Singular Initial Value Problems for some Quasi-Linear Second-Order Ordinary Differential Equations

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Abstract

We study existence, (non-)uniqueness and regularity of one- and two-sided solutions of singular initial value problems for second-order quasi-linear differential equations of the form $g(u)u'' = f(x, u, u')$ with initial conditions $u(y) = c_0$ and $u'(y) = c_1$ where c_0 is a simple zero of g and $f(y, c_0, c_1) = 0$. Our approach is based on the geometric theory of differential equations and in particular on singularity theory.

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1. Introduction

In the article [1], we studied with a geometric approach singular initial value problems for scalar quasi-linear equations of order $q \geq 1$

$$g(x, u, u', \dots, u^{(q-1)})u^{(q)} = f(x, u, u', \dots, u^{(q-1)}) \tag{1.1}$$

for a function $u(x)$, i. e. initial value problems where g vanishes for the prescribed initial data. The main emphasis was on the second-order case with the assumptions that $g = g(x)$ and that the initial data are prescribed at a simple zero of g, for which we provided an almost complete classification of all possible situations extending slightly analytic results by Liang [2].

The goal of this article is to provide a similar classification for the family of singular initial value problems consisting on the one hand of a scalar second-order equation of the form

$$g(u)u'' = f(x, u, u') \tag{1.2a}$$

where the two functions f, g are assumed to be smooth in all their arguments and everywhere defined and on the other hand of initial conditions

$$u(y) = c_0 , \qquad u'(y) = c_1 \tag{1.2b}$$

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where c_0 is a *simple* zero of g and $f(y, c_0, c_1) = 0$. From a classical analytical point of view, this problem is considerably harder than the one treated in [1], as we now face a truely nonlinear principal part whereas there the mainly considered equations were actually semilinear. Somewhat surprisingly, it turns out that for our geometric approach the two situations behave very similar. The main cases are essentially identical, but the special case $c_1 = 0$ leads to new phenomena.

We use a geometric approach to the theory of singularities of ordinary differential equations going back at least to Poincaré (see [3, 4] for an introduction into its basic ideas). We emphasise that the topic here are singularities of the differential equations themselves (in a sense that will be made precise below) and not singularities of solutions. Quasi-linear equations possess their own special geometry allowing us to analyse them at a lower order. We call their singularities impasse points and an initial value problem is singular, if the initial data correspond to an impasse point. We associate with any quasi-linear equation a vector field and any proper impasse point is a stationary point of this vector field. Questions of the existence, (non-)uniqueness and regularity of solutions can then be answered with methods from the theory of dynamical systems.

This article is structured as follows. The next section provides a brief introduction into the geometric theory of differential equations. In Section 3, we discuss the geometry of the problem studied here: we determine explicit expressions for the prolongations, identify all singularities and introduce the notion of a resonant initial value problem. In the following two sections, we consider the existence, (non)uniqueness and regularity of solutions to initial value problems without and with resonances, respectively. Some special cases are considered in Section 6. After some conclusions, we briefly discuss in an appendix how our geometric approach can be used for numerically determining solutions even in cases of non-unique solutions and/or solutions of finite regularity.

2. Geometric Theory of Differential Equations

Our approach is based on the geometric theory of differential equations where differential equations are represented by an intrinsic object: a surface in a suitable jet bundle. As it suffices for our purposes, we will restrict here to the case of scalar ordinary differential equations; see [5, 6] and references therein for an in-depth treatment of the general case.

Given a smooth function[1] $\varphi\colon \mathbb{R} \to \mathbb{R}$, the q-jet of it at a point $x \in \mathbb{R}$ is defined as the equivalence class $[\varphi]_x^{(q)}$ of all smooth functions possessing at x the same Taylor expansion up to order q as φ and can be identified with the corresponding Taylor polynomial. The qth order *jet bundle* $\mathcal{J}_q = \mathcal{J}_q(\mathbb{R}, \mathbb{R})$ consists of all such q-jets and defines an $(q+2)$-dimensional manifold. We identify $\mathcal{J}_0 = \mathbb{R}^{1+1}$ with the space of the independent variable x and the dependent variable u. By the theorem of Taylor, coordinates on $\mathcal{J}_q$ are given by $(x, u, u', \dots, u^{(q)}) = (x, \mathbf{u}_{(q)})$ where $u^{(q)}$ denotes the derivative of order q

[1] For notational simplicity, we will use a global notation, although our results are of a local nature. Strictly speaking, φ is only defined on some open subset of $\mathbb{R}$ which we suppress.

and $\mathbf{u}_{(q)}$ the collection of all derivatives from order 0 up to q. For orders $q > r$, there are canonical projection maps $\pi^q_r\colon \mathcal{J}_q \to \mathcal{J}_r$ between the corresponding jet bundles simply "forgetting" the higher derivatives. In addition, we have the projection $\pi^q\colon \mathcal{J}_q \to \mathbb{R}$ to the base space where everything except the expansion point x is "forgotten".

We define a *differential equation of order q* as a submanifold $\mathcal{R}_q \subseteq \mathcal{J}_q$ such that its projection $\pi^q(\mathcal{R}_q)$ lies dense in $\mathbb{R}$. In the standard geometric definition, it is required that $\mathcal{R}_q$ is a fibred submanifold and that the restriction of π^q to $\mathcal{R}_q$ defines a surjective submersion. However, these stronger assumptions exclude the appearance of any kind of singularity. Our relaxed condition still suffices to ensure that x may indeed be considered as an independent variable which is the main point. In practice, the set $\mathcal{R}_q$ is usually given as the zero set of some smooth function $\mathbf{F}\colon \mathcal{J}_q \to \mathbb{R}^k$ with $k = 1$ for a scalar equation and we will always assume that this is the case.

We identify a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ with its graph or, more precisely, we prefer to consider instead of the function φ the section $\sigma_\varphi\colon \mathbb{R} \to \mathcal{J}_0,\ x \mapsto (x, \varphi(x))$ whose image is the graph of φ. It induces naturally a section of any higher jet bundle $\mathcal{J}_q$ with $q \geq 1$, namely the *prolonged section*

$$j_q\sigma_\varphi\colon \mathbb{R} \to \mathcal{J}_q, \quad x \mapsto (x, \varphi(x), \varphi'(x), \dots, \varphi^{(q)}(x)) .$$

Obviously, $j_q\sigma_\varphi$ can be defined only at points x where φ is at least q times differentiable. A *(strong) solution* of a differential equation $\mathcal{R}_q \subseteq \mathcal{J}_q$ is a function φ such that the image of $j_q\sigma_\varphi$ lies completely in the manifold $\mathcal{R}_q$. This represents a natural geometric formulation of the usual notion of a solution.

A very important geometric structure on the jet bundle $\mathcal{J}_q$ for $q \geq 1$ is provided by the *contact distribution* $C^{(q)} \subset T\mathcal{J}_q$ which encodes geometrically the chain rule and thus the different roles played by the various jet variables. In our case of scalar ordinary differential equations, the contact distribution on the qth order jet bundle $\mathcal{J}_q$ is spanned by two vector fields:

$$C^{(q)}_{\text{trans}} = \partial_x + u'\partial_u + \cdots + u^{(q)}\partial_{u^{(q-1)}} , \qquad C^{(q)}_{\text{vert}} = \partial_{u^{(q)}} . \tag{2.1}$$

The field $C^{(q)}_{\text{vert}}$ lies in the vertical bundles of the canonical projections π^q_{q-1} and π^q. By contrast, the field $C^{(q)}_{\text{trans}}$ is transversal to the projection π^q. The tangent vectors to the image of any prolonged section $j_q\sigma_\varphi$ lie in $C^{(q)}$, i. e. they can be written as a linear combination $aC^{(q)}_{\text{trans}} + bC^{(q)}_{\text{vert}}$. The function φ is $q + 1$ times differentiable, if and only if the coefficient a does not vanish anywhere, and the derivative $\varphi^{(q+1)}$ is then given by the ratio b/a.

To avoid case distinctions, we introduce the vectors fields $C^{(0)}_{\text{trans}} = \partial_x$ and $C^{(0)}_{\text{vert}} = \partial_u$. By abuse of notation, we will use the vector fields $C^{(q)}_{\text{trans}}$ and $C^{(q)}_{\text{vert}}$ on any jet bundle $\mathcal{J}_r$ with $r \geq q$ without writing out the required pull-backs. As most of the time we are interpreting the contact vector fields as derivations, this should not lead to any confusion.

