Quasi-Stability versus Genericity

Amir Hashemi[1], Michael Schweinfurter[2], and Werner M. Seiler[2]

[1] Department of Mathematical Sciences, Isfahan University of Technology
Isfahan, 84156-83111, Iran; Amir.Hashemi@cc.iut.ac.ir
[2] Institut für Mathematik, Universität Kassel
Heinrich-Plett-Straße 40, 34132 Kassel, Germany
[michael.schweinfurter,seiler]@mathematik.uni-kassel.de

Abstract. Quasi-stable ideals appear as leading ideals in the theory of Pommaret bases. We show that quasi-stable leading ideals share many of the properties of the generic initial ideal. In contrast to genericity, quasi-stability is a characteristic independent property that can be effectively verified. We also relate Pommaret bases to some invariants associated with local cohomology, exhibit the existence of linear quotients in Pommaret bases and prove some results on componentwise linear ideals.

1 Introduction

The generic initial ideal of a polynomial ideal $0 \neq \mathcal{I} \trianglelefteq \mathcal{P} = \Bbbk[\mathcal{X}] = \Bbbk[x_1, \dots, x_n]$ was defined by Galligo [10] for the reverse lexicographic order and $\operatorname{char} \Bbbk = 0$; the extension to arbitrary term orders and characteristics is due to Bayer and Stillman [5]. Extensive discussions can be found in [9, Sect. 15.9], [17, Chapt. 4] and [13]. A characteristic feature of the generic initial ideal is that it is Borel-fixed, a property depending on the characteristics of $\Bbbk$.

Quasi-stable ideals are known under many different names like ideals of nested type [6], ideals of Borel type [19] or weakly stable ideals [7]. They appear naturally as leading ideals in the theory of Pommaret bases [25], a special class of Gröbner bases with additional combinatorial properties. The notion of quasi-stability is characteristic independent.

The generic initial ideal has found quite some interest, as many invariants take the same value for $\mathcal{I}$ and $\operatorname{gin} \mathcal{I}$, whereas arbitrary leading ideals generally lead to larger values. However, there are several problems with $\operatorname{gin} \mathcal{I}$: it depends on $\operatorname{char} \Bbbk$; there is no effective test known to decide whether a given leading ideal is $\operatorname{gin} \mathcal{I}$ and thus one must rely on expensive random transformations for its construction. The main point of the present work is to show that quasi-stable leading ideals enjoy many of the properties of $\operatorname{gin} \mathcal{I}$ and can nevertheless be effectively detected and deterministically constructed.

Throughout this article, $\mathcal{P} = \Bbbk[\mathcal{X}]$ denotes a polynomial ring in the variables $\mathcal{X} = \{x_1, \dots, x_n\}$ over an infinite field $\Bbbk$ of arbitrary characteristic and $0 \neq \mathcal{I} \lhd \mathcal{P}$ a proper homogeneous ideal. When considering bases of $\mathcal{I}$, we will always assume that these are homogeneous, too. $\mathfrak{m} = \langle \mathcal{X} \rangle \lhd \mathcal{P}$ is the homogeneous maximal ideal. In order to be consistent with [24, 25], we will use a non-standard convention for the reverse lexicographic order: given two arbitrary terms x^μ, x^ν of the same degree, $x^\mu \prec_{\text{revlex}} x^\nu$ if the first non-vanishing entry of $\mu - \nu$ is positive. Compared with the usual convention, this corresponds to a reversion of the numbering of the variables $\mathcal{X}$.

2 Pommaret Bases

Pommaret bases are a special case of *involutive bases*; see [24] for a general survey. The algebraic theory of Pommaret bases was developed in [25] (see also [26, Chpts. 3-5]). Given an exponent vector $\mu = [\mu_1, \dots, \mu_n] \neq 0$ (or the term x^μ or a polynomial $f \in \mathcal{P}$ with $\operatorname{lt} f = x^\mu$ for some fixed term order), we call $\min\{i \mid \mu_i \neq 0\}$ the *class* of μ (or x^μ or f), denoted by $\operatorname{cls}\mu$ (or $\operatorname{cls} x^\mu$ or $\operatorname{cls} f$). Then the *multiplicative variables* of x^μ or f are $\mathcal{X}_P(x^\mu) = \mathcal{X}_P(f) = \{x_1, \dots, x_{\operatorname{cls}\mu}\}$. We say that x^μ is an *involutive divisor* of another term x^ν, if $x^\mu \mid x^\nu$ and $x^{\nu-\mu} \in \Bbbk[x_1, \dots, x_{\operatorname{cls}\mu}]$. Given a finite set $\mathcal{F} \subset \mathcal{P}$, we write $\deg \mathcal{F}$ for the maximal degree and $\operatorname{cls}\mathcal{F}$ for the minimal class of an element of $\mathcal{F}$.

Definition 1. *Assume first that the finite set $\mathcal{H} \subset \mathcal{P}$ consists only of terms. $\mathcal{H}$ is a* Pommaret basis *of the monomial ideal $\mathcal{I} = \langle \mathcal{H} \rangle$, if as a $\Bbbk$-linear space*

$$\bigoplus_{h \in \mathcal{H}} \Bbbk[\mathcal{X}_P(h)] \cdot h = \mathcal{I} \tag{1}$$

(in this case each term $x^\nu \in \mathcal{I}$ has a unique involutive divisor $x^\mu \in \mathcal{H}$). A finite polynomial set $\mathcal{H}$ is a Pommaret basis *of the polynomial ideal $\mathcal{I}$ for the term order $\prec$, if all elements of $\mathcal{H}$ possess distinct leading terms and these terms form a Pommaret basis of the leading ideal $\operatorname{lt}\mathcal{I}$.*

Pommaret bases can be characterised similarly to Gröbner bases. However, involutive standard representations are unique. Furthermore, the existence of a Pommaret basis implies a number of properties that usually hold only generically.

