (denoted by \mathcal{P} -multiplicative) and the other variables as Pommaret non-multiplicative. For $u = 1$ all the variables are multiplicative. The integer k is called the *class* of u and is denoted by $\text{cls}(u)$.

Definition 8. The set $F \subset \mathcal{P}$ is called involutively head autoreduced if for each $f \in F$ there is no $h \in F \setminus \{f\}$ with $\text{LM}(h) \mid_{\mathcal{L}} \text{LM}(f)$.

Definition 9. Let $I \subset \mathcal{P}$ be an ideal and \mathcal{L} an involutive division. An involutively head autoreduced subset $H \subset \mathcal{I}$ is an involutive basis for \mathcal{I} if for all $f \in \mathcal{I}$ there exists $h \in H$ so that $\text{LM}(h) \mid_{\mathcal{L}} \text{LM}(f)$.

Example 10. For the ideal $\mathcal{I} = \langle xy, y^2, z \rangle \subset \mathbb{k}[x, y, z]$ the set $\{xy, y^2, z, xz, yz\}$ is a Janet basis, but there exists only an infinite Pommaret basis of the form $\{xy, y^2, z, xz, yz, x^2y, x^2z, \dots, x^ky, x^kz, \dots\}$. One can show that every ideal has a finite Janet basis, i. e. the Janet division is Noetherian.

Gerdt [10] proposed an efficient algorithm to construct involutive bases using a completion process where prolongations of given elements by non-multiplicative variables are reduced. This process terminates in finitely many steps for any Noetherian division. In addition, Seiler [19] characterized the ideals having finite Pommaret bases by relating them to the notion of quasi stability. More precisely, a given ideal has a finite Pommaret basis iff it is in *quasi stable position* (or equivalently if the coordinates are δ -regular) see [19, Prop. 4.4].

Definition 11. *A monomial ideal \mathcal{I} is called quasi stable if for any monomial $m \in \mathcal{I}$ and all integers i, j, s with $1 \leq j < i \leq n$ and $s > 0$, if $x_i^s \mid m$ there exists an integer $t \geq 0$ such that $x_j^t m / x_i^s \in \mathcal{I}$. A homogeneous ideal \mathcal{I} is in quasi stable position if $\text{LM}(\mathcal{I})$ is quasi stable.*

3 Computation of Involutive Basis for Syzygy Module

We present now an effective approach to compute, for a given ideal, simultaneously involutive bases of the ideal and of its syzygy module. We first recall some related concepts and facts from [19]. In loc. cit., an involutive version of Schreyer's theorem is stated where S -polynomials are replaced by non-multiplicative prolongations and an involutive normal form algorithm is used.

More precisely, let $H \subset \mathcal{P}^t$ be a finite set for some $t \in \mathbb{N}$, \prec_s the corresponding Schreyer ordering and \mathcal{L} an involutive division. We divide H into t disjoint subsets $H_i = \{\mathbf{h} \in H \mid \text{LM}(\mathbf{h}) = x^\alpha \mathbf{e}_i, x^\alpha \in \mathcal{M}\}$. In addition, for each i , let $B_i = \{x^\alpha \in \mathcal{M} \mid x^\alpha \mathbf{e}_i \in \text{LM}(H_i)\}$. We assign to each $\mathbf{h} \in H_i$ the multiplicative variables $M_{\mathcal{L}, H, \prec}(\mathbf{h}) = \{x_i \mid x_i \in M_{\mathcal{L}, B_i}(x^\alpha) \text{ with } \text{LM}(\mathbf{h}) = x^\alpha \mathbf{e}_i\}$. Then, the definition of involutive bases for sub-modules proceeds as for ideals.

Let $H = \{h_1, \dots, h_t\} \subset \mathcal{P}$ be an involutive basis. Let $h_i \in H$ be an arbitrary element and x_k a non-multiplicative variable of it. From the definition of involutive bases, there exists a unique j such that $\text{LM}(h_j) \mid x_k \text{LM}(h_i)$. We order the elements of H in such a way that $i < j$ (which is always possible for a continuous division [19, Lemma 5.5]). Then we find a unique involutive standard representation $x_k h_i = \sum_{j=1}^t p_j^{(i,k)} h_j$ where $p_j^{(i,k)} \in \mathbb{k}[M_{\mathcal{L}, H, \prec}(h_j)]$ and the corresponding syzygy $\mathbf{S}_{i,k} = x_k \mathbf{e}_i - \sum_{j=1}^t p_j^{(i,k)} \mathbf{e}_j \in \mathcal{P}^t$. We denote the set of all thus obtained syzygies by $H_{\text{Syz}} = \{\mathbf{S}_{i,k} \mid 1 \leq i \leq t; x_k \in \text{NM}_{\mathcal{L}, H, \prec}(h_i)\}$. An involutive division \mathcal{L} is of *Schreyer type* if all sets $\text{NM}_{\mathcal{L}, H, \prec}(h)$ with $h \in H$ are again involutive bases for the ideals defined by them. Both the Janet and the Pommaret divisions are of Schreyer type.