Given a qth order differential equation $\mathcal{R}_q \subseteq \mathcal{J}_q$ and a (smooth) point $\rho = (\bar{x}, \bar{\mathbf{u}}_{(q)}) \in \mathcal{R}_q$ on it, that part of the contact distribution which is tangential to $\mathcal{R}_q$ is the *Vessiot space* $\mathcal{V}_\rho[\mathcal{R}_q] = T_\rho\mathcal{R}_q \cap C^{(q)}|_\rho$ at ρ. Nonzero elements of the Vessiot space may be interpreted as infinitesimal solutions (or integral elements in the language of Cartan). The family of all Vessiot spaces is called the *Vessiot distribution* $\mathcal{V}[\mathcal{R}_q]$ of $\mathcal{R}_q$; in general, it is neither a smooth nor a regular distribution.

Computing the Vessiot space $\mathcal{V}_\rho[\mathcal{R}_q]$ at a point $\rho \in \mathcal{R}_q$ requires only linear algebra. Any vector $X \in \mathcal{V}_\rho[\mathcal{R}_q]$ must be a linear combination of the basic contact fields: $X = aC^{(q)}_{\text{trans}}|_\rho + bC^{(q)}_{\text{vert}}|_\rho$. Furthermore, X must be tangent to $\mathcal{R}_q$ and hence must satisfy $X(\mathbf{F})(\rho) = 0$ assuming that $\mathcal{R}_q$ is given as the zero set of a function $\mathbf{F}\colon \mathcal{J}_q \to \mathbb{R}^k$. Entering our ansatz yields then the following linear system for the two coefficients a and b:

$$C^{(q)}_{\text{trans}}(\mathbf{F})(\rho)a + C^{(q)}_{\text{vert}}(\mathbf{F})(\rho)b = 0\,. \tag{2.2}$$

We are mainly interested in scalar equations and their prolongations. Thus we assume that the original equation $\mathcal{R}_q$ is a hypersurface defined by a scalar function $f\colon \mathcal{J}_q \to \mathbb{R}$. Its *prolongations* are the submanifolds $\mathcal{R}_{q+k} \subset \mathcal{J}_{q+k}$ defined by the systems $f = D_x f = \cdots = D_x^k f = 0$ where D_x denotes total differentiation with respect to the independent variable x and $k \geq 0$. One can show that for determining the Vessiot spaces on any equation $\mathcal{R}_{q+k}$ it suffices to consider only the highest order equation $D_x^k f = 0$ in the defining system, i. e. we can replace in (2.2) q by $q+k$ and $\mathbf{F}$ by $D_x^k f$. Hence in our situation, we always obtain a single equation for the two unknown coefficients a, b. Hence, the Vessiot spaces $\mathcal{V}_\rho[\mathcal{R}_{q+k}]$ are one-dimensional at almost all points $\rho \in \mathcal{R}_{q+k}$ and, at least locally, the Vessiot distribution $\mathcal{V}[\mathcal{R}_{q+k}]$ can almost everywhere be spanned by a single vector field $X^{(q+k)}$ (solutions of (2.2) depend smoothly on the coordinates of the point ρ).

The properties of the Vessiot spaces decide whether or not a point is a singularity of a given differential equation $\mathcal{R}_q \subset \mathcal{J}_q$. At singularities, the standard existence and uniqueness theorems fail (see e. g. [3] for a discussion of mainly scalar first-order equations and [7] for a more general discussion). Slightly extending the terminology of [3], we call a point $\rho \in \mathcal{R}_q$ an *s-fold irregular singularity*, if $\dim \mathcal{V}_\rho[\mathcal{R}_q] = s+1$ with $s > 0$. In the case of a one-dimensional Vessiot space, we further distinguish whether or not it lies transversally to the canonical projection π^q. If $\mathcal{V}_\rho[\mathcal{R}_q]$ is vertical (i. e. if all solutions of (2.2) satisfy $a = 0$), then the point ρ is a *regular singularity*. Otherwise, ρ is a *regular point*.

The Vessiot distribution allows us to extend the notion of a solution. A *generalised solution* of the differential equation $\mathcal{R}_q \subset \mathcal{J}_q$ is a one-dimensional integral manifold $\mathcal{N} \subseteq \mathcal{R}_q$ of the Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$, i. e. $T_\rho\mathcal{N} \subseteq \mathcal{V}_\rho[\mathcal{R}_q]$ at every point $\rho_q \in \mathcal{N}$. A generalised solution is *proper*, if there does not exist a point $\bar{x} \in \mathbb{R}$ such $\mathcal{N} \subseteq (\pi^q)^{-1}(\bar{x})$. The projection $\pi^q_0(\mathcal{N}) \subset \mathcal{J}_0$ of a proper generalised solution is called a *geometric solution*.

If φ is a strong solution, then $\operatorname{im} \sigma_\varphi$ is a geometric solution stemming from the generalised solution $\operatorname{im} j_q\sigma_\varphi$. However, a geometric solution is not necessarily the graph of a function; it may not even be a smooth curve, as it arises from a projection. An improper generalised solution is of no interest for an existence theory, but sometimes it can be useful as a separatrix.

An *initial value problem* for a differential equation $\mathcal{R}_q \subset \mathcal{J}_q$ is described by providing a point $\rho_q \in \mathcal{R}_q$ and asks for (generalised) solutions $\mathcal{N}$ such that $\rho_q \in \overline{\mathcal{N}}$ where $\overline{\mathcal{N}}$ denotes the (Euclidean) closure of $\mathcal{N}$. We call such a solution *two-sided*, if $\rho_q \in \mathcal{N}$, and *one-sided* otherwise. An initial value problem is *singular*, if ρ_q is not a regular point of $\mathcal{R}_q$.

Let $\rho_q \in \mathcal{R}_q$ be a regular singularity. Then generically the corresponding initial value problem possesses two one-sided solutions either both starting in ρ_q or both ending in ρ_q. If a two-sided solution exists, then it is unique and its derivative of order $q+1$ blows up at $\bar{x}$ [1]. If ρ_q is an irregular singularity, then the initial value problem typically possess several (possibly even infinitely many) solutions. In many cases, one has intersecting two-sided solutions. But it may also happen that solutions cannot be continued through the singularity.

If we consider the fibre $\mathcal{F}_{q+k} = (\pi^{q+k}_q)^{-1}(\rho_q) \cap \mathcal{R}_{q+k}$ of all points lying above ρ_q in the prolongation $\mathcal{R}_{q+k}$ for some $k \in \mathbb{N}$, then it is not difficult to show that $\mathcal{F}_{q+k}$ consists entirely of regular points, if and only if ρ_q is a regular point. If ρ_q is a regular singularity, then $\mathcal{F}_{q+k}$ is empty implying that no two-sided solution through such a point can stem from a smooth function. If ρ_q is an irregular singularity, then all points in $\mathcal{F}_{q+k}$ are either regular or irregular singularities. A necessary condition for the existence of two-sided solution stemming from a smooth function is thus the existence of an infinite tower $(\rho_q, \rho_{q+1}, \rho_{q+2}, \dots)$ of irregular singularities $\rho_{q+k} \in \mathcal{R}_{q+k}$ lying above each other, i. e. satisfying $\pi^{q+k+1}_{q+k}(\rho_{q+k+1}) = \rho_{q+k}$.

As first observed in [8], quasilinear equations possess their own special theory leading to specific phenomena not present in fully nonlinear equations (see also [1] for a more detailed discussion). The key point is that for a qth order quasilinear equation $\mathcal{R}_q \subset \mathcal{J}_q$ the Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ – or more precisely the above mentioned vector field $X^{(q)}$ spanning it – can be projected down to the set $\pi^q_{q-1}(\mathcal{R}_q) \subseteq \mathcal{J}_{q-1}$. In general, the vector field $Y^{(q-1)}$ obtained by projecting $X^{(q)}$ is defined only on a proper subset of $\mathcal{J}_{q-1}$. However, under very modest assumptions it can be extended to the whole jet bundle $\mathcal{J}_{q-1}$.

A point $\tilde{\rho} \in \mathcal{J}_{q-1}$ is an *impasse point* for the quasi-linear differential equation $\mathcal{R}_q \subset \mathcal{J}_q$, if the vector field $Y^{(q-1)}$ is not transversal to the fibration π^{q-1} at $\tilde{\rho}$ (i. e. if its ∂_x-component vanishes at $\tilde{\rho}$). Otherwise it is a *regular point*. An impasse point is *proper*, if the field $Y^{(q-1)}$ vanishes at $\tilde{\rho}$. Otherwise it is *improper*. It is shown in [1] that if $\tilde{\rho}$ is a proper impasse point, then all points in the fibre $(\pi^q_{q-1})^{-1}(\tilde{\rho})$ lie in $\mathcal{R}_q$ and are singularities. But if $\tilde{\rho}$ is an improper impasse point, then no point exists in $\mathcal{R}_q$ that lies over it and hence these points lead to phenomena specific to quasilinear equations.