Proposition 2 ([24, Thm. 5.4]). *The finite set $\mathcal{H} \subset \mathcal{I}$ is a Pommaret basis of the ideal $\mathcal{I} \lhd \mathcal{P}$ for the term order $\prec$, if and only if every polynomial $0 \neq f \in \mathcal{I}$ possesses a unique involutive standard representation $f = \sum_{h \in \mathcal{H}} P_h h$ where each non-zero coefficient $P_h \in \Bbbk[\mathcal{X}_P(h)]$ satisfies $\operatorname{lt}(P_h h) \preceq \operatorname{lt}(f)$.*

Proposition 3 ([24, Cor. 7.3]). *Let $\mathcal{H}$ be a finite set of polynomials and $\prec$ a term order such that no leading term in $\operatorname{lt}\mathcal{H}$ is an involutive divisor of another one. The set $\mathcal{H}$ is a Pommaret basis of the ideal $\langle \mathcal{H} \rangle$ with respect to $\prec$, if and only if for every $h \in \mathcal{H}$ and every non-multiplicative index $\operatorname{cls} h < j \leq n$ the product $x_j h$ possesses an involutive standard representation with respect to $\mathcal{H}$.*

Theorem 4 ([25, Cor. 3.18, Prop. 3.19, Prop. 4.1]). *Let $\mathcal{H}$ be a Pommaret basis of the ideal $\mathcal{I} \lhd \mathcal{P}$ for an order $\prec$.*

(i) *If $D = \dim(\mathcal{P}/\mathcal{I})$, then $\{x_1, \dots, x_D\}$ is the unique maximal strongly independent set modulo $\mathcal{I}$ (and thus $\operatorname{lt}\mathcal{I} \cap \Bbbk[x_1, \dots, x_D] = \{0\}$).*

(ii) *The restriction of the canonical map $\mathcal{P} \to \mathcal{P}/\mathcal{I}$ to the subring $\Bbbk[x_1, \dots, x_D]$ defines a Noether normalisation.*
(iii) *If $d = \min_{h\in\mathcal{H}} \operatorname{cls} h$ is the minimal class of a generator in $\mathcal{H}$ and $\prec$ is the reverse lexicographic order, then $x_1, \dots, x_{d-1}$ is a maximal $\mathcal{P}/\mathcal{I}$-regular sequence and thus* $\operatorname{depth} \mathcal{P}/\mathcal{I} = d - 1$.

The involutive standard representations of the non-multiplicative products $x_j h$ appearing in Proposition 3 induce a basis of the first syzygy module. This observation leads to a stronger version of Hilbert's syzygy theorem.

Theorem 5 ([25, Thm. 6.1]). *Let $\mathcal{H}$ be a Pommaret basis of the ideal $\mathcal{I} \subseteq \mathcal{P}$. If we denote by $\beta_0^{(k)}$ the number of generators $h \in \mathcal{H}$ with* $\operatorname{cls} \operatorname{lt} h = k$ *and set* $d = \operatorname{cls} \mathcal{H}$*, then $\mathcal{I}$ possesses a finite free resolution*

$$0 \longrightarrow \mathcal{P}^{r_{n-d}} \longrightarrow \cdots \longrightarrow \mathcal{P}^{r_1} \longrightarrow \mathcal{P}^{r_0} \longrightarrow \mathcal{I} \longrightarrow 0 \tag{2}$$

of length $n - d$ where the ranks of the free modules are given by

$$r_i = \sum_{k=d}^{n-i} \binom{n-k}{i} \beta_0^{(k)} . \tag{3}$$

We denote by $\operatorname{reg} \mathcal{I}$ the *Castelnuovo-Mumford regularity* of $\mathcal{I}$ (considered as a graded module) and by $\operatorname{pd} \mathcal{I}$ its *projective dimension*. The *satiety* $\operatorname{sat} \mathcal{I}$ is the lowest degree from which on the ideal $\mathcal{I}$ and its *saturation* $\mathcal{I}^{\text{sat}} = \mathcal{I} : \mathfrak{m}^\infty$ coincide. These objects can be easily read off from a Pommaret basis for $\prec_{\text{revlex}}$.

Theorem 6 ([25, Thm. 8.11,Thm. 9.2, Prop. 10.1, Cor. 10.2]). *Let $\mathcal{H}$ be a Pommaret basis of the ideal $\mathcal{I} \lhd \mathcal{P}$ for the order $\prec_{\text{revlex}}$. We denote by $\mathcal{H}_1 = \{h \in \mathcal{H} \mid \operatorname{cls} h = 1\}$ the subset of generators of class 1.*

(i) $\operatorname{reg} \mathcal{I} = \deg \mathcal{H}$.
(ii) $\operatorname{pd} \mathcal{I} = n - \operatorname{cls} \mathcal{H}$.
(iii) *Let $\tilde{\mathcal{H}}_1 = \{h/x_1^{\deg_{x_1} \operatorname{lt} h} \mid h \in \mathcal{H}_1\}$. Then the set $\bar{\mathcal{H}} = (\mathcal{H} \setminus \mathcal{H}_1) \cup \tilde{\mathcal{H}}_1$ is a weak Pommaret basis*[3] *of the saturation $\mathcal{I}^{\text{sat}}$. Thus $\mathcal{I}^{\text{sat}} = \mathcal{I} : x_1^\infty$ and the ideal $\mathcal{I}$ is saturated, if and only if $\mathcal{H}_1 = \emptyset$.*
(iv) $\operatorname{sat} \mathcal{I} = \deg \mathcal{H}_1$.

Remark 7. Bayer et al. [3] call a non-vanishing Betti number β_{ij} *extremal*, if $\beta_{k\ell} = 0$ for all $k \geq i$ and $\ell > j$. In [25, Rem. 9.7] it is shown how the positions and the values of all extremal Betti numbers can be obtained from the Pommaret basis $\mathcal{H}$ for $\prec_{\text{revlex}}$. Let $h_{\gamma_1} \in \mathcal{H}$ be of minimal class among all generators of maximal degree in $\mathcal{H}$ and set $i_1 = n - \operatorname{cls} h_{\gamma_1}$ and $q_1 = \deg h_{\gamma_1}$. Then β_{i_1, q_1+i_1} is an extremal Betti number and its value is given by the number of generators of degree q_1 and class $n - i_1$. If $\operatorname{cls} h_{\gamma_1} = \operatorname{depth} \mathcal{I}$, it is the only one. Otherwise let h_{γ_2} be of minimal class among all generators of maximal degree in $\{h \in \mathcal{H} \mid \operatorname{cls} h < \operatorname{cls} h_{\gamma_1}\}$. Defining i_2, q_2 analogous to above, β_{i_2, q_2+i_2} is a further extremal Betti number and its value is given by the number of generators of degree q_2 and class $n - i_2$ and so on.

[3] Thus elimination of redundant generators yields a Pommaret basis [24, Prop. 5.7].

3 δ-Regularity and Quasi-Stable Ideals

Not every ideal $\mathcal{I} \lhd \mathcal{P}$ possesses a finite Pommaret basis. One can show that this is solely a problem of the chosen variables $\mathcal{X}$; after a suitable linear change of variables $\tilde{\mathcal{X}} = A\mathcal{X}$ with a non-singular matrix $A \in \Bbbk^{n\times n}$ the transformed ideal $\tilde{\mathcal{I}} \lhd \tilde{\mathcal{P}} = \Bbbk[\tilde{\mathcal{X}}]$ has a finite Pommaret basis (for the same term order which we consider as being defined on exponent vectors) [25, Sect. 2].