Theorem 12. *([19, Thm. 5.10]) With the above notations, let \mathcal{L} be a continuous involutive division of Schreyer type w.r.t. \prec and H an involutive basis. Then H_{Syz} is an \mathcal{L} -involutive basis for $\text{Syz}(H)$ w.r.t. \prec_s .*

We now present a non-trivial variant of Gerdt's algorithm [10] computing simultaneously a minimal involutive basis for the input ideal and an involutive basis for the syzygy module of this basis. It uses an analogous idea as the algorithm given in [1]. However, since we aim at determining also a syzygy module,

we must save the traces of all reductions and for this reason we cannot use the syzygies to remove useless reductions.

Algorithm 1 INVBASIS

Input: A finite set $F \subset \mathcal{P}$; an involutive division \mathcal{L} ; a monomial ordering \prec

Output: A minimal \mathcal{L} -basis for $\langle F \rangle$ and an \mathcal{L} -basis for syzygy module of this basis.

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1:  $F := \text{sort}(F, \prec)$ 
2:  $T := \{(F[1], F[1], \emptyset, \mathbf{e}_1, \text{false})\}$ 
3:  $Q := \{(F[i], F[i], \emptyset, \mathbf{e}_i, \text{false}) \mid i = 2, \dots, |F|\}$ 
4:  $S := \{\}$  and  $j := |F|$ 
5: while  $Q \neq \emptyset$  do
6:    $Q := \text{sort}(Q, \prec_s)$ 
7:   select and remove  $p := Q[1]$  from  $Q$ 
8:    $h := \text{INVNORMALFORM}(p, T, \mathcal{L}, \prec)$ 
9:   if  $h[1] = 0$  then
10:      $S := S \cup \{h[2]\}$ 
11:   end if
12:   if  $h[1] = 0$  and  $\text{LM}(\text{Poly}(p)) = \text{LM}(\text{Anc}(p))$  then
13:      $Q := \{q \in Q \mid \text{Anc}(q) \neq \text{Poly}(p) \text{ or } q[5] = \text{true}\}$ 
14:   end if
15:   if  $p[5] = \text{true}$  then
16:      $q := \text{UPDATE}(q, p)$  for each  $q \in T$ 
17:   end if
18:   if  $h[1] \neq 0$  and  $\text{LM}(\text{Poly}(p)) \neq \text{LM}(h)$  then
19:     for  $q \in T$  with proper conventional division  $\text{LM}(h[1]) \mid \text{LM}(\text{Poly}(q))$  do
20:        $Q := Q \cup \{[q[1], q[2], q[3], q[4], \text{true}]\}$ 
21:        $T := T \setminus \{q\}$ 
22:     end for
23:      $j := j + 1$  and  $T := T \cup \{(h[1], h[1], \emptyset, \mathbf{e}_j, \text{false})\}$ 
24:   else
25:      $T := T \cup \{(h[1], \text{Anc}(p), \text{NM}(p), h[2], \text{false})\}$ 
26:   end if
27:   for  $q \in T$  and  $x \in \text{NM}_{\mathcal{L}}(\text{LM}(\text{Poly}(q)), \text{LM}(\text{Poly}(T)) \setminus \text{NM}(q))$  do
28:      $Q := Q \cup \{(x, \text{Poly}(q), \text{Anc}(q), \emptyset, x, \text{Rep}(q), \text{false})\}$ 
29:      $\text{NM}(q) := \text{NM}(q) \cup \text{NM}_{\mathcal{L}}(\text{LM}(\text{Poly}(q)), \text{LM}(\text{Poly}(T))) \cup \{x\}$ 
30:   end for
31: end while
32: return  $(\text{Poly}(T), \{\text{Rep}(p) - \mathbf{e}_{\text{index}(p)} \mid p \in T\} \cup S)$ 

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The algorithm INVBASIS relies on the following data structure for polynomials. To each polynomial f , we associate a quintuple $p = (f, g, V, \mathbf{q}, \text{flag})$. The first entry $f = \text{Poly}(p)$ is the polynomial itself, $g = \text{Anc}(p)$ is the ancestor of f (realised as a pointer to the quintuple associated with the ancestor) and $V = \text{NM}(p)$ is its list of already processed non-multiplicative variables. The fourth entry $\mathbf{q} = \text{Rep}(p)$ denotes the representation of f in our current basis, i.e. if $\mathbf{q} = \sum_{r \in T \cup Q} h_r \mathbf{e}_{\text{index}(r)}$ then $f = \sum_{r \in T \cup Q} h_r \text{Poly}(r)$ where $h_r \in \mathcal{P}$ and $\text{index}(r)$