Following [1], we introduce a weaker notion of solutions. A *weak generalised solution* of the quasi-linear differential equation $\mathcal{R}_q$ is a one-dimensional invariant manifold $\mathcal{N} \subset \mathcal{J}_{q-1}$ of the vector field $Y^{(q-1)}$, i. e. we have at every point $\tilde{\rho} \in \mathcal{N}$ that $Y_{\tilde{\rho}} \in T_{\tilde{\rho}}\mathcal{N}$. A weak generalised solution $\mathcal{N}$ is *proper*, if in addition $T\mathcal{N} \subseteq C^{(q-1)}$ and there does not exist a point $x \in \mathbb{R}$ such that $\mathcal{N} \subseteq (\pi^{q-1})^{-1}(x)$. The projection $\pi^{q-1}_0(\mathcal{N}) \subset \mathcal{J}_0$ of a proper weak generalised solution is a *weak geometric solution*.

If φ is a strong solution of $\mathcal{R}_q$ as defined above, then $\operatorname{im}\sigma_\varphi$ is a weak geometric solution and $\operatorname{im} j_{q-1}\sigma_\varphi$ is weak generalised solution. If the differential equation $\mathcal{R}_q$ does not possess any impasse points, then all weak generalised or geometric solutions are of this form. Otherwise, further weak generalised solutions may exist which cannot be interpreted as prolonged graphs of functions. The qualification "weak" refers to the fact that even if a weak generalised solution is the prolonged graph of a function, it is only guaranteed that this function is $q-1$ times differentiable, although we are dealing with a qth order differential equation.

3. Geometry of our Initial Value Problem

We first determine the prolongations of the given equation (1.2a) to arbitrary order. We introduce $F_2(x, \mathbf{u}_{(2)}) = g(u)u'' - f(x, u, u')$ as its implicit form defining the hypersurface $\mathcal{R}_2 \subset \mathcal{J}_2$. The first prolongation is then the surface $\mathcal{R}_3 \subset \mathcal{J}_3$ defined by the simultaneous vanishing of F_2 and of

$$F_3(x, \mathbf{u}_{(3)}) = g(u)u''' + \big[g'(u)u' - f_{u'}(x, \mathbf{u}_{(1)})\big]u'' - h_3(x, \mathbf{u}_{(1)}) \tag{3.1}$$

with $h_3(x, \mathbf{u}_{(1)}) = C^{(1)}_{\text{trans}}(f)$. An iteration yields that the differential equation $\mathcal{R}_q \subset \mathcal{J}_q$ is defined by the simultaneous vanishing of F_k for $i = 2, \dots, q$ where we define for any order $k > 2$

$$\begin{aligned} F_k(x, \mathbf{u}_{(k)}) = g(u)u^{(k)} &+ \\ &\big[(k-2)g'(u)u' - f_{u'}(x, \mathbf{u}_{(1)})\big]u^{(k-1)} - h_k(x, \mathbf{u}_{(k-2)}) \end{aligned} \tag{3.2}$$

with a remainder term recursively given by

$$\begin{aligned} h_k(x, \mathbf{u}_{(k-2)}) = C^{(k-2)}_{\text{trans}}\Big(&h_{k-1}(x, \mathbf{u}_{(k-3)}) - \\ &\big[(k-3)g'(u)u' - f_{u'}(x, \mathbf{u}_{(1)})\big]u^{(k-2)}\Big) . \end{aligned} \tag{3.3}$$

For notational simplicity, we introduce for any $k > 2$ the function $\tilde{h}_k(x, \mathbf{u}_{(k-1)})$ by requiring that $F_k = g(u)u^{(k)} - \tilde{h}_k(x, \mathbf{u}_{(k-1)})$, i. e.

$$\tilde{h}_k(x, \mathbf{u}_{(k-1)}) = h_k(x, \mathbf{u}_{(k-2)}) - \big[(k-2)g'(u)u' - f_{u'}(x, \mathbf{u}_{(1)})\big]u^{(k-1)} . \tag{3.4}$$

For $k = 2$, we set $\tilde{h}_2(x, u, u') = h_2(x, u, u') = f(x, u, u')$.

Lemma 3.1. *A point $\rho_q = (\bar{x}, \bar{\mathbf{u}}_{(q)}) \in \mathcal{R}_q$ (for any $q \geq 2$) is a singularity, if and only if $g(\bar{u}) = 0$. It is an irregular singularity, if and only if in addition $\tilde{h}_{q+1}(\bar{x}, \bar{\mathbf{u}}_{(q)}) = 0$.*

Proof. The linear system (2.2) defining the Vessiot distribution consists for $\mathcal{R}_q \subset \mathcal{J}_q$ with $q \geq 2$ of the single equation

$$-\tilde{h}_{q+1}(x, \mathbf{u}_{(q)})a + g(u)b = 0 . \tag{3.5}$$

At an irregular singularity, both coefficients must vanish. At a regular one, only the coefficient of b vanishes (entailing that $a = 0$). □

Off the irregular singularities, the Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ is smooth and one-dimensional. It follows from (3.5) that it is there spanned by the vector field

$$X^{(q)} = g(u)C^{(q)}_{\text{trans}} + \tilde{h}_{q+1}(x, \mathbf{u}_{(q)})C^{(q)}_{\text{vert}} . \tag{3.6}$$

Note that $X^{(q)}$ vanishes at any irregular singularity. Furthermore, it is projectable to $\mathcal{R}_{q-1} \subset \mathcal{J}_{q-1}$ (where we identify $\mathcal{R}_1 = \mathcal{J}_1$) defining there the vector field

$$\begin{aligned} Y^{(q-1)} &= g(u)C^{(q-1)}_{\text{trans}} + \tilde{h}_q(x, \mathbf{u}_{(q-1)})C^{(q-1)}_{\text{vert}} \\ &= g(u)\partial_x + \sum_{k=0}^{q-2} g(u)u^{(k+1)}\partial_{u^{(k)}} + \tilde{h}_q(x, \mathbf{u}_{(q-1)})\partial_{u^{(q-1)}} \ . \end{aligned} \tag{3.7}$$

Obviously, we find $Y^{(q-1)} = X^{(q-1)}$ and the proper impasse points of $\mathcal{R}_q$ are exactly the irregular singularities of $\mathcal{R}_{q-1}$.

It is important to note that for $q \geq 2$ the vector field $X^{(q)} = Y^{(q)}$ is uniquely defined only on the three-dimensional submanifold $\mathcal{R}_q \subset \mathcal{J}_q$. Of course, the coordinate expressions given in (3.6) and (3.7), respectively, can be continued everywhere in $\mathcal{J}_q$, but outside of $\mathcal{R}_q$ this continuation is not unique. For example, the vector field $X^{(q)} = Y^{(q)}$ contains coefficients of the form $g(u)u^{(k)}$ with $2 \leq k < q$ and – according to (3.2) – we may replace these coefficients by $\tilde{h}_k(x, \bar{\mathbf{u}}_{(q-1)})$ without modifying the field on $\mathcal{R}_q$. This yields the vector field

$$\widehat{Y}^{(q)} = g(u)\partial_x + g(u)u'\partial_u + \sum_{k=1}^{q} \tilde{h}_{k+1}(x, \mathbf{u}_{(k)})\partial_{u^{(k)}} \ , \tag{3.8}$$

which will be useful later.

From now on, we assume that we are given a point $\rho_1 = (y, c_0, c_1) \in \mathcal{J}_1$ such that c_0 is a simple zero of g and $f(y, c_0, c_1) = 0$. We introduce

$$\delta = g'(c_0) \ , \qquad \gamma = f_{u'}(y, c_0, c_1) \ . \tag{3.9}$$

By the made assumptions, $\delta \neq 0$. Without loss of generality, we can set the sign of δ as we wish by simply multiplying (1.2a) by -1, if necessary. If $c_1 \neq 0$, then we choose the sign such that $\delta c_1 > 0$. It will turn out that the properties of the singular initial value problem (1.2) are to a large extent determined by the two values δc_1 and γ.

Definition 3.2. The point ρ_1 and the initial value problem defined by it, respectively, are *resonant at order q* for some $q \in \mathbb{N}$, if $q\delta c_1 = \gamma$. In this case, given some point $\rho_q \in (\pi^q_1)^{-1}(\rho_1) \subset \mathcal{J}_q$ projecting on ρ_1, we define the *resonance parameter* $A_q = h_{q+2}(\rho_q)$ and the resonance is *critical* for $A_q \neq 0$ and *smooth* for $A_q = 0$.

In this definition, we are speaking of "the" resonance parameter A_q, although it obviously depends on the point ρ_q. However, it will turn out later that there exists a natural unique choice for this point and that we will exclusively work with this particular point.