Definition 8. *The variables $\mathcal{X}$ are* δ-regular *for $\mathcal{I} \lhd \mathcal{P}$ and the order $\prec$, if $\mathcal{I}$ has a finite Pommaret basis for $\prec$.*

In [25, Sect. 2] a method is presented to detect effectively whether given variables are δ-singular and, if this is the case, to produce deterministically δ-regular variables. Furthermore, it is proven there that generic variables are δ-regular so that one can also employ probabilistic approaches although these are usually computationally disadvantageous.

It seems to be rather unknown that Serre implicitly presented already in 1964 a version of δ-regularity. In a letter appended to [14], he introduced the notion of a *quasi-regular* sequence and related it to Koszul homology.[4] Let $\mathcal{V}$ be a finite-dimensional vector space, $S\mathcal{V}$ the symmetric algebra over $\mathcal{V}$ and $\mathcal{M}$ a finitely generated graded $S\mathcal{V}$-module. A vector $v \in \mathcal{V}$ is called quasi-regular at degree q for $\mathcal{M}$, if $vm = 0$ for an $m \in \mathcal{M}$ implies $m \in \mathcal{M}_{<q}$. A sequence $(v_1, \dots, v_k)$ of vectors $v_i \in \mathcal{V}$ is quasi-regular at degree q for $\mathcal{M}$, if each v_i is quasi-regular at degree q for $\mathcal{M}/\langle v_1, \dots, v_{i-1}\rangle\mathcal{M}$.

Given a basis $\mathcal{X}$ of $\mathcal{V}$, we can identify $S\mathcal{V}$ with the polynomial ring $\mathcal{P} = \Bbbk[\mathcal{X}]$. Then it is shown in [15, Thm. 5.4] that the variables $\mathcal{X}$ are δ-regular for a homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$ and the reverse lexicographic order, if and only if they form a quasi-regular sequence for the module $\mathcal{P}/\mathcal{I}$ at degree $\operatorname{reg}\mathcal{I}$.

Our first result describes the degrees appearing in the Pommaret basis for the reverse lexicographic order in an intrinsic manner and generalises [29, Lemma 2.3] where only Borel-fixed monomial ideals for $\operatorname{char}\Bbbk = 0$ are considered.

Proposition 9. *Let the variables $\mathcal{X}$ be δ-regular for the ideal $\mathcal{I}$ and the reverse lexicographic order. If $\mathcal{H}$ denotes the corresponding Pommaret basis and $\mathcal{H}_i \subseteq \mathcal{H}$ the subset of generators of class i, then the integer*

$$q_i = \max\left\{\, q \in \mathbb{N}_0 \mid (\langle \mathcal{I}, x_1, \dots, x_{i-1}\rangle : x_i)_q \neq \langle \mathcal{I}, x_1, \dots, x_{i-1}\rangle_q \,\right\} \tag{4}$$

satisfies $q_i = \deg\mathcal{H}_i - 1$ (with the convention that $\deg\emptyset = \max\emptyset = -\infty$).

Proof. Set $\tilde{\mathcal{P}} = \Bbbk[x_i, \dots, x_n]$ and $\tilde{\mathcal{I}} = \mathcal{I}|_{x_1=\cdots=x_{i-1}=0} \lhd \tilde{\mathcal{P}}$. Then it is easy to see that $q_i = \max\{q \mid (\tilde{\mathcal{I}} : x_i)_q \neq \tilde{\mathcal{I}}_q\}$. Furthermore, the variables $x_i, \dots, x_n$ are δ-regular for $\tilde{\mathcal{I}}$ and the reverse lexicographic order—the Pommaret basis of $\tilde{\mathcal{I}}$ is given by $\tilde{\mathcal{H}} = \bigcup_{k\geq i} \tilde{\mathcal{H}}_k$ with $\tilde{\mathcal{H}}_k = \mathcal{H}_k|_{x_1=\cdots=x_{i-1}=0}$ (cf. [27, Lemma 3.1]).

[4] Quasi-regular sequences were rediscovered by Schenzel et al. [23] under the name *filter-regular* sequences and by Aramova and Herzog [1] as *almost regular* sequences.

Assume first that $\tilde{\mathcal{H}}_i = \emptyset$. In this case $x_i f \in \tilde{\mathcal{I}}$ implies $f \in \tilde{\mathcal{I}}$, as one can immediately see from the involutive standard representation of $x_i f$ with respect to $\tilde{\mathcal{H}}$ (all coefficients must lie in $\langle x_i \rangle$). If $\tilde{\mathcal{H}}_i \neq \emptyset$, then we choose a generator $\tilde{h}_{\max} \in \tilde{\mathcal{H}}_i$ of maximal degree. By the properties of $\prec_{\text{revlex}}$, we find $\tilde{h}_{\max} \in \langle x_i \rangle$ and hence may write $\tilde{h}_{\max} = x_i \tilde{g}$. By definition of a Pommaret basis, $\tilde{g} \notin \tilde{\mathcal{I}}$ and thus $q_i \geq \deg \tilde{g} = \deg \mathcal{H}_i - 1$.

Assume now that $q_i > \deg \mathcal{H}_i - 1$. Then there exists a polynomial $\tilde{f} \in \tilde{\mathcal{P}} \setminus \tilde{\mathcal{I}}$ with $\deg \tilde{f} = q_i$ and $x_i \tilde{f} \in \tilde{\mathcal{I}}$. Consider the involutive standard representation $x_i \tilde{f} = \sum_{\tilde{h} \in \tilde{\mathcal{H}}} P_{\tilde{h}} \tilde{h}$ with respect to $\tilde{\mathcal{H}}$. If $\operatorname{cls} \tilde{h} > i$, then we must have $P_{\tilde{h}} \in \langle x_i \rangle$. If $\operatorname{cls} \tilde{h} = i$, then by definition $P_{\tilde{h}} \in \Bbbk[x_i]$. Since $\deg (x_i \tilde{f}) > \deg \tilde{\mathcal{H}}_i$, any non-vanishing coefficient $P_{\tilde{h}}$ must be of positive degree in this case. Thus we can conclude that all non-vanishing coefficients $P_{\tilde{h}}$ lie in $\langle x_i \rangle$. But then we may divide the involutive standard representation of $x_i \tilde{f}$ by x_i and obtain an involutive standard representation of $\tilde{f}$ itself so that $\tilde{f} \in \tilde{\mathcal{I}}$ in contradiction to the assumptions we made. □

Consider the following invariants related to the local cohomology of $\mathcal{P}/\mathcal{I}$ (with respect to the maximal graded ideal $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$):

$$\begin{aligned} a_i(\mathcal{P}/\mathcal{I}) &= \max \{ q \mid H^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I})_q \neq 0 \}, & 0 \leq i \leq \dim(\mathcal{P}/\mathcal{I}), \\ \operatorname{reg}_t(\mathcal{P}/\mathcal{I}) &= \max \{ a_i(\mathcal{P}/\mathcal{I}) + i \mid 0 \leq i \leq t \}, & 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}), \\ a_t^*(\mathcal{P}/\mathcal{I}) &= \max \{ a_i(\mathcal{P}/\mathcal{I}) \mid 0 \leq i \leq t \}, & 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}). \end{aligned}$$

Trung [29, Thm. 2.4] related them for monomial Borel-fixed ideals and $\operatorname{char} \Bbbk = 0$ to the degrees of the minimal generators. We can now generalise this result to arbitrary homogeneous polynomial ideals.