gives the position of r in the current list $T \cup Q$. The final entry is a boolean flag. If $flag = true$ then at some stage of the algorithm p has been moved from T to Q , otherwise $flag = false$. We denote by $Sig(p) = LM_{\prec_s}(\text{Rep}(p))$ the signature of p . By an abuse of notation, $Sig(f)$ also denotes $Sig(p)$. The same holds for the Rep function. If P is a set of quintuples, we denote by $\text{Poly}(P)$ the set $\{\text{Poly}(p) \mid p \in P\}$. In addition, the functions $\text{sort}(X, \prec)$ and $\text{sort}(X, \prec_s)$ sort X in increasing order according to $\text{LM}(X)$ w.r.t. \prec and $\{Sig(p) \mid p \in X\}$ w.r.t. \prec_s , respectively. We remark that in the original form of Gerdt's algorithm [10] the function $\text{sort}(Q, \prec)$ was applied to sort the set of all non-multiplicative prolongations, however, in our experiments we observed that using $\text{sort}(Q, \prec_s)$ increased the performance of the algorithm.

Obviously, the representation of each polynomial must be updated whenever the set $T \cup Q$ changes in a non-trivial way. We remark that elements of Q can appear non-trivially in the representations of polynomials only if they have been elements of T at an earlier stage of the algorithm (recall that such a move is noted in the flag of each quintuple), as all reductions are performed w.r.t. T only. If updates are necessary, then they are performed by the function UPDATE . Involutive normal forms are computed with the help of the following subalgorithm taking care of the representations.

Algorithm 2 INVNORMALFORM

Input: A quintuple p ; a set of quintuples T ; a division \mathcal{L} ; a monomial ordering \prec

Output: A normal form of p w.r.t. T and its new representation.

$h := \text{Poly}(p)$ and $G := \text{Poly}(T)$ and $\mathbf{q} := \text{Rep}(p)$

while h contains a monomial m which is \mathcal{L} -divisible by $g \in G$ **do**

if $m = \text{LM}(\text{Poly}(p))$ and $\text{C1}(h, g)$ **then**

return $([0, \text{Anc}(p) \text{Rep}(\text{Anc}(g)) - \text{Anc}(g) \text{Rep}(\text{Anc}(p))])$

end if

$h := h - (cm / \text{LT}(g)).g$ where c is the coefficient of m in h

$\mathbf{q} := \mathbf{q} - (cm / \text{LT}(g)) \text{Rep}(g)$

end while

return $([h, \mathbf{q}])$

Here we apply the involutive form of Buchberger's first criterion [10]. We say that $\text{C1}(p, g)$ is true if $\text{LM}(\text{Anc}(p)) \text{LM}(\text{Anc}(g)) = \text{LM}(\text{Poly}(p))$.

Theorem 13. *If \mathcal{L} is a Noetherian continuous involutive division of Schreyer type then INVBASIS terminates in finitely many steps and returns a minimal involutive basis for its input ideal and also an involutive basis for the syzygy module of the constructed basis.*

Proof. The termination of the algorithm is ensured by the termination of Gerdt's algorithm, see [10]. Let us now deal with its correctness. We first note that if an element p is removed by Buchberger's criteria, then it is superfluous and by [10, Thm. 2] the set $\text{Poly}(T)$ forms a minimal involutive basis for $\langle F \rangle$. Thus,

it remains to show that $R = \{\text{Rep}(p) - \mathbf{e}_{\text{index}(p)} \mid p \in T\} \cup S$ is an involutive basis for $\text{Poly}(T) = \{h_1, \dots, h_t\}$ w.r.t. \prec_s . Using Thm. 12, we must show that the representation of each non-multiplicative prolongation of the elements of $\text{Poly}(T)$ appears in R . Let us consider $h_i \in \text{Poly}(T)$ and a non-multiplicative variable x_k for it. Then, due to the structure of the algorithm, $x_k h_i$ is created and studied in the course of the algorithm.

Now, four cases can occur. If $x_k h_i$ reduces to zero then we can write $x_k h_i = \sum_{j=1}^t p_j^{(i,k)} h_j$ where $p_j^{(i,k)} \in \mathbb{k}[M_{\mathcal{L}, H, \prec}(h_j)]$. Therefore the representation $x_k \mathbf{e}_i - \sum_{j=1}^t p_j^{(i,k)} \mathbf{e}_j \in \mathcal{P}^t$ is added to S and consequently it appears in R . If the involutive normal form of $x_k h_i$ is non-zero then we can write $x_k h_i = \sum_{j=1}^t p_j^{(i,k)} h_j + h_\ell$ where $p_j^{(i,k)} \in \mathbb{k}[M_{\mathcal{L}, H, \prec}(h_j)]$. In this case, we add h_ℓ into T and the representation component of $x_k h_i$ is updated to $x_k \mathbf{e}_i - \sum_{j=1}^t p_j^{(i,k)} \mathbf{e}_j$. Then, as we can see in the output of the algorithm, $x_k \mathbf{e}_i - \sum_{j=1}^t p_j^{(i,k)} \mathbf{e}_j - \mathbf{e}_\ell$ appears in R as the syzygy corresponding to $x_k h_i$.