Proposition 3.3. *Let $\rho_q \in \mathcal{R}_q$ (for some $q \geq 2$) be an irregular singularity and $\mathcal{F}_{q+1} = (\pi^{q+1}_q)^{-1}(\rho_q) \subset \mathcal{J}_{q+1}$ the fibre above it. Then $\mathcal{F}_{q+1} \subset \mathcal{R}_{q+1}$ and all points in it are singularities of $\mathcal{R}_{q+1}$. If $\rho_1 = \pi^q_1(\rho_q)$ is not resonant at order q, then $\mathcal{F}_{q+1}$ contains exactly one irregular singularity. If we have a critical (smooth) resonance at order q, then all points in $\mathcal{F}_{q+1}$ are regular (irregular) singular.*

Proof. The first assertion follows immediately from the quasilinearity of (3.2) and Lemma 3.1. The remaining statements are a simple consequence of the definition of the resonance parameter A_q and again of Lemma 3.1. □

A proper impasse point $\rho_q = (y, c_0, c_1, \dots, c_q) \in \mathcal{R}_q$ of the prolonged equation $\mathcal{R}_{q+1}$ is a stationary point of the vector field $Y^{(q)}$. Its Jacobian at ρ_q is the $(q+2)\times(q+2)$-matrix

$$J^{(q)} = \begin{pmatrix} 0 & \delta & 0 & \cdots & 0 & 0 \\ 0 & \delta c_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \delta c_q & 0 & \cdots & 0 & 0 \\ a^{(q)} & b_0^{(q)} & b_1^{(q)} & \cdots & b_{q-1}^{(q)} & \gamma - (q-1)\delta c_1 \end{pmatrix} \tag{3.10}$$

where $a^{(q)} = \partial\tilde{h}_{q+1}/\partial x\,(\rho_q)$ and $b_i^{(q)} = \partial\tilde{h}_{q+1}/\partial u^{(i)}\,(\rho_q)$. As the Jacobian is triangular except for its first row, one readily determines its spectrum to consists of 0 ($q-1$ times), δc_1, and $\gamma - (q-1)\delta c_1$.

Recall from above that $Y^{(q)}$ is uniquely defined only on the three-dimensional manifold $\mathcal{R}_q$. Hence, only three of these eigenvalues are relevant, namely those possessing (generalised) eigenvectors tangent to $\mathcal{R}_q$. A direct search for them would be rather cumbersome requiring either to construct a parametrisation of $\mathcal{R}_q$ or to check a number of tangency conditions. Hence, we use a little trick by looking instead at the alternative vector field $\widehat{Y}^{(q)}$ introduced in (3.8). Its Jacobian at ρ_q is the matrix

$$\widehat{J}^{(q)} = \begin{pmatrix} 0 & \delta & 0 & & & & 0 \\ 0 & \delta c_1 & 0 & & & & \\ a^{(1)} & b_0^{(1)} & \gamma & & & & \\ & & b_1^{(2)} & \gamma - \delta c_1 & & & \\ \vdots & & & b_2^{(3)} & \gamma - 2\delta c_1 & & \\ & & & & \ddots & \ddots & \\ a^{(q)} & b_0^{(q)} & & \cdots & & b_{q-1}^{(q)} & \gamma - (q-1)\delta c_1 \end{pmatrix}. \tag{3.11}$$

Again, the Jacobian is triangular except for its first row and its spectrum consists of 0, δc_1, γ, $\gamma - \delta c_1$,..., $\gamma - (q-1)\delta c_1$. The three relevant eigenvalues are now easily identified, as they must be simultanously eigenvalues of $J^{(q)}$ and $\widehat{J}^{(q)}$. If we assume that $\gamma \neq 0$ and that no resonance occurs at some order less than q, then this is the case for 0, δc_1 and $\gamma - (q-1)\delta c_1$.

Generically, all three eigenvalues are distinct. It is easy to see from the matrix $J^{(q)}$ that in this case the eigenspace for the eigenvalue $\gamma - (q-1)\delta c_1$ is spanned by the vector $(0, \dots, 0, 1)^T$ and the eigenspace for δc_1 by the vector (we are assuming here that we do not have a resonance at order q)

$$\left(1, c_1, \dots, c_q, -\frac{a^{(q)} + b_0^{(q)} c_1 + \cdots + b_{q-1}^{(q)} c_q}{\gamma - q\delta c_1}\right)^T. \tag{3.12}$$

Looking at the definition of the values $a^{(q)}$ and $b_i^{(q)}$, one sees that the numerator of the last entry may also be written as $C_{\text{trans}}^{(q)}(\tilde{h}_{q+1})(\rho_q) = h_{q+2}(\rho_q)$. This eigenvector is transversal to the projection π^q. Just from the matrices, it is not so easy to find the relevant eigenvector for the eigenvalue 0. But it is obvious because of the first row that its second entry (the ∂_u-component) vanishes.

Special cases arise if either $c_1 = 0$ or if the initial point ρ_1 is resonant at order $q-1$ or q. In the first two cases, 0 is a double eigenvalue; in the last case δc_1 is a double eigenvalue. Finally, we have the case $c_1 = \gamma = 0$ leading to a triple eigenvalue 0. We will treat the special cases $\gamma = 0$ or $c_1 = 0$ in Sect. 6.

If there is a resonance at order q, we have to look closer at the eigenspace of the double eigenvalue δc_1. It is trivial that the vector $(0,\dots,0,1)^T$ is contained in it and generically it spans the complete eigenspace. Now consider the vector $(1, c_1, \dots, c_q, 0)^T$. Multiplying it by $J^{(q)} - \delta c_1 E$ where E denotes the identity matrix yields the vector $(0,\dots,0,a^{(q)} + b_0^{(q)} c_1 + \dots + b_{q-1}^{(q)} c_q)^T$. We noted already above that the last entry equals $h_{q+2}(\rho_q)$ which according to Def. 3.2 is also the resonance parameter A_q. Thus in the case of a critical resonance ($A_q \neq 0$), the eigenvalue δc_1 is not semisimple and we have a Jordan block of length 2 for it. If the resonance is smooth ($A_q = 0$), then the eigenspace is two-dimensional.

Example 3.4. Throughout this article, we will consider some instances of the following two-parameter family of equations

$$uu'' = \frac{1}{2}(u' - x)^2 - \frac{1}{2}(r + su)^2 \tag{3.13}$$

with initial conditions $\rho_1 = (y, c_0, c_1)$. Here $g(u) = u$ entailing $c_0 = 0$. For such a choice of g, our singular initial value problem may be interpreted as an analysis of zeros of solutions of the given differential equation and the special case $c_1 = 0$ corresponds to a multiple zero. Entering $u = 0$ and $x = y$ into (3.13) yields two admissable values for c_1 in the initial conditions (1.2b), namely $c_1 = y \pm r$. One easily checks that here $\delta = 1$ and $\gamma = c_1 - y = \pm r$. For finding an irregular singularity on $\mathcal{R}_2$ above the initial point $\rho_1 = (y, 0, c_1)$, we must solve the equation $\tilde{h}_3(y, 0, c_1, c_2) = 0$ for c_2 according to Lemma 3.1. We find $\tilde{h}_3 = -xu'' - \sigma^2 uu' - (1 + rs)u' + x$ by differentiating (3.13) and hence the unique solution $c_2 = 1 - (1 + rs)c_1/y$ (for $y \neq 0$).

For $y = 0$, the point $\rho_1 = (0, 0, r)$ is resonant at order $k = 1$. Since for our family $h_3(x, u, u') = x - u' - (r + su)su'$, the resonance parameter is uniquely given by $A_1 = h_3(0, 0, r) = -r(1 + rs)$. The Jacobian at ρ_1 is

$$J^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & r & 0 \\ -r & -rs & r \end{pmatrix} \tag{3.14}$$

with a simple eigenvalue 0 and a double eigenvalue r. For the choice $r = s = 1$ the resonance is critical and indeed in this case the Jacobian $J^{(1)}$ is not diagonalisable. The eigenspace for 0 is spanned by $(1, 0, 1)^T$ and the one for r by $(0, 0, 1)^T$. The choice $r = -s = 1$ leads to a smooth resonance. We find the same kernel, but an additional, linearly independent eigenvector for r is given by $(1, 1, 0)^T$. Hence the Jacobian $J^{(1)}$ is now diagonalisable.

If we take $r = 1$, then the choices $y = (1 - n)/n$ for an arbitrary natural number $n \geq 1$ and $c_1 = y + r$ leads to a resonance at order $k = n$, i. e. we can achieve resonances at arbitrary high orders. Consider the point $\rho_1 = (-1/2, 0, 1/2)$ obtained for $r = 1$ and $n = 2$, thus resonant at order 2. The unique irregular singularity above ρ_1 is the point $\rho_2 = (-1/2, 0, 1/2, s + 2)$. Since $h_4 = 1 - (s^2 u + rs + 2)u'' - (su')^2$, the resonance parameter is $A_2 = -(s+2)^2 - s^2/4 + 1$. Thus the resonance is smooth only for $s = -2$ or $s = -6/5$.

4. Initial Value Problems Without a Resonance

Let again $\rho_1 = (y, c_0, c_1) \in \mathcal{J}_1$ be the point corresponding to the singular initial conditions (1.2b). We study now the case that ρ_1 is *not* resonant at any order $k \geq 1$ and prove first that under this assumption the initial value problem (1.2) possesses a unique formal power series solution.