Corollary 10. *Let the variables $\mathcal{X}$ be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the reverse lexicographic order. Denote again by $\mathcal{H}_i$ the subset of the Pommaret basis $\mathcal{H}$ of $\mathcal{I}$ consisting of the generators of class i and set $q_i = \deg \mathcal{H}_i - 1$. Then*

$$\begin{aligned} \operatorname{reg}_t(\mathcal{P}/\mathcal{I}) &= \max \{ q_1, q_2, \dots, q_{t+1} \}, & 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}), \\ a_t^*(\mathcal{P}/\mathcal{I}) &= \max \{ q_1, q_2 - 1, \dots, q_{t+1} - t \}, & 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}). \end{aligned}$$

Proof. This follows immediately from [29, Thm. 1.1] and Proposition 9. □

For monomial ideals it is in general useless to transform to δ-regular variables, as the transformed ideal is no longer monomial. Hence it is a special property of a monomial ideal to possess a finite Pommaret basis: such an ideal is called *quasi-stable*. The following theorem provides several purely algebraic characterisations of quasi-stability independent of Pommaret bases. It combines ideas and results from [4, Def. 1.5], [6, Prop. 3.2/3.6], [19, Prop. 2.2] and [25, Prop. 4.4].

Theorem 11. *Let $\mathcal{I} \lhd \mathcal{P}$ be a monomial ideal and $D = \dim(\mathcal{P}/\mathcal{I})$. Then the following statements are equivalent.*

(i) *$\mathcal{I}$ is quasi-stable.*

(ii) *The variable* x_1 *is not a zero divisor for* $\mathcal{P}/\mathcal{I}^{\mathrm{sat}}$ *and for all* $1 \leq k < D$ *the variable* x_{k+1} *is not a zero divisor for* $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_k\rangle^{\mathrm{sat}}$.
(iii) *We have* $\mathcal{I} : x_1^\infty \subseteq \mathcal{I} : x_2^\infty \subseteq \cdots \subseteq \mathcal{I} : x_D^\infty$ *and for all* $D < k \leq n$ *an exponent* $e_k \geq 1$ *exists such that* $x_k^{e_k} \in \mathcal{I}$.
(iv) *For all* $1 \leq k \leq n$ *the equality* $\mathcal{I} : x_k^\infty = \mathcal{I} : \langle x_k, \dots, x_n\rangle^\infty$ *holds.*
(v) *For every associated prime ideal* $\mathfrak{p} \in \mathrm{Ass}(\mathcal{P}/\mathcal{I})$ *an integer* $1 \leq j \leq n$ *exists such that* $\mathfrak{p} = \langle x_j, \dots, x_n\rangle$.
(vi) *If* $x^\mu \in \mathcal{I}$ *and* $\mu_i > 0$ *for some* $1 \leq i < n$, *then for each* $0 < r \leq \mu_i$ *and* $i < j \leq n$ *an integer* $s \geq 0$ *exists such that* $x_j^s x^\mu / x_i^r \in \mathcal{I}$.

The terminology "quasi-stable" stems from a result of Mall. The minimality assumption is essential here, as the simple example $\langle x^2, y^2\rangle \lhd \Bbbk[x, y]$ shows.

Lemma 12 ([21, Lemma 2.13], [26, Prop. 5.5.6]). *A monomial ideal is stable,*[5] *if and only if its minimal basis is a Pommaret basis.*

Thus already in the monomial case Pommaret bases are generally not minimal. The following result of Mall characterises those polynomial ideals for which the reduced Gröbner basis is simultaneously a Pommaret basis. We provide here a much simpler proof due to a more suitable definition of Pommaret bases.

Theorem 13 ([21, Thm. 2.15]). *The reduced Gröbner basis of the ideal* $\mathcal{I} \lhd \mathcal{P}$ *is simultaneously a Pommaret basis, if and only if* $\operatorname{lt} \mathcal{I}$ *is stable.*

Proof. By definition, the leading terms $\operatorname{lt} \mathcal{G}$ of a reduced Gröbner basis $\mathcal{G}$ form the minimal basis of $\operatorname{lt} \mathcal{I}$. The assertion is now a trivial corollary to Lemma 12 and the definition of a Pommaret basis. □

4 The Generic Initial Ideal

If we fix an order $\prec$ and perform a linear change of variables $\tilde{\mathcal{X}} = A\mathcal{X}$ with a non-singular matrix $A \in \Bbbk^{n\times n}$, then, according to Galligo's Theorem [10, 5], for almost all matrices A the transformed ideal $\tilde{\mathcal{I}} \lhd \tilde{\mathcal{P}} = \Bbbk[\tilde{\mathcal{X}}]$ has the same leading ideal, the *generic initial ideal* $\operatorname{gin} \mathcal{I}$ for the used order. By a further result of Galligo [11, 5], $\operatorname{gin} \mathcal{I}$ is Borel fixed, i. e. invariant under the natural action of the Borel group. For $\operatorname{char} \Bbbk = 0$, the Borel fixed ideals are precisely the stable ones; in positive characteristics the property of being Borel fixed has no longer such a simple combinatorial interpretation.

We will show in this section that many properties of the generic initial ideal $\operatorname{gin} \mathcal{I}$ also hold for the ordinary leading ideal $\operatorname{lt} \mathcal{I}$—provided the used variables are δ-regular. This observation has a number of consequences. While there does not exist an effective criterion for deciding whether a given leading ideal is actually $\operatorname{gin} \mathcal{I}$, δ-regularity is simply proven by the existence of a finite Pommaret

[5] In our "reverse" conventions, a monomial ideal $\mathcal{I}$ is called *stable*, if for every term $t \in \mathcal{I}$ and every index $k = \operatorname{cls} t < i \leq n$ also $x_i t / x_k \in \mathcal{I}$.

basis. Furthermore, $\operatorname{gin}\mathcal{I}$ can essentially be computed only by applying a random change of variables which has many disadvantages from a computational point of view. By contrast, [25, Sect. 2] presents a deterministic approach for the construction of δ-regular variables which in many case will lead to fairly sparse transformations.