The third case that may occur is that $x_k h_i$ is removed by Buchberger's first criterion. Assume that p is the quintuple associated to $x_k h_i$ and g is another quintuple so that $\text{C1}(p, g)$ is true. It follows that $\text{LM}(\text{Anc}(p)) \text{LM}(\text{Anc}(g)) = \text{LM}(\text{Poly}(p))$ holds. We may let $x_k h_i = u \text{Anc}(p)$, $\text{Poly}(g) = v \text{Anc}(g)$ and $\text{LM}(x_k h_i) = m \text{LM}(g)$ for some monomials u and v and term m (assume that the polynomials are monic). Thus,

$$x_k h_i - m \text{Poly}(g) = u \text{Anc}(p) - mv \text{Anc}(g).$$

As $\text{LM}(\text{Anc}(p)) \text{LM}(\text{Anc}(g)) = \text{LCM}(\text{LM}(\text{Anc}(p)), \text{LM}(\text{Anc}(g)))$, Buchberger's first criterion applied to $\text{Anc}(p)$ and $\text{Anc}(g)$ yields that $\text{Anc}(p) \text{Rep}(\text{Anc}(g)) - \text{Anc}(g) \text{Rep}(\text{Anc}(p))$ is the corresponding syzygy which is added to S .

The last case to be considered is that $x_k h_i$ is removed by the second **if**-loop in the main algorithm. In this case, we conclude that $\text{Anc}(p)$ is reduced to zero and in consequence h_i is reduced to zero. So, h_i is a useless polynomial and we do not need to keep $x_k h_i$ which ends the proof. \square

Remark 14. There also exists an involutive version of Buchberger's second criterion [10]: $\text{C2}(p, g)$ is true if $\text{LCM}(\text{LM}(\text{Anc}(p)), \text{LM}(\text{Anc}(g)))$ properly divides $\text{LM}(\text{Poly}(p))$. We cannot use this criterion in the `INVNORMALFORM` algorithm. A non-multiplicative prolongation $x_k h_i$ removed by it is surely useless in the sense that is not needed for determining the involutive basis of \mathcal{I} , but it can nevertheless be necessary for the construction of its syzygy module.

Example 15. Let us consider the ideal \mathcal{I} generated by $F = \{f_1 = z^2, f_2 = zy, f_3 = xz - y, f_4 = y^2, f_5 = xy - y, f_6 = x^2 - x + z\} \subset \mathbb{k}[x, y, z]$ from [19, Ex. 5.6]. Then, F is a Janet basis w.r.t. $z \prec y \prec x$. Since x, y are non-multiplicative variables for f_1, f_2, f_3 and x is non-multiplicative variable for f_4, f_5 then the following set is a Janet basis for the syzygy module of F : $\{y\mathbf{e}_1 - z\mathbf{e}_2, x\mathbf{e}_1 - z\mathbf{e}_3 - \mathbf{e}_2, y\mathbf{e}_2 - z\mathbf{e}_4, x\mathbf{e}_2 - z\mathbf{e}_5 - \mathbf{e}_2, y\mathbf{e}_3 - z\mathbf{e}_5 + \mathbf{e}_4 - \mathbf{e}_2, x\mathbf{e}_3 - z\mathbf{e}_6 + \mathbf{e}_5 - \mathbf{e}_3 + \mathbf{e}_1, x\mathbf{e}_4 - y\mathbf{e}_5 - \mathbf{e}_4, x\mathbf{e}_5 - y\mathbf{e}_6 + \mathbf{e}_2\}$.

4 Application to Pommaret Basis Computation

In this section we show how to apply the approach presented in the preceding section in the computation of Pommaret bases. The Pommaret division is not Noetherian and thus a given ideal may not have a finite Pommaret basis. However, a generic linear change of variables transforms the ideal into quasi stable position where a finite Pommaret basis exists. Seiler [19] proposed a deterministic algorithm to compute such a linear change by performing repeatedly an *elementary* linear change and then a test on the Janet basis of the transformed ideal. Now, to apply the method presented in this paper, we use the INV BASIS algorithm to compute a minimal Janet basis H for the input ideal and at the same time a Janet basis for $\text{Syz}(H)$. Then, for each $h \in H$ we check whether there exists a variable which is Janet but not Pommaret multiplicative. If not, H is a Pommaret basis and we are done. Otherwise, we make an elementary linear change of variables, say ϕ . Then, we apply the following algorithm, NEXTINV BASIS, to compute a minimal Janet basis for the ideal generated by $\phi(H)$ by applying $\phi(\text{Syz}(H))$ to remove superfluous reductions. We describe first the main procedure.