Proposition 4.1. *Assume that the point ρ_1 corresponding to the singular initial conditions* (1.2b) *is not resonant at any order $k \geq 1$. Then there exists a uniquely determined tower of points $\rho_q = (y, c_0, c_1, \dots, c_q) \in \mathcal{R}_q$ for $q = 2, 3, \dots$ such that ρ_q is the sole irregular singularity of $\mathcal{R}_q$ in the fibre above $\rho_{q-1} \in \mathcal{R}_{q-1}$ (i. e. $\pi^q_{q-1}(\rho_q) = \rho_{q-1}$). For any $q > 1$, the q-jet at y of any C^q-solution of the initial value problem* (1.2) *must be ρ_q.*

Proof. By the same arguments as in the proof of Proposition 3.3, the fibre $\mathcal{F}_2 = (\pi^2_1)^{-1}(\rho_1)$ is completely contained in $\mathcal{R}_2$ and contains a unique irregular singularity $\rho_2 = (y, c_0, c_1, c_2)$, as ρ_1 is assumed to be not resonant. Through any other point in $\mathcal{F}_2$, there exists a unique generalised solution which is $\mathcal{F}_2$ itself. Hence the prolonged graph of any solution of (1.2) of regularity at least C^2 must go through ρ_2.

Applying Proposition 3.3, we can repeat these arguments order by order and construct thus in a unique manner the asserted infinite tower. Furthermore, for any solution of regularity at least C^q, its prolonged graph must go through ρ_q. The infinite sequence $c_0, c_1, c_2, \dots$ can be considered as the coefficients of a unique formal power series solution of (1.2) around y. □

We now show that this formal power series belongs to a unique two-sided smooth solution of the initial value problem (1.2). Note, however, that the uniqueness holds only within the space of smooth functions defined in a neighbourhood of y. As we will show afterwards, depending on the relative signs of δc_1 and γ further solutions – either one-sided or of finite regularity – may exist.

Theorem 4.2. *In the absence of resonances and for $c_1 \neq 0$, the initial value problem* (1.2) *possesses a unique two-sided smooth solution φ_∞.*

Proof. We choose an order $q \geq 1$ so large that $\gamma - (q-1)\delta c_1 < 0$. Since we assume that $\delta c_1 > 0$, this is always possible. Then, at the – by Proposition 4.1 uniquely determined – stationary point $\rho_q \in \mathcal{R}_q$ of the vector field $Y^{(q)}$, all three eigenvalues of the Jacobian $J^{(q)}$ are of different signs by the made assumptions. According to the classical Centre Manifold Theorem (see e. g. [9, Thm. 3.2.1]), $Y^{(q)}$ possesses thus at ρ_q three one-dimensional invariant manifolds, a stable, an unstable and a centre manifold, each of which defines a (for $q = 1$ weak) generalised solution.

The conditions for a stationary point, $g(u) = 0$ and $\tilde{h}_{q+1}(x, u, u', \dots, u^{(q)}) = 0$, define a whole curve on the three-dimensional manifold $\mathcal{R}_q$ which must be part of any centre manifold [10, Cor. 3.3]. As in our case, any centre manifold is one-dimensional, we have a unique centre manifold given by this curve. Again it is cumbersome to derive an explicit expression for the tangent vector to this curve. But from above we know that any vector in the kernel of $J^{(q)}$ has a vanishing second entry. This implies, by our assumption $c_1 \neq 0$, that the eigenvector cannot be written as a linear combination of contact vectors and that hence this curve cannot be the prolonged graph of a function and does not define a proper weak generalised solution.

The stable manifold is also easy to find. As it must be tangent to the eigenvector $(0, \dots, 0, 1)^T$, it lies vertical at least in the point ρ_q. One easily verifies that the whole fibre $\mathcal{F}_q = (\pi^q_{q-1})^{-1}(\rho_{q-1})$ remains invariant under $Y^{(q)}$ and hence $\mathcal{F}_q$ can be identified as the stable manifold. As it lies in the fibre $(\pi^q)^{-1}(y)$, it does not define a proper weak generalised solution.

The unstable manifold is tangent to the eigenvector to δc_1 given by (3.12). Under the made assumptions, this vector is transversal to the projection π^q and a multiple of it can be written as $(\gamma - q\delta c_1)C^{(q)}_{\text{trans}} - (a^{(q)} + b^{(q)}_0 c_1 + \cdots + b^{(q)}_{q-2} c_{q-1})C^{(q)}_{\text{vert}}$. As the unstable manifold is invariant unter the vector field $Y^{(q)}$ lying in the contact distribution $C^{(q)}$, it follows that it is a proper weak generalised solution which is in fact the prolonged graph of a function (in some neighbourhood of ρ_q).

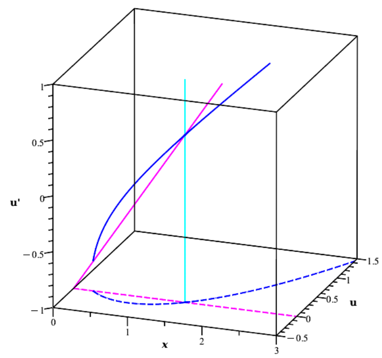
We now proceed to order $q+1, q+2, \dots$ and always find the same situation: a stationary point with three one-dimensional invariant manifolds only one of which defines a proper generalised solution. These proper generalised solutions must project on each other, as at each order only one proper generalised solution exists and the projection of a proper generalised solution is again a proper generalised solution. Thus they all correspond to different prolonged graphs of the same smooth function φ_∞ defining a unique two-sided solution of the initial value problem (1.2). □

It is now easy to characterise when φ_∞ is the sole solution of the considered initial value problem. It is simply a matter of the signs of γ and δc_1

Corollary 4.3. *Assume that there are no resonances, that $\gamma c_1 \neq 0$ and that γ and δc_1 have different signs. Then the initial value problem* (1.2) *possesses no other weak geometric solutions (and thus in particular no other strong solutions) besides φ_∞, not even one-sided ones or ones of finite regularity.*

Proof. By the made assumption on the signs of γ and δc_1, we can choose in the proof of Theorem 4.2 $q = 1$. The considerations there then imply that $\operatorname{im} j_1\sigma_{\varphi_\infty}$ is the sole weak generalised solution of (1.2). □

Example 4.4. In the family from Ex. 3.4, we consider the instance $r = 1$ and $s = 0$. Furthermore, we choose in the initial condition (1.2b) $y = 3/2$. Then the admissible choice $c_1 = y - r = 1/2$ leads to $\gamma = -1$ and hence to an initial value problem covered by Cor. 4.3, i. e. with a unique smooth two-sided solution. A numerically computed plot of this solution and its first derivative is shown as blue curve in the left part of Fig. 1 (how such numerical computations can be performed is briefly discussed in Appendix Appendix A). It is realised as the unstable manifold at the initial point $\rho_1 = (3/2, 0, 1/2)$. The plot shows in addition in cyan the stable manifold at ρ_1, i. e. the vertical fibre, and in magenta the centre manifold at ρ_1 containing the stationary points of $Y^{(1)}$. Furthermore, dashed lines show the projections of these manifolds to $\mathcal{J}_0$ (the fibre projects of course simply on a point). In other words, the dashed blue line is the graph of the solution. One might think that the dashed magenta line shows a second solution which is constant. However, the solid magenta line is *not* the graph of a prolonged constant function. This proves graphically that the tangent vectors to the centre manifold are not contact vectors.



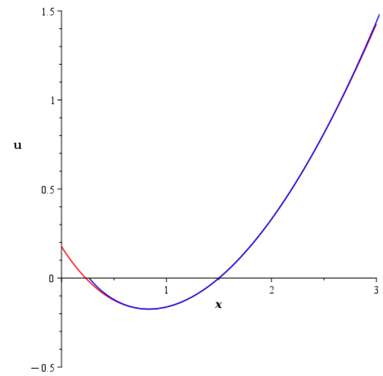


Figure 1: Left: Prolonged graph of unique smooth solution in Ex. 4.4. Right: Comparison with Taylor polynomial of order 5.

It would require some efforts to continue the blue lines towards the left, as obviously the solution encounters at approximately $x = 0.264$ the next zero which would have to be analysed for its properties. Using Prop. 4.1, it is also straightforward to determine a Taylor approximation of the solution. With a computer algebra system, one can easily compute explicit expressions for some prolongations and then solve them for the highest derivative. One obtains for the next coefficients

$$c_2 = \frac{2}{3}\,, \quad c_3 = -\frac{1}{6}\,, \quad c_4 = \frac{11}{45}\,, \quad c_5 = -\frac{269}{540}\,. \tag{4.1}$$

A graphical comparison of the corresponding Taylor polynomial with our numerical solution shows an excellent agreement (in the right part of Fig. 1 the Taylor approximation is plotted in red).