From a theoretical point of view, the following trivial lemma which already appeared in [5, 10] implies that proving a statement about quasi-stable leading ideals immediately entails the analogous statement about $\operatorname{gin}\mathcal{I}$.

Lemma 14. *The generic initial ideal* $\operatorname{gin}\mathcal{I}$ *is quasi-stable.*

Proof. For $\operatorname{char}\Bbbk = 0$, the assertion is trivial, since then $\operatorname{gin}\mathcal{I}$ is even stable, as mentioned above. For arbitrary $\operatorname{char}\Bbbk$, it follows simply from the fact that generic variables[6] are δ-regular and thus yield a quasi-stable leading ideal. $\square$

The next corollary is a classical result [13, Cor. 1.33] for which we provide here a simple alternative proof. The subsequent theorem extends many well-known statements about $\operatorname{gin}\mathcal{I}$ to the leading ideal in δ-regular variables (for $\prec_{\text{revlex}}$); they are all trivial consequences of the properties of a Pommaret basis.

Corollary 15. *Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and* $\operatorname{char}\Bbbk = 0$*. Then all bigraded Betti numbers satisfy the inequality $\beta_{i,j}(\mathcal{P}/\mathcal{I}) \leq \beta_{i,j}(\mathcal{P}/\operatorname{gin}\mathcal{I})$.*

Proof. We choose variables $\mathcal{X}$ such that $\operatorname{lt}\mathcal{I} = \operatorname{gin}\mathcal{I}$. By Lemma 14, these variables are δ-regular for the given ideal $\mathcal{I}$. As $\operatorname{char}\Bbbk = 0$, the generic initial ideal is stable and hence the bigraded version of (3) applied to $\operatorname{lt}\mathcal{I}$ yields the bigraded Betti number $\beta_{i,j}(\mathcal{P}/\operatorname{gin}\mathcal{I})$. Now the claim follows immediately from analysing the resolution (2) degree by degree. $\square$

Theorem 16. *Let the variables $\mathcal{X}$ be δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the reverse lexicographic order $\prec_{\text{revlex}}$.*

(i) $\operatorname{pd}\mathcal{I} = \operatorname{pd}\operatorname{lt}\mathcal{I}$.
(ii) $\operatorname{sat}\mathcal{I} = \operatorname{sat}\operatorname{lt}\mathcal{I}$.
(iii) $\operatorname{reg}\mathcal{I} = \operatorname{reg}\operatorname{lt}\mathcal{I}$.
(iv) $\operatorname{reg}_t\mathcal{I} = \operatorname{reg}_t\operatorname{lt}\mathcal{I}$ *for all* $0 \leq t \leq \dim(\mathcal{P}/\mathcal{I})$.
(v) $a_t^*(\mathcal{I}) = a_t^*(\operatorname{lt}\mathcal{I})$ *for all* $0 \leq t \leq \dim(\mathcal{P}/\mathcal{I})$.
(vi) *The extremal Betti numbers of $\mathcal{I}$ and $\operatorname{lt}\mathcal{I}$ occur at the same positions and have the same values.*
(vii) $\operatorname{depth}\mathcal{I} = \operatorname{depth}\operatorname{lt}\mathcal{I}$.
(viii) *$\mathcal{P}/\mathcal{I}$ is Cohen-Macaulay, if and only if $\mathcal{P}/\operatorname{lt}\mathcal{I}$ is Cohen-Macaulay.*

Proof. The assertions (i-v) are trivial corollaries of Theorem 6 and Corollary 10, respectively, where it is shown for all considered quantities that they depend only on the leading terms of the Pommaret basis of $\mathcal{I}$. Assertion (vi) is a consequence of Remark 7 and the assertions (vii) and (viii) follow from Theorem 4. $\square$

[6] Recall that we assume throughout that $\Bbbk$ is an infinite field, although a sufficiently large finite field would also suffice [26, Rem. 4.3.19].

Remark 17. In view of Part (viii), one may wonder whether a similar statement holds for Gorenstein rings. In [27, Ex. 5.5] the ideal $\mathcal{I} = \langle z^2 - xy, yz, y^2, xz, x^2\rangle \lhd \Bbbk[x, y, z]$ is studied. The used coordinates are δ-regular for $\prec_{\text{revlex}}$, as a Pommaret basis is obtained by adding the generator x^2y. It follows from [27, Thm. 5.4] that $\mathcal{P}/\mathcal{I}$ is Gorenstein, but $\mathcal{P}/\operatorname{lt}\mathcal{I}$ not. A computation with CoCoA [8] gives here $\operatorname{gin}\mathcal{I} = \langle z^2, yz, y^2, xz, xy, x^3\rangle$ (assuming $\operatorname{char}\Bbbk = 0$) and again one may conclude with [27, Thm. 5.4] that $\mathcal{P}/\operatorname{gin}\mathcal{I}$ is not Gorenstein.

5 Componentwise Linear Ideals

Given an ideal $\mathcal{I} \lhd \mathcal{P}$, we denote by $\mathcal{I}_{\langle d\rangle} = \langle \mathcal{I}_d\rangle$ the ideal generated by the homogeneous component $\mathcal{I}_d$ of degree d. Herzog and Hibi [16] called $\mathcal{I}$ *componentwise linear*, if for every degree $d \geq 0$ the ideal $\mathcal{I}_{\langle d\rangle} = \langle \mathcal{I}_d\rangle$ has a linear resolution. For a connection with Pommaret bases, we need a refinement of δ-regularity.

Definition 18. *The variables $\mathcal{X}$ are* componentwise δ-regular *for the ideal $\mathcal{I}$ and the order $\prec$, if all ideals $\mathcal{I}_{\langle d\rangle}$ for $d \geq 0$ have finite Pommaret bases for $\prec$.*

It follows from the proof of [25, Thm. 9.12] that for the definition of componentwise δ-regularity it suffices to consider the finitely many degrees $d \leq \operatorname{reg}\mathcal{I}$. Thus trivial modificiations of any method for the construction of δ-regular variables allow to determine effectively componentwise δ-regular variables.