Algorithm 3 QUASISTABLE

Input: A finite set $F \subset \mathcal{P}$ of homogeneous polynomials and a monomial ordering \prec

Output: A linear change Φ so that $\langle \Phi(F) \rangle$ has a finite Pommaret basis

```

 $\Phi :=$ the identity map
 $J, S :=$ INVBASIS( $F, \mathcal{J}, \prec$ ) and  $A :=$ TEST(LM( $J$ ))
while  $A \neq \text{true}$  do
     $\phi := A[3] \mapsto A[3] + cA[2]$  for a random choice of  $c \in \mathbb{k}$ 
     $Temp :=$ NEXTINVBASIS( $\Phi \circ \phi(J), \Phi \circ \phi(S), \mathcal{J}, \prec$ )
     $B :=$ TEST(LM( $Temp$ ))
    if  $B \neq A$  then
         $\Phi := \Phi \circ \phi$  and  $A := B$ 
    end if
end while
return ( $\Phi$ )

```

The function TEST receives a set of monomials forming a minimal Janet basis and returns true if it is a Pommaret basis, too. Otherwise, by [19, Prop. 2.10], there exists a monomial m in the set for which a Janet multiplicative variable (say x_ℓ) is not Pommaret multiplicative. In this case, the function returns (*false*, $x_\ell, \text{cls}(m)$). Using these variables, we construct an elementary linear change of variables.

The NEXTINV BASIS algorithm is similar to the INVBASIS algorithm given above. However, the new algorithm computes only the involutive basis of the input ideal generated by a set H . In addition, in the new algorithm, we use $\text{Syz}(H)$ to remove useless reductions. Below, only the differences between the two algorithms are exhibited.

Algorithm 4 NEXTINVBASIS

Input: A finite set $F \subset \mathcal{P}$; a generating set S for $\text{Syz}(F)$; an involutive division \mathcal{L} ; a monomial ordering \prec

Output: A minimal involutive basis for $\langle F \rangle$

```
⋮      {Lines 1–6 of INVBASIS}
select and remove  $p := Q[1]$  from  $Q$ 
if  $\nexists \mathbf{s} \in S$  s.t.  $\text{LM}_{\prec_s}(\mathbf{s}) \mid \text{Sig}(p)$  then
    ⋮      {Lines 8–30 of INVBASIS}
end if
⋮      {Lines 31/32 of INVBASIS}
```

Lemma 16. *Let $H \subset \mathcal{P}$ and S be a generating set for $\text{Syz}(H)$. For any invertible linear change of variables ϕ , $\phi(S)$ generates $\text{Syz}(\phi(H))$.*

Proof. Suppose that $H = \{h_1, \dots, h_t\}$ and $S = \{\mathbf{s}_1, \dots, \mathbf{s}_\ell\} \subset \mathcal{P}^t$. Let $\mathbf{s}_i = (p_{i1}, \dots, p_{it})$. Since $p_{i1}h_1 + \dots + p_{it}h_t = 0$ and ϕ is a ring homomorphism then $\phi(p_{i1})\phi(h_1) + \dots + \phi(p_{it})\phi(h_t) = 0$ and therefore $\phi(\mathbf{s}_i) \in \text{Syz}(\phi(H))$. Conversely, assume that $\mathbf{s} = (p_1, \dots, p_t) \in \text{Syz}(\phi(H))$. This shows that $p_1\phi(h_1) + \dots + p_t\phi(h_t) = 0$. By invertibility of ϕ we have $(\phi^{-1}(p_1), \dots, \phi^{-1}(p_t)) \in \text{Syz}(H)$. From assumptions, we conclude that $(\phi^{-1}(p_1), \dots, \phi^{-1}(p_t)) = g_1\mathbf{s}_1 + \dots + g_\ell\mathbf{s}_\ell$ for some $g_i \in \mathcal{P}$. By applying ϕ on both sides of this equality, we can deduce that \mathbf{s} is generated by $\phi(S)$ and the proof is complete. \square

Theorem 17. *The algorithm QUASISTABLE terminates in finitely many steps and returns for a given homogeneous ideal a linear change of variables s.t. the transformed ideal possesses a finite Pommaret basis.*

Proof. Seiler [19, Prop. 2.9] proved that for a generic linear change of variables ϕ , the ideal $\langle \phi(F) \rangle$ has a finite Pommaret basis. He also showed that the process of finding such a linear change, by applying elementary linear changes, terminates in finitely many steps, see [19, Remark 9.11] (or [13]). These arguments establish the finite termination of the algorithm. To prove the correctness, using Thm. 13, we must only show that if $p \in Q$ is removed by $\mathbf{s} \in S$ then it is superfluous. To this end, assume that $F = \{f_1, \dots, f_k\}$ and $\mathbf{s} = (p_1, \dots, p_k)$. Thus, we have $p_1f_1 + \dots + p_kf_k = 0$. On the other hand, we know that $\text{LM}_{\prec_s}(\mathbf{s}) \mid \text{Sig}(p)$. W.l.o.g., we may assume that $\text{LM}_{\prec_s}(\mathbf{s}) = \text{LM}(p_1)\mathbf{e}_1$. Therefore, $\text{Poly}(p)$ can be written as a combination $g_1f_1 + \dots + g_kf_k$ such that $\text{LM}(g_1)$ divides $\text{LM}(p_1)$. Let $t = \text{LM}(p_1)/\text{LM}(g_1)$. We can write $\text{LM}(g_1)f_1$ as a linear combination of some multiplications mf_i where m is a monomial such that $m\mathbf{e}_i$ is strictly smaller than $\text{LM}(g_1)\mathbf{e}_1$. It follows that p has an involutive representation provided that we study tmf_i for each m and i . Since the signature of tmf_i is strictly smaller than $t\text{LM}(g_1)\mathbf{e}_1 = \text{Sig}(p)$, we are sure that no loop is performed and therefore p can be omitted. \square