There remains to study the case that γ and δc_1 possess the same sign. We introduce the natural number $k = \lceil \gamma/\delta c_1 \rceil$. For any order $q \leq k$ also $\gamma - (q-1)\delta c_1$ and δc_1 possess the same sign, whereas for any $q > k$ their signs differ. In the proof of Theorem 4.2, $k + 1$ is thus the smallest possible choice for q. The considerations there immediately imply that any additional solution besides φ_∞ can be at most of regularity C^k and we will now show that such solutions indeed exist.

Theorem 4.5. *Assume that there is no resonance, that $c_1 \neq 0$ and that γ and δc_1 possess the same sign. Then the initial value problem* (1.2) *has besides the smooth solution φ_∞ infinitely many one-sided solutions $\varphi \in C^k \setminus C^{k+1}$ where $k = \lceil \gamma/\delta c_1 \rceil$. Each of these solutions is uniquely characterised by the (one-sided) limit*

$$\lim_{x \to y} \frac{\varphi^{(k)}(x) - c_k}{|x - y|^\lambda} \tag{4.2}$$

where $\lambda = \frac{\gamma-(k-1)\delta c_1}{\delta c_1}$. If one takes two one-sided solutions, one defined to the left and one to the right of y, then they define together a two-sided solution of regularity $C^k \setminus C^{k+1}$. A Taylor expansion to order k about y yields for all these two-sided solutions (and for φ_∞) the same Taylor polynomial $\sum_{i=0}^{k} \frac{c_i}{i!}(x-y)^i$ and their kth derivative is Hölder continuous with Hölder exponent λ.

We emphasise that in the case $k = 1$, i. e. for $0 < \gamma < \delta c_1$, the solutions besides φ_∞ are only weak solutions, as they are not twice differentiable in y. For $k > 1$, all solutions are strong.

Proof. In any order $q \leq k$, the vector field $Y^{(q)}$ possesses under the made assumptions at the stationary point ρ_q a two-dimensional unstable manifold $\mathcal{W}^u_q \subset \mathcal{R}_q$. As we have no resonance at any order, the two positive eigenvalues δc_1 and $\gamma - (q-1)\delta c_1$ of the Jacobian $J^{(q)}$ are different. Hence, ρ_q is an unstable, two-tangent node for the dynamics of the field $Y^{(q)}$ restricted to $\mathcal{W}^u_q$. This immediately implies that we have infinitely many one-sided generalised solutions at ρ_q, each defined by the union of $\{\rho_q\}$ with a trajectory leaving ρ_q. It also follows from the phase portrait of a two-tangent node that one can combine trajectories leaving in opposite directions to two-sided generalised solutions.

We distinguish two cases: $q < k$ and $q = k$. In the first case, $\gamma - (q-1)\delta c_1$ is the larger eigenvalue. Hence almost all trajectories leave ρ_q tangential to the eigenspace of δc_1 which lies transversal to the projection π^q. This implies that two-sided generalised solutions constructed out of these trajectories are proper and locally the prolonged graph of a two-sided solution which is at least $(q+1)$-times differentiable. Only two trajectories leave tangential to the eigenspace of $\gamma - (q-1)\delta c_1$. As above, it is easy to see that their union with $\{\rho_q\}$ is just the fibre $\mathcal{F}_q = (\pi^q_{q-1})^{-1}(\rho_{q-1})$ and hence an improper generalised solution without relevance for an existence theory.

If $q = k$, then δc_1 is the larger eigenvalue. Thus now almost all trajectories ρ_k are tangential to the vertical eigenspace of $\gamma-(k-1)\delta c_1$. One of the two-sided generalised solutions constructed out of them is again the fibre $\mathcal{F}_k = (\pi^k_{k-1})^{-1}(\rho_{k-1})$ and hence an improper generalised solution without relevance for an existence theory. All the other ones are the prolonged graphs of two-sided solutions of regularity C^k. Because of the vertical tangent, the solutions are, however, not $(k+1)$-time differentiable. In addition, we find one two-sided generalised solution tangent to the eigenspace of δc_1. It stems from the smooth solution φ_∞.

Obviously, in the above argumentation we implicitly applied the Hartman–Grobman Theorem to the linearisation of the restriction of $Y^{(q)}$ to $\mathcal{W}^u_q$ around ρ_q. Its standard form (see e. g. [11, Sect. 2.8]) asserts only the existence of a local homeomorphism h between the nonlinear and the linear dynamics. However, for a smooth dynamical system as in our case, the homeomorphism h is differentiable [12]. This implies that tangent information are preserved and our above arguments based on them are indeed valid.

We take now a closer look at the linearisation of the restriction of $Y^{(k)}$ to $\mathcal{W}^u_k$ around ρ_k. It lives in the plane spanned by the above computed eigenvectors to the positive eigenvalues δc_1 and $\gamma - (k-1)\delta c_1$. If we use coefficients α and β of these vectors as coordinates on the plane, then it is straightforward to write down the solution of the linear system through the point $(\bar{\alpha}, \bar{\beta})$:

$$\beta = \bar{\beta}\Big(\frac{\alpha}{\bar{\alpha}}\Big)^{\lambda} . \tag{4.3}$$

We are only interested in the case $\bar{\alpha} \neq 0$, as otherwise we obtain vertical lines. It suffices to consider $\bar{\alpha} = \pm 1$ for uniquely describing all remaining trajectories. The sign of $\bar{\alpha}$ decides whether the trajectory approaches the origin from the left or from the right. Since $0 < \lambda < 1$, these trajectories have all a vertical tangent in the origin. Hence, the function $\beta(\alpha)$ defined by (4.3) is not differentiable at $\alpha = 0$, but it is there Hölder continuous with Hölder exponent λ.

Going back to jet coordinates, the trajectories (4.3) of the linearisation may be written in the following parametrised form:

$$\begin{gathered} x(t) = y + \bar{\alpha}e^{\delta c_1 t} \ , \ u(t) = c_0 + \bar{\alpha}c_1 e^{\delta c_1 t} \ , \ldots, \ u^{(k-1)}(t) = c_{k-1} + \bar{\alpha}c_k e^{\delta c_1 t} \ , \\ u^{(k)}(t) = c_k - \bar{\alpha}\frac{h_{k+2}(\rho_k)}{\gamma - k\delta c_1}e^{\delta c_1 t} + \bar{\beta}e^{(\gamma-(k-1)\delta c_1)t} \ . \end{gathered} \tag{4.4}$$

The limit $x \to y$ corresponds to the limit $t \to -\infty$. Using the first equation in (4.4) to eliminate t, we see from the last equation that $\bar{\beta}$ equals the limit (4.2) which thus uniquely characterises each trajectory. Sufficiently close to ρ_k, the trajectories of the linearisation are mapped by the differentiable homeomorphism h to those of $Y^{(k)}$ restricted to $\mathcal{W}^u_k$. Hence the latter ones are also characterised by the limit (4.2) and they correspond to the prolonged graph of a C^k function φ for which $\varphi^{(k)}$ is Hölder continuous with Hölder exponent λ.

Finally, the statement about the Taylor expansion is trivial, as it simply reflects that the prolonged graphs of all the two-sided solutions define generalised solutions going through $\rho_k = (y, c_0, c_1, \ldots, c_k)$ and the definition of jet bundles. □

Example 4.6. In the family from Ex. 3.4, we consider again the instance $r = 1$ and $s = 0$. But this time we choose $y = 1/3$ and $c_1 = y - r = -2/3$. Now $\gamma = -1$ has the same sign as δc_1 and additional solutions exist, which according to Thm. 4.5 lie in $C^2 \setminus C^3$ since $k = \lceil \gamma/\delta c_1 \rceil = 2$. Some solutions together with their second derivatives are shown in Fig. 2. The red curves show the unique smooth solution and its derivative. For the other curves, it is clearly visible how the second derivatives approach a vertical tangent implying that the third derivatives become infinite there. Furthermore, one can see that the third derivatives are of opposite signs to the left and to the right of y.

5. Initial Value Problems with a Resonance

We proceed to the case that a resonance occurs at some order $k > 0$ assuming that neither γ nor c_1 vanishes. Then we always find infinitely many solutions. Their regularity depends on whether or not the resonance is smooth.