Theorem 19 ([25, Thm. 8.2, Thm. 9.12]). *Let the variables $\mathcal{X}$ be componentwise δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the reverse lexicographic order. If $\mathcal{I}$ is componentwise linear, then the free resolution (2) of $\mathcal{I}$ induced by the Pommaret basis $\mathcal{H}$ is minimal and the Betti numbers of $\mathcal{I}$ are given by (3). Conversely, if the resolution (2) is minimal, then the ideal $\mathcal{I}$ is componentwise linear.*

The following corollary generalises the analogous result for stable ideals to componentwise linear ideals (Aramova et al. [2, Thm. 1.2(a)] noted a version for $\operatorname{gin}\mathcal{I}$). It is an immediate consequence of the linear construction of the resolution (2) in [25, Thm. 6.1] and its minimality for componentwise linear ideals.

Corollary 20. *Let $\mathcal{I} \lhd \mathcal{P}$ be componentwise linear. If the Betti number $\beta_{i,j}$ does not vanish, then also all Betti numbers $\beta_{i',j}$ with $i' < i$ do not vanish.*

As a further corollary, we obtain a simple proof of an estimate given by Aramova et al. [2, Cor. 1.5] (based on [18, Thm. 2]).

Corollary 21. *Let $\mathcal{I} \lhd \mathcal{P}$ be a componentwise linear ideal with $\operatorname{pd}\mathcal{I} = p$. Then the Betti numbers satisfy $\beta_i \geq \binom{p+1}{i+1}$.*

Proof. Let $\mathcal{H}$ be the Pommaret basis of $\mathcal{I}$ for $\prec_{\text{revlex}}$ in componentwise δ-regular variables and $d = \operatorname{cls}\mathcal{H}$. By Theorem 19, (2) is the minimal resolution of $\mathcal{I}$ and hence (3) gives us β_i. By Theorem 4, $p = n - d$. We also note that δ-regularity implies that $\beta_0^{(k)} > 0$ for all $d \leq k \leq n$. Now we compute

$$\beta_i = \sum_{k=d}^{n-i} \binom{n-k}{i} \beta_0^{(k)} = \sum_{\ell=i}^{p} \binom{\ell}{i} \beta_0^{(n-\ell)} \geq \sum_{\ell=i}^{p} \binom{\ell}{i} = \binom{p+1}{i+1}$$

by a well-known identity for binomial coefficients. □

Example 22. The estimate in Corollary 21 is sharp. It is realised by any componentwise linear ideal whose Pommaret basis satisfies $\beta_0^{(i)} = 0$ for $i < d$ and $\beta_0^{(i)} = 1$ for $i \geq d$. As a simple monomial example consider the ideal $\mathcal{I}$ generated by the d terms $h_1 = x_n^{\alpha_n+1}$, $h_2 = x_n^{\alpha_n} x_{n-1}^{\alpha_{n-1}+1}$,…, $h_d = x_n^{\alpha_n} \cdots x_{d+1}^{\alpha_{d+1}} x_d^{\alpha_d+1}$ for arbitrary exponents $\alpha_i \geq 0$. One easily verifies that $\mathcal{H} = \{h_1, \dots, h_d\}$ is indeed simultaneously the Pommaret and the minimal basis of $\mathcal{I}$.

Recently, Nagel and Römer [22, Thm. 2.5] provided some criteria for componentwise linearity based on $\operatorname{gin} \mathcal{I}$ (see also [2, Thm 1.1] where the case $\operatorname{char} \Bbbk = 0$ is treated). We will now show that again $\operatorname{gin} \mathcal{I}$ may be replaced by $\operatorname{lt} \mathcal{I}$, if one uses componentwise δ-regular variables. Furthermore, our proof is considerably simpler than the one by Nagel and Römer.

Theorem 23. *Let the variables $\mathcal{X}$ be componentwise δ-regular for the ideal $\mathcal{I} \lhd \mathcal{P}$ and the reverse lexicographic order. Then the following statements are equivalent:*

(i) *$\mathcal{I}$ is componentwise linear.*
(ii) *$\operatorname{lt} \mathcal{I}$ is stable and all bigraded Betti numbers β_{ij} of $\mathcal{I}$ and $\operatorname{lt} \mathcal{I}$ coincide.*
(iii) *$\operatorname{lt} \mathcal{I}$ is stable and all total Betti numbers β_i of $\mathcal{I}$ and $\operatorname{lt} \mathcal{I}$ coincide.*
(iv) *$\operatorname{lt} \mathcal{I}$ is stable and $\beta_0(\mathcal{I}) = \beta_0(\operatorname{lt} \mathcal{I})$.*

Proof. The implication "(i) ⇒ (ii)" is a simple consequence of Theorem 19. Since our variables are componentwise δ-regular, the resolution (2) is minimal. This implies immediately that $\operatorname{lt} \mathcal{I}$ is stable. Applying Theorem 5 to the Pommaret basis $\operatorname{lt} \mathcal{H}$ of $\operatorname{lt} \mathcal{I}$ yields the minimal resolution of $\operatorname{lt} \mathcal{I}$. In both cases, the leading terms of all syzygies are determined by $\operatorname{lt} \mathcal{H}$ and hence the bigraded Betti numbers of $\mathcal{I}$ and $\operatorname{lt} \mathcal{I}$ coincide.

The implications "(ii) ⇒ (iii)" and "(iii) ⇒ (iv)" are trivial. Thus there only remains to prove "(iv) ⇒ (i)". Let $\mathcal{H}$ be the Pommaret basis of $\mathcal{I}$. Since $\operatorname{lt} \mathcal{I}$ is stable by assumption, $\operatorname{lt} \mathcal{H}$ is its minimal basis by Lemma 12 and $\beta_0(\operatorname{lt} \mathcal{I})$ equals the number of elements of $\mathcal{H}$. The assumption $\beta_0(\mathcal{I}) = \beta_0(\operatorname{lt} \mathcal{I})$ implies that $\mathcal{H}$ is a minimal generating system of $\mathcal{I}$. Hence, none of the syzygies obtained from the involutive standard representations of the non-multiplicative products yh with $h \in \mathcal{H}$ and $y \in \overline{\mathcal{X}}_P(h)$ may contain a non-vanishing constant coefficients. By [25, Lemma 8.1], this observation implies that the resolution (2) induced by $\mathcal{H}$ is minimal and hence the ideal $\mathcal{I}$ is componentwise linear by Theorem 19. □

6 Linear Quotients

Linear quotients were introduced by Herzog and Takayama [20] in the context of constructing iteratively a free resolution via mapping cones. As a special case, they considered monomial ideals where certain colon ideals defined by an ordered minimal basis are generated by variables. Their definition was generalised by Sharifan and Varabaro [28] to arbitrary ideals.