We have implemented the algorithm QUASISTABLE in MAPLE 17⁴ and compared its performance with our implementation of the HDQUASISTABLE algorithm presented in [1] (it is a similar procedure applying a Hilbert driven technique). For this, we used some well-known examples from computer algebra literature. All computations were done over \mathbb{Q} using the degree reverse lexicographical monomial ordering. The results are represented in the following tables where the time and memory columns indicate the consumed CPU time in second and amount of megabytes of used memory, respectively. The dim column refers to the dimension of the corresponding ideal. The columns corresponding to C_1 and C_2 show, respectively, the number of polynomials removed by C_1 and C_2 criteria. The seventh column denotes the number of polynomials eliminated by the criterion related to signature applied in NEXTINVBASIS algorithm (see [1] for more details). The eighth column shows the number of polynomials eliminated by the Hilbert driven technique which may be applied in NEXTINVBASIS algorithm to remove useless reductions, (see [1] for more details). The ninth column shows the number of polynomials eliminated by the syzygy criterion described in NEXTINVBASIS algorithm. The last three columns represent, respectively, the number of reductions to zero, the number of performed elementary linear changes and the maximum degree attained in the computations. The computations in this paper are performed on a personal computer with 2.60 GHz Pentium(R) Core(TM) Dual-Core CPU, 2 GB of RAM, 32 bits under the Windows 7 operating system.

Weispfenning94	time	memory	dim	C_1	C_2	SC	HD	Syz	redz	lin	deg
QUASISTABLE	4.5	255.5	2	0	0	0	34	10	41	1	14
HDQUASISTABLE	5.3	261.4	2	0	1	9	46	-	29	1	14

Liu	time	memory	dim	C_1	C_2	SC	HD	Syz	redz	lin	deg
QUASISTABLE	6.1	246.7	2	8	0	10	71	47	44	4	6
HDQUASISTABLE	8.9	346.0	2	6	3	25	125	-	60	4	6

Noon	time	memory	dim	C_1	C_2	SC	HD	Syz	redz	lin	deg
QUASISTABLE	74.1	3653.2.2	1	6	7	10	213	83	215	4	10
HDQUASISTABLE	72.3	3216.9.7	1	4	24	10	351	-	105	4	10

Katsura5	time	memory	dim	C_1	C_2	SC	HD	Syz	redz	lin	deg
QUASISTABLE	95.7	4719.2	5	49	0	0	257	56	115	3	8
HDQUASISTABLE	120.8	5527.7	5	44	4	6	420	-	122	3	8

Vermeer	time	memory	dim	C_1	C_2	SC	HD	Syz	redz	lin	deg
QUASISTABLE	175.5	8227.9	3	5	3	101	158	139	343	3	13
HDQUASISTABLE	192.5	8243.7	3	3	28	157	343	-	190	3	13

Butcher	time	memory	dim	C_1	C_2	SC	HD	Syz	redz	lin	deg
QUASISTABLE	290.6	12957.8	3	135	89	73	183	86	534	3	8
HDQUASISTABLE	433.1	17005.5	3	178	178	219	355	-	386	3	8

As one sees for some examples, some columns are different. It is worth noting that this difference may be due to the fact that the coefficients in the linear changes are chosen randomly and this may affect the behavior of the algorithm.

⁴ The MAPLE code of the implementations of our algorithms and examples are available at <http://amirhashemi.iut.ac.ir/software>s

5 Involutive Variant of the GVW Algorithm

Gao et al. [8] described recently a new algorithm, the GVW algorithm, to compute simultaneously Gröbner bases for a given ideal and for the syzygy module of the given ideal basis. In this section, we present an involutive variant of this approach and compare its efficiency with the existing algorithms to compute involutive bases. For a review of the general setting of the signature based structure that we use in this paper, we refer to [8]. Let $\{f_1, \dots, f_k\} \subset \mathcal{P}$ be a finite set of non-zero polynomials and $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ the standard basis for \mathcal{P}^k . Let us fix an involutive division \mathcal{L} and a monomial ordering \prec . Our goal is to compute an involutive basis for $\mathcal{I} = \langle f_1, \dots, f_k \rangle$ and a Gröbner basis for $\text{Syz}(f_1, \dots, f_k)$ w.r.t. \prec_s . Let us consider