Theorem 5.1. *Assume that $\gamma c_1 \neq 0$ and that the initial point ρ_1 is resonant at order $k > 0$. Then the initial value problem* (1.2) *possesses infinitely many one-sided solutions, pairs of which can be combined into two-sided solutions possessing the same Taylor polynomial of degree k around y.*

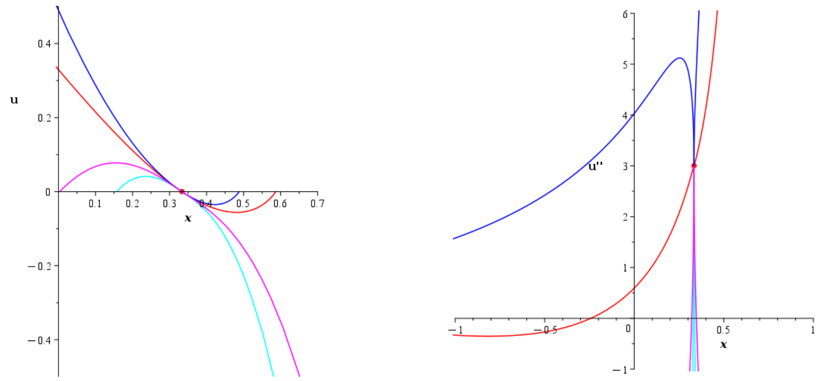


Figure 2: Solutions in Ex. 4.6. Left: functions; right: second derivatives.

In the case of a smooth resonance, to almost every one-sided solution there exists a unique second one-sided solution with which it can be combined to a smooth two-sided solution uniquely determined by the value of its $(k+1)$*st derivative in* y.

In the case of a critical resonance, each one-sided solution φ *is uniquely characterised by the one-sided limit*

$$\lim_{x\to y} \epsilon\Big[\frac{\varphi^{(k)}(x) - c_k}{A_k(x-y)} - \frac{1}{\delta c_1}\ln\Big(\frac{x-y}{\epsilon}\Big)\Big] \tag{5.1}$$

where $A_k \neq 0$ *is the resonance parameter and* $\epsilon = \pm 1$ *is chosen such that* $\frac{x-y}{\epsilon} > 0$. *Any two one-sided solutions approaching* y *from different sides can be combined to a two-sided solution living in* $\mathcal{C}^k \setminus \mathcal{C}^{k+1}$.

Proof. Using the same arguments as in the proof of Prop. 4.1, we can construct a unique finite sequence of irregular singularities $\rho_j \in \mathcal{R}_j$ for $j = 2, \dots, k$ lying above each other and our initial point ρ_1. The final point ρ_k is a stationary point of $Y^{(k)}$ and its Jacobian $J^{(k)}$ at ρ_k has a double eigenvalue δc_1. Hence, we find there besides a one-dimensional centre manifold a two-dimensional unstable manifold $\mathcal{W}_k^u$ (recall that we assume that $\delta c_1 > 0$). Restricting $Y^{(k)}$ to $\mathcal{W}_k^u$, it follows from our above analysis of the eigenspace that ρ_k is a star node in the case of a smooth resonance and a one-tangent node in the case of a critical resonance.

Our assertions will now be proven by firstly analysing the respective local phase portraits around ρ_k of the restricted flow. In the smooth case, it will turn out that we can continue the sequence $(\rho_j \in \mathcal{R}_j)_{j=2,\dots,k}$ beyond order k obtaining smooth solutions, whereas in the critical case this is not possible.

We consider first the smooth case. To each trajectory approaching a star node, there exists a "partner" trajectory approach the node from the opposite side such that the two trajectories are tangent to the same line in the node. Two such trajectories define together with the node ρ_k a two-sided weak generalised solution. For one of these generalised solutions the tangent at ρ_k is vertical and again it is easy to see that we obtain simply the fibre $\mathcal{F}_k$, i.e. an improper generalised solution. All the other ones possess a transversal tangent at ρ_k and hence are locally the prolonged graph of a solution of regularity at least C^{k+1}. By Prop. 3.3, the fibre $\mathcal{F}_{k+1} = (\pi_k^{k+1})^{-1}(\rho_k)$ consists entirely of irregular singularities of $\mathcal{R}_{k+1}$. We have a bijection between the set of all proper generalised solutions through ρ_k and the fibre $\mathcal{F}_{k+1}$ mapping each generalised solution to that point in $\mathcal{F}_{k+1}$ which has as $u^{(k+1)}$-coordinate the "slope" of its tangent at ρ_k (for a generalised solution, the tangent direction at ρ_k is given by $Y^{(k)}|_{\rho_k}$; expressing $Y^{(k)}|_{\rho_k} = aC^{(k)}_{\mathrm{trans}}|_{\rho_k} + bC^{(k)}_{\mathrm{vert}}|_{\rho_k}$, its "slope" is defined as b/a).

Let $\rho_{k+1} \in \mathcal{F}_{k+1}$ be the fibre point corresponding to a proper generalised solution $\mathrm{im}\, j_k\sigma_\varphi$ through ρ_k. The Jacobian $J^{(k+1)}$ of the vector field $Y^{(k+1)}$ at ρ_{k+1} has 0 as a double and δc_1 as a simple eigenvalue. The same reasoning as in the proof of Thm. 4.2 shows that the one-dimensional unstable manifold defines a proper generalised solution. The two-dimensional centre manifold is not necessarily unique. However, the tangent space of any centre manifold at ρ_{k+1} is spanned by the vector $(0, \dots, 0, 1)^T$ and the tangent vector to the curve of irregular singularities of $\mathcal{R}_{k+1}$. By the discussion in the proof of Thm. 4.2, no vector in this tangent space can have a non-vanishing u-component and hence no trajectory on a centre manifold can be a proper two-sided generalised solution, since we are assuming $c_1 \neq 0$. We thus conclude that the unstable manifold corresponds to the generalised solution $\mathrm{im}\, j_{k+1}\sigma_\varphi$. There also cannot be any proper one-sided generalised solution approaching ρ_{k+1}, as its projection to $\mathcal{R}_k$ would have already shown up in our analysis of ρ_k.

Again by Prop. 3.3, we have a unique irregular singularity $\rho_{k+2} \in \mathcal{R}_{k+2}$ above ρ_{k+1} and the same is true in any higher order. Furthermore, from order $q \geq k+2$ on, we have again the situation that at the unique irregular singularity $\rho_q \in \mathcal{R}_q$ all eigenvalues of $J^{(q)}$ have different signs and hence that all invariant manifolds are one-dimensional. With the same arguments as in the proof of Thm. 4.2, we conclude that φ is a smooth solution.

In the case of a critical resonance, our analysis of the eigenspace in Section 3 showed that it is one-dimensional and vertical. Again two trajectories approaching ρ_k from different sides define together with ρ_k a generalised solution which is locally of the form $\mathrm{im}\, j_k\sigma_\varphi$. However, since its tangent in ρ_k is vertical, we have $\varphi \in C^k \setminus C^{k+1}$. For characterising the one-sided solutions, we consider again the linearisation of the restriction of $Y^{(k)}$ to $\mathcal{W}^u_k$. We have seen that the vectors

$$\mathbf{v}_1 = (0, \dots, A_k)^T\ , \quad \mathbf{v}_2 = (1, c_1, \dots, c_k, 0)^T \tag{5.2}$$

define a Jordan basis of the generalised eigenspace. Denoting again by α and β coordinates with respect to this basis, the trajectory through the point $(\bar\alpha, \bar\beta)$ is given in parametrised form by

$$\alpha(t) = (\bar\alpha + \bar\beta t)e^{\delta c_1 t}\ , \quad \beta(t) = \bar\beta e^{\delta c_1 t}\ . \tag{5.3}$$

If $\bar\beta = 0$, the trajectory is just the α-axis corresponding to the vertical fibre in the jet bundle. Obviously, this is not a proper generalised solution and we thus exclude this case. It suffices to consider $\bar\beta = \pm 1$ for uniquely describing all remaining trajectories. The sign of $\bar\beta$ decides whether the trajectory approaches the origin from the left or from the right. We proceed now as in the proof of Thm. 4.5: we first express the linearised dynamics in jet coordinates and then eliminate the parameter t using the equation $x(t) = y + \beta(t)$; if one enters the result into the equation $u^{(k)}(t) = c_k + A_k\alpha(t)$ and solves for $\bar{\alpha}$, one finally obtains

$$\frac{\bar{\alpha}}{\bar{\beta}} = \frac{u^{(k)} - c_k}{A_k(x-y)} - \frac{1}{\delta c_1}\ln\Big(\frac{x-y}{\bar{\beta}}\Big)\,. \tag{5.4}$$

Again, the trajectory reaches ρ_k in the limit $t \to -\infty$ corresponding to a one-sided limit $x \to y$. □

If we compare the smooth case with Prop. 4.1, then we construct again towers $\rho_1, \rho_2, \dots, \rho_k, \rho_{k+1}, \rho_{k+2}, \dots$ of irregular singularities lying above each other. The difference is that now ρ_{k+1} can be chosen arbitrarily. All points before it are uniquely determined by ρ_1 and all points after it are again uniquely determined once ρ_{k+1} has been chosen. Thus we find a one-parameter family of smooth solutions. Note that there exist further two-sided solutions of finite regularity C^k. Indeed, we may also combine two one-sided solutions approaching ρ_k from different sides which have *not* the same tangent at ρ_k. The arising generalised solution is then locally still of the form im $j_k\sigma_\varphi$ for some function φ. This function possesses at y both a left and a right $(k+1)$st derivative, but their values differ so that $\varphi \notin C^{k+1}$.