Definition 24. *Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $\mathcal{F} = \{f_1, \dots, f_r\}$ an ordered basis of it. Then $\mathcal{I}$ has* linear quotients *with respect to $\mathcal{F}$, if for each $1 < k \leq r$ the ideal $\langle f_1, \dots, f_{k-1}\rangle : f_k$ is generated by a subset $\mathcal{X}_k \subseteq \mathcal{X}$ of variables.*

We show first that in the monomial case this concept captures the essence of a Pommaret basis. For this purpose, we "invert" some notions introduced in [25]. We associate with a monomial Pommaret basis $\mathcal{H}$ a directed graph, its *P-graph*. Its vertices are the elements of $\mathcal{H}$. Given a non-multiplicative variable $x_j \in \overline{\mathcal{X}}_P(h)$ for a generator $h \in \mathcal{H}$, there exists a unique involutive divisor $\bar{h} \in \mathcal{H}$ of $x_j h$ and we include a directed edge from h to $\bar{h}$.

An ordering of the elements of $\mathcal{H}$ is called an *inverse P-ordering*, if $\alpha > \beta$ whenever the P-graph contains a path from h_α to h_β. It is straightforward to describe explicitly an inverse P-ordering: we set $\alpha > \beta$, if $\operatorname{cls} h_\alpha < \operatorname{cls} h_\beta$ or if $\operatorname{cls} h_\alpha = \operatorname{cls} h_\beta$ and $h_\alpha \prec_{\text{lex}} h_\beta$, i. e. we sort the generators h_α first by their class and then within each class lexicographically (according to our reverse conventions!). One easily verifies that this defines an inverse P-ordering.

Example 25. Consider the monomial ideal $\mathcal{I} \subset \Bbbk[x, y, z]$ generated by the six terms $h_1 = z^2$, $h_2 = yz$, $h_3 = y^2$, $h_4 = xz$, $h_5 = xy$ and $h_6 = x^2$. One easily verifies that these terms form a Pommaret basis of $\mathcal{I}$. The P-graph in (5) shows that the generators are already inversely P-ordered, namely according to the description above.

$$\begin{tikzcd}
 & h_5 \arrow[r] \arrow[dr] & h_3 \arrow[d] \\
h_6 \arrow[ur] \arrow[dr] & & h_2 \arrow[d] \\
 & h_4 \arrow[r] \arrow[ur] & h_1
\end{tikzcd} \tag{5}$$

Proposition 26. *Let $\mathcal{H} = \{h_1, \dots, h_r\}$ be an inversely P-ordered monomial Pommaret basis of the quasi-stable monomial ideal $\mathcal{I} \lhd \mathcal{P}$. Then the ideal $\mathcal{I}$ possesses linear quotients with respect to the basis $\mathcal{H}$ and*

$$\langle h_1, \dots, h_{k-1}\rangle : h_k = \langle \overline{\mathcal{X}}_P(h_k)\rangle \qquad k = 1, \dots r\ . \tag{6}$$

Conversely, assume that $\mathcal{H} = \{h_1, \dots, h_r\}$ is a monomial generating set of the monomial ideal $\mathcal{I} \lhd \mathcal{P}$ such that (6) is satisfied. Then $\mathcal{I}$ is quasi-stable and $\mathcal{H}$ its Pommaret basis.

Proof. Let $y \in \overline{\mathcal{X}}_P(h_k)$ be a non-multiplicative variable for $h_k \in \mathcal{H}$. Since $\mathcal{H}$ is a Pommaret basis, the product yh_k possesses an involutive divisor $h_i \in \mathcal{H}$ and, by definition, the P-graph of $\mathcal{H}$ contains an edge from k to i. Thus $i < k$ for an inverse P-ordering, which proves the inclusion "$\supseteq$".

The following argument shows that the inclusion cannot be strict. Consider a term $t \in \Bbbk[\mathcal{X}_P(h_k)]$ consisting entirely of multiplicative variables and assume that $th_k \in \langle h_1, \dots, h_{k-1}\rangle$, i. e. $th_k = s_1 h_{i_1}$ for some term $s_1 \in \Bbbk[\mathcal{X}]$ and some index $i_1 < k$. By definition of a Pommaret basis, s_1 must contain at least one non-multiplicative variable y_1 of h_{i_1}. But now we may rewrite $y_1 h_{i_1} = s_2 h_{i_2}$ with $i_2 < i_1$ and $s_2 \in \Bbbk[\mathcal{X}_P(h_{i_2})]$. Since this implies $\operatorname{cls} h_2 \geq \operatorname{cls} h_1$, we find $\mathcal{X}_P(h_{i_1}) \subseteq \mathcal{X}_P(h_{i_2})$. Hence after a finite number of iterations we arrive at a representation $th_k = sh_i$ where $s \in \Bbbk[\mathcal{X}_P(h_i)]$ which is, however, not possible for a Pommaret basis.

For the converse, we show by a finite induction over k that every non-multiplicative product yh_k with $y \in \overline{\mathcal{X}}_P(h_k)$ possesses an involutive divisor h_i with $i < k$ which implies our assertion by Proposition 3. For $k = 1$ nothing is to be shown, since (6) implies in this case that all variables are multiplicative for h_1 (and thus this generator is of the form $h_1 = x_n^\ell$ for some $\ell > 0$), and $k = 2$ is trivial. Assume that our claim was true for $h_1, h_2, \dots, h_{k-1}$. Because of (6), we may write $yh_k = t_1 h_{i_1}$ for some $i_1 < k$. If $t_1 \in \Bbbk[\mathcal{X}_P(h_{i_1})]$, we set $i = i_1$ and are done. Otherwise, t_1 contains a non-multiplicative variable $y_1 \in \overline{\mathcal{X}}_P(h_{i_1})$. By our induction assumption, $y_1 h_{i_1}$ has an involutive divisor h_{i_2} with $i_2 < i_1$ leading to an alternative representation $yh_k = t_2 h_{i_2}$. Now we iterate and find after finitely many steps an involutive divisor h_i of yh_k, since the sequence $i_1 > i_2 > \cdots$ is strictly decreasing and h_1 has no non-multiplicative variables. $\square$

Remark 27. As we are here exclusively concerned with Pommaret bases, we formulated and proved the above result only for this special case. However, Proposition 26 remains valid for any involutive basis with respect to a *continuous* involutive division L (and thus for all divisions of practical interest). The continuity of L is needed here for two reasons. Firstly, it guarantees the existence of an L-ordering, as for such divisions the L-graph is always acyclic [25, Lemma 5.5]. Secondly, the above argument that finitely many iterations lead to a representation $th_k = sh_i$ where s contains only multiplicative variables for h_i is specific for the Pommaret division and cannot be generalised. However, the very definition of continuity [12, Def. 4.9] ensures that for continuous divisions such a rewriting cannot be done infinitely often.