$$\mathcal{V} = \{(\mathbf{u}, v) \in \mathcal{P}^k \times \mathcal{P} \mid u_1 f_1 + \dots + u_k f_k = v \text{ with } \mathbf{u} = (u_1, \dots, u_k)\}$$

as an \mathcal{P} -submodule of \mathcal{P}^{k+1} . For any pair $p = (\mathbf{u}, v) \in \mathcal{P}^k \times \mathcal{P}$, $\text{LM}_{\prec_s}(\mathbf{u})$ is called the *signature* of p and is denoted by $\text{Sig}(p)$. We define the involutive version of top-reduction defined in [8]. Let $p_1 = (\mathbf{u}_1, v_1)$, $p_2 = (\mathbf{u}_2, v_2) \in \mathcal{P}^k \times \mathcal{P}$. When v_2 is non-zero, we say p_1 is *involutively top-reducible* by p_2 if:

- v_1 is non-zero and $\text{LM}(v_2)$ \mathcal{L} -divides $\text{LM}(v_1)$ and
- $\text{LM}(t\mathbf{u}_2) \preceq_s \text{LM}(\mathbf{u}_1)$ where $t = \text{LM}(v_1)/\text{LM}(v_2)$.

The corresponding top-reduction is $p_1 -ctp_2 = (\mathbf{u}_1 - ct\mathbf{u}_2, v_1 - ctv_2)$ where $c = \text{LC}(v_1)/\text{LC}(v_2)$. Such a top-reduction is called *regular*, if $\text{LM}(\mathbf{u}_1 - ct\mathbf{u}_2) = \text{LM}(\mathbf{u}_1)$, and *super* otherwise.

Definition 18. A finite subset $G \subset \mathcal{V}$ is called a strong involutive basis for \mathcal{I} if every pair in \mathcal{V} is involutively top-reducible by some pair in G . A strong involutive basis G is minimal if any other strong involutive basis G' of \mathcal{I} satisfies $\text{LM}(G) \subseteq \text{LM}(G')$.

Proposition 19. Suppose that $G = \{(\mathbf{u}_1, v_1), \dots, (\mathbf{u}_m, v_m)\}$ is a strong involutive basis for \mathcal{I} . Then $G_0 = \{\mathbf{u}_i \mid v_i = 0, 1 \leq i \leq m\}$ is a Gröbner basis for $\text{Syz}(f_1, \dots, f_k)$, and $G_1 = \{v_1, \dots, v_m\}$ is an involutive basis for \mathcal{I} .

Proof. The proof is an easy consequence of the proof of [8, Prop. 2.2]. □

Let $p_1 = (\mathbf{u}_1, v_1)$ and $p_2 = (\mathbf{u}_2, v_2)$ be two pairs in \mathcal{V} . We say that p_1 is *covered* by p_2 if $\text{LM}(\mathbf{u}_2)$ divides $\text{LM}(\mathbf{u}_1)$ and $t\text{LM}(v_2) \prec \text{LM}(v_1)$ (strictly smaller) where $t = \text{LM}(\mathbf{u}_1)/\text{LM}(\mathbf{u}_2)$. Also, p is covered by G if it is covered by some pair in G . A pair $p \in \mathcal{V}$ is *eventually super reducible* by G if there is a sequence of regular top-reductions of p by G leading to (\mathbf{u}', v') which is no longer regularly reducible by G but super reducible by G .

Theorem 20. Let $G \subset \mathcal{V}$ be a finite set such that, for any module monomial $\mathbf{m} \in \mathcal{P}^k$, there is a pair $(\mathbf{u}, v) \in G$ such that $\text{LM}(\mathbf{u}) \mid \mathbf{m}$. Then the following conditions are equivalent:

1. G is a strong involutive basis for \mathcal{I} ,
2. any non-multiplicative prolongation of any element of G is eventually super top-reducible by G ,
3. any non-multiplicative prolongation of any element in G is covered by G .

Proof. The proof of all implications are similar to the proofs of the corresponding statements in [8, Thm. 2.4] except that we need some slight changes in the proof of (3 \Rightarrow 1). We proceed by reductio ad absurdum. Assume that there is a pair $p = (\mathbf{u}, v) \in \mathcal{V}$ which is not involutively top-reducible by G and has minimal signature. Then, by assumption, there exists $p_1 = (\mathbf{u}_1, v_1) \in G$ such that $\text{LM}(\mathbf{u}) = t \text{LM}(\mathbf{u}_1)$ for some t . Select p_1 such that $t \text{LM}(v_1)$ is minimal. Let us now consider tp_1 . Two cases may happen: If all variables in t are multiplicative for p_1 then, $p - tp_1$ has a signature smaller than p and by assumption it has a standard representation leading to a standard representation for p which is a contradiction. Otherwise, t has a non-multiplicative variable. Then, tp_1 is covered by a pair $p_3 = (\mathbf{u}_3, v_3) \in G$. This shows that $t_3 \text{LM}(v_3) \prec t \text{LM}(v_1)$ with $t_3 = t \text{LM}(\mathbf{u}_1) / \text{LM}(\mathbf{u}_3)$. Therefore, the polynomial part of $t_3 p_3$ is smaller than tp_1 which contradicts the choice of p_1 , and this ends the proof. \square

Based on this theorem and similar to the structure of the GVW algorithm, we describe a variant of Gerdt's algorithm for computing strong involutive bases. The structure of the new algorithm is similar to the INVBASIS algorithm and therefore we omit the identical parts.

Algorithm 5 STINVBASIS

Input: A finite set $F \subset \mathcal{P}$; an involutive division \mathcal{L} ; a monomial ordering \prec

Output: A minimal strong involutive basis for $\langle F \rangle$

$F := \text{sort}(F, \prec)$ and $T := \{(F[1], F[1], \emptyset, \mathbf{e}_1)\}$

$Q := \{(F[i], F[i], \emptyset, \mathbf{e}_i) \mid i = 2, \dots, |F|\}$ and $H := \{\}$

while $Q \neq \emptyset$ **do**

$Q := \text{sort}(Q, \prec_s)$ and select/remove the first element p from Q

if p is not covered by G , T or H **then**

$h := \text{INVTOPREDUCE}(p, T, \mathcal{L}, \prec)$

if $\text{Poly}(h) = 0$ **then**

$H := H \cup \{\text{Sig}(p)\}$

end if

if $\text{Poly}(h) = 0$ and $\text{LM}(\text{Poly}(p)) = \text{LM}(\text{Anc}(p))$ **then**

$Q := \{q \in Q \mid \text{Anc}(q) \neq \text{Poly}(p)\}$

end if

if $\text{Poly}(h) \neq 0$ and $\text{LM}(\text{Poly}(p)) \neq \text{LM}(\text{Poly}(h))$ **then**

\vdots {Lines 19–25 of INVBAS}

end if

\vdots {Lines 27–30 of INVBAS}

end if

end while

return $(\text{Poly}(T), H)$

Algorithm 6 INVTOPREDUCE

Input: A quadruple p ; a set of quadruples T ; a division \mathcal{L} ; a monomial ordering \prec
Output: A top-reduced form of p modulo T
 $h := p$
while Poly(h) has a term am with $a \in \mathbb{k}$ and $\text{LM}(\text{Poly}(q)) \mid_{\mathcal{L}} m$ with $q \in T$ **do**
 if $m/\text{LM}(\text{Poly}(q)) \text{Sig}(q) \prec_s \text{Sig}(p)$ **then**
 Poly(h) := Poly(h) – $am/\text{LT}(\text{Poly}(q)) \cdot \text{Poly}(q)$
 Rep(h) := Rep(h) – $am/\text{LT}(\text{Poly}(q)) \cdot \text{Rep}(q)$
 end if
end while
return (h)

The proof of the next theorem is a consequence of Thm. 20 and the termination and correctness of Gerdt’s algorithm.

Theorem 21. *If \mathcal{L} is Noetherian, then STINVBASIS terminates in finitely many steps returning a minimal strong involutive basis for its input ideal.*

We have implemented the STINVBASIS algorithm in MAPLE 17 and compared its performance with our implementation of INVOLUTIVEBASIS algorithm (see [1]) and VARGERDT algorithm (a variant of Gerdt’s algorithm, see [12]).

Liu	time	memory	C_1	C_2	SC	cover	redz	deg
STINVBASIS	.390	14.806	-	-	-	17	20	6
INVOLUTIVEBASIS	.748	23.830	4	3	2	-	18	6
VARGERDT	1.653	64.877	6	3	-	-	18	19

Noon	time	memory	C_1	C_2	SC	cover	redz	deg
STINVBASIS	1.870	75.213	-	-	-	54	42	10
INVOLUTIVEBASIS	2.620	105.641	4	15	6	-	50	10
VARGERDT	12.32	454.573	6	9	-	-	56	10

Haas3	time	memory	C_1	C_2	SC	cover	redz	deg
STINVBASIS	157.623	6354.493	-	-	-	490	8	33
INVOLUTIVEBASIS	22.345	833.0	0	0	83	-	152	33
VARGERDT	137.733	5032.295	0	98	-	-	255	33

Sturmfels-Eisenbud	time	memory	C_1	C_2	SC	cover	redz	deg
STINVBASIS	2442.414	120887.953	-	-	-	634	29	8
INVOLUTIVEBASIS	24.70	951.070	28	103	95	-	81	6
VARGERDT	59.32	2389.329	43	212	-	-	91	6

Weispfenning94	time	memory	C_1	C_2	SC	cover	redz	deg
STINVBASIS	183.129	8287.044	-	-	-	588	28	18
INVOLUTIVEBASIS	1.09	45.980	0	1	9	-	28	10
VARGERDT	4.305	168.589	0	9	-	-	38	15

As we observe, the performance of the new algorithm is not in general better than that of the others. This is due to the signature-based structure of the new algorithm which does not allow to perform full normal forms.

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