In general, we cannot expect that the second part of Proposition 26 remains true, when we consider arbitrary polynomial ideals. However, for the first part we find the following variation of [28, Thm. 2.3].

Proposition 28. *Let $\mathcal{H}$ be a Pommaret basis of the polynomial ideal $\mathcal{I} \lhd \mathcal{P}$ for the term order $\prec$ and $h' \in \mathcal{P}$ a polynomial with $\operatorname{lt} h' \notin \operatorname{lt} \mathcal{H}$. If $\mathcal{I} : h' = \langle \overline{\mathcal{X}}_P(h') \rangle$, then $\mathcal{H}' = \mathcal{H} \cup \{h'\}$ is a Pommaret basis of $\mathcal{J} = \mathcal{I} + \langle h' \rangle$. If furthermore $\mathcal{I}$ is componentwise linear, the variables $\mathcal{X}$ are componentwise δ-regular and $\mathcal{H}'$ is a minimal basis of $\mathcal{J}$, then $\mathcal{J}$ is componentwise linear, too.*

Proof. If $\mathcal{I} : h' = \langle \overline{\mathcal{X}}_P(h') \rangle$, then all products of h' with one of its non-multiplicative variables lie in $\mathcal{I}$ and hence possess an involutive standard representation with respect to $\mathcal{H}$. This immediately implies the first assertion.

In componentwise δ-regular variables all syzygies obtained from the involutive standard representations of products yh with $h \in \mathcal{H}$ and $y \in \overline{\mathcal{X}}_P(h)$ are free of constant coefficients, if $\mathcal{I}$ is componentwise linear. If $\mathcal{H}'$ is a minimal basis of $\mathcal{J}$, the same is true for all syzygies obtained from products yh' with $y \in \overline{\mathcal{X}}_P(h')$. Hence we can again conclude with [25, Lemma 8.1] that the resolution of $\mathcal{J}$ induced by $\mathcal{H}'$ is minimal and $\mathcal{J}$ componentwise linear by Theorem 19. $\square$

References

1. Aramova, A., Herzog, J.: Almost regular sequences and Betti numbers. Amer. J. Math. 122, 689–719 (2000)
2. Aramova, A., Herzog, J., Hibi, T.: Ideals with stable Betti numbers. Adv. Math. 152, 72–77 (2000)
3. Bayer, D., Charalambous, H., Popescu, S.: Extremal Betti numbers and applications to monomial ideals. J. Alg. 221, 497–512 (1999)
4. Bayer, D., Stillman, M.: A criterion for detecting m-regularity. Invent. Math. 87, 1–11 (1987)
5. Bayer, D., Stillman, M.: A theorem on refining division orders by the reverse lexicographic orders. Duke J. Math. 55, 321–328 (1987)
6. Bermejo, I., Gimenez, P.: Saturation and Castelnuovo-Mumford regularity. J. Alg. 303, 592–617 (2006)
7. Caviglia, G., Sbarra, E.: Characteristic-free bounds for the Castelnuovo-Mumford regularity. Compos. Math. 141, 1365–1373 (2005)
8. CoCoATeam: CoCoA: a system for doing Computations in Commutative Algebra. Available at `http://cocoa.dima.unige.it`
9. Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Mathematics 150, Springer-Verlag, New York (1995)
10. Galligo, A.: A propos du théorème de préparation de Weierstrass. In: Norguet, F. (ed.) Fonctions de Plusieurs Variables Complexes, pp. 543–579. Lecture Notes in Mathematics 409, Springer-Verlag, Berlin (1974)
11. Galligo, A.: Théorème de division et stabilité en géometrie analytique locale. Ann. Inst. Fourier 29(2), 107–184 (1979)
12. Gerdt, V., Blinkov, Y.: Involutive bases of polynomial ideals. Math. Comp. Simul. 45, 519–542 (1998)
13. Green, M.: Generic initial ideals. In: Elias, J., Giral, J., Miró-Roig, R., Zarzuela, S. (eds.) Six Lectures on Commutative Algebra, pp. 119–186. Progress in Mathematics 166, Birkhäuser, Basel (1998)
14. Guillemin, V., Sternberg, S.: An algebraic model of transitive differential geometry. Bull. Amer. Math. Soc. 70, 16–47 (1964), (With a letter of Serre as appendix)
15. Hausdorf, M., Sahbi, M., Seiler, W.: δ- and quasi-regularity for polynomial ideals. In: Calmet, J., Seiler, W., Tucker, R. (eds.) Global Integrability of Field Theories, pp. 179–200. Universitätsverlag Karlsruhe, Karlsruhe (2006)
16. Herzog, J., Hibi, T.: Componentwise linear ideals. Nagoya Math. J. 153, 141–153 (1999)
17. Herzog, J., Hibi, T.: Monomial Ideals. Graduate Texts in Mathematics 260, Springer-Verlag, London (2011)

18. Herzog, J., Kühl, M.: On the Bettinumbers of finite pure and linear resolutions. Comm. Alg. 12, 1627–1646 (1984)
19. Herzog, J., Popescu, D., Vladoiu, M.: On the Ext-modules of ideals of Borel type. In: Commutative Algebra, pp. 171–186. Contemp. Math. 331, Amer. Math. Soc., Providence (2003)
20. Herzog, J., Takayama, Y.: Resolutions by mapping cones. Homol. Homot. Appl. 4, 277–294 (2002)
21. Mall, D.: On the relation between Gröbner and Pommaret bases. Appl. Alg. Eng. Comm. Comp. 9, 117–123 (1998)
22. Nagel, U., Römer, T.: Criteria for componentwise linearity. Preprint arXiv:1108.3921 (2011)
23. Schenzel, P., Trung, N., Cuong, N.: Verallgemeinerte Cohen-Macaulay-Moduln. Math. Nachr. 85, 57–73 (1978)
24. Seiler, W.: A combinatorial approach to involution and δ-regularity I: Involutive bases in polynomial algebras of solvable type. Appl. Alg. Eng. Comm. Comp. 20, 207–259 (2009)
25. Seiler, W.: A combinatorial approach to involution and δ-regularity II: Structure analysis of polynomial modules with Pommaret bases. Appl. Alg. Eng. Comm. Comp. 20, 261–338 (2009)
26. Seiler, W.: Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra. Algorithms and Computation in Mathematics 24, Springer-Verlag, Berlin (2009)
27. Seiler, W.: Effective genericity, δ-regularity and strong Noether position. Comm. Alg. (to appear)
28. Sharifan, L., Varbaro, M.: Graded Betti numbers of ideals with linear quotients. Matematiche 63, 257–265 (2008)
29. Trung, N.: Gröbner bases, local cohomology and reduction number. Proc. Amer. Math. Soc. 129, 9–18 (2001)