Example 5.2. In our family (3.13), we choose $r = 1$ and consider the initial point with $y = -1/2$ and $c_1 = 1/2$ yielding $\gamma = 1$. As already mentioned in Ex. 3.4, this point is resonant at order $k = 2$. According to Thm. 5.1, we can therefore expect infinitely many solutions at least of regularity C^2. The initial part of our tower consists here of one more point, namely $\rho_2 = (-1/2, 0, 1/2, s+2)$. In Ex. 3.4, we already determined the resonance parameter to be $A_2 = -(s+2)^2 - s^2/4 + 1$. For the choice $s = -2$, we therefore find a smooth resonance with infinitely many smooth solutions, whereas for $s = 2$ all solutions lie in $C^2 \setminus C^3$.

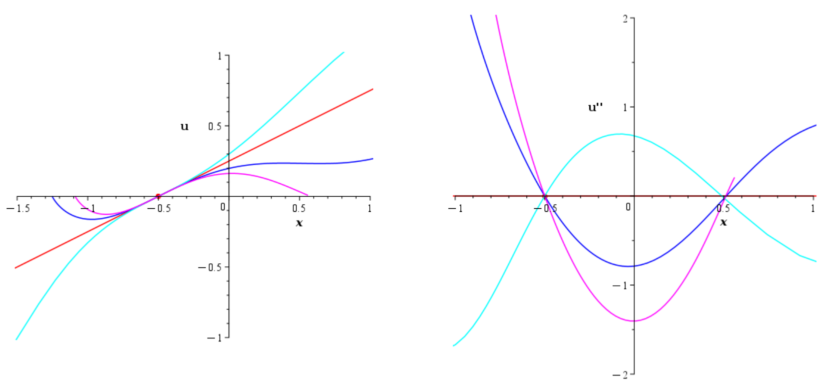


Figure 3: Smooth resonance in Ex. 5.2. Left: functions; right: second derivatives

Fig. 3 shows some numerical solutions and their second derivatives for the case of the smooth resonance. The red curves belong here to an easily verified exact solution: $u(x) = \frac{1}{2}(x + \frac{1}{2})$ (for which the second derivative is of course the zero function). One obtains here $c_2 = 0$, so that our initial point is also an inflexion point of the graph of any solution through it. The numerical results seem to indicate that the graphs of all solutions also have a second inflexion point at $x = -y = -\frac{1}{2}$, but there is no obvious explanation for this observation.

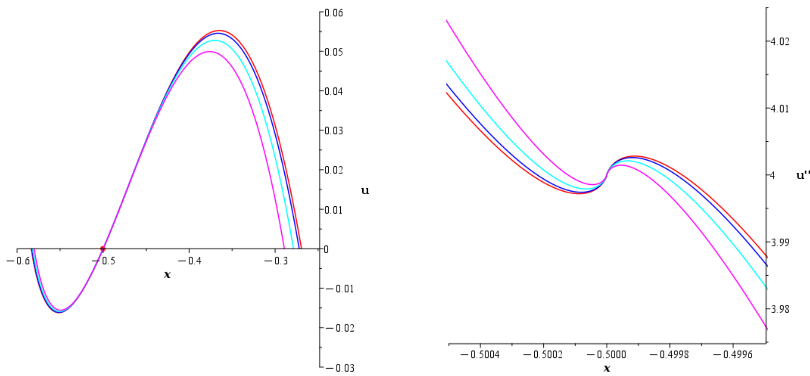


Figure 4: Critical resonance in Ex. 5.2. Left: functions; right: second derivatives

In the case of the critical resonance, the solutions encounter in both directions very soon further zeros, as one can see in Fig. 4. One has to zoom in very closely to the singularity to see that the graphs of the second derivatives have a vertical tangent at $x = 1/2$. So close to the singularity, the graphs also exhibit the typical shape of trajectories near a one-tangent node with vertical tangent.

6. The Special Cases $\gamma = 0$ or $c_1 = 0$

In these special cases, typically further subcases arise and a complete classification of all possibilities becomes rather tedious. We will thus concentrate on generic situations. We consider first the case $\gamma = 0$ assuming that still $c_1 \neq 0$. Again, it can be handled fairly similar to the treatment of the corresponding case in [1], but becomes in fact even simpler.

Theorem 6.1. *Assume that $\gamma = 0$, but $c_1 \neq 0$. Then the initial value problem* (1.2) *possesses a unique smooth two-sided solution φ_∞. There do not exist any additional one-sided solutions, not even proper weak generalised ones.*

Proof. For any $q > 1$, the three relevant eigenvalues of the matrix $J^{(q)}$ are 0, δc_1 and $-(q-1)\delta c_1$. The assumption $c_1 \neq 0$ implies that they have all different signs and hence we can argue as in the proofs of Thm. 4.2 and Cor. 4.3 obtaining the existence and uniqueness of a two-sided smooth solution φ_∞ and the absence of any additional proper generalised solutions.

This only leaves the possibility that weak generalised solutions exist on $\mathcal{J}_1$. We must distinguish two cases depending on whether or not the entry $a^{(1)}$ in the Jacobian $J^{(1)}$ vanishes. If $a^{(1)} \neq 0$, the rank of $J^{(1)}$ is still two, although it possesses a double eigenvalue 0. A Jordan basis for the generalised eigenspace of it is given by the vectors $(0, 0, a^{(1)})^T$ and $(1, 0, 0)^T$. If $a^{(1)} = 0$, then the Jacobian $J^{(1)}$ has a two-dimensional nullspace generated by the vectors $(1,0,0)^T$ and $(0,0,1)^T$. Hence there exist many transversal (generalised) eigenvectors in both cases. However, we still find that the second entry of any (generalised) eigenvector vanishes implying under the assumption $c_1 \neq 0$ that none of them can be a contact vector, i. e. a linear combination of the vectors $C^{(1)}_{\text{trans}}|_{\rho_1}$ and $C^{(1)}_{\text{vert}}|_{\rho_1}$. Hence the two-dimensional centre manifolds cannot lead to any proper weak generalised solutions. □

We continue with the case $c_1 = 0$, but $\gamma \neq 0$. Note that the latter condition entails that the rank of $\hat{J}^{(q)}$ is $q+1$. It follows from the matrices $J^{(q)}$ and $\hat{J}^{(q)}$ that our relevant eigenvalues are 0 (double) and γ (simple). Because of our rank of $\hat{J}^{(q)}$, the eigenspace of 0 can only be one-dimensional and we thus find a Jordan block of length 2 for 0. One can recursively describe an eigenvector $\mathbf{v}^{(q)}$ spanning this eigenspace as follows. If we write

$$\mathbf{v}^{(q)} = \left(-\gamma^q, 0, \gamma^{q-1}\chi^{(1)}, \dots, \gamma\chi^{(q-1)}, \chi^{(q)}\right)^T , \tag{6.1}$$

then the coefficients $\chi^{(i)}$ satisfy the recursion

$$\chi^{(1)} = a^{(1)} , \qquad \chi^{(q)} = a^{(q)}\gamma^{q-1} - \sum_{j=1}^{q-1} b^{(q)}_j \gamma^{q-1-j}\chi^{(j)} \quad \text{for } q > 1 , \tag{6.2}$$

as one can easily verify by direct substitution. For a Jordan basis, one needs a second (generalised) eigenvector $\mathbf{w}^{(q)}$. Explicit formulae for its entries are even more cumbersome than (6.2). Fortunately, they are not needed for our analysis. We only note that each vector $\mathbf{v}^{(q)}$ has a non-vanishing first entry. It follows immediately from the first row of the matrix $J^{(q)}$ (or $\hat{J}^{(q)}$) that any vector $\mathbf{w}^{(q)}$ with $J^{(q)}\mathbf{w}^{(q)} = \mathbf{v}^{(q)}$ must have a non-vanishing second entry.

Theorem 6.2. *Assume that $c_1 = 0$, but $\gamma \neq 0$. Then the initial value problem* (1.2) *has either zero, one or infinitely many smooth solutions. In the second case, the solution is the constant function $\varphi(x) \equiv c_0$ and thus trivially two-sided. In the last case, the solutions show a saddle-node like behaviour, i. e. they are unique on one side of the initial point. No additional one-sided proper weak generalised solutions exist.*

Proof. Prop. 4.1 remains valid also in the case $c_1 = 0$. Thus we find above the initial point ρ_1 a unique tower of irregular singularities $\rho_2 \in \mathcal{R}_2$, $\rho_3 \in \mathcal{R}_3$,... which are also stationary points of the vector fields $Y^{(q)}$. Hence the considered initial value problem always possesses a unique formal power series solution.

According to our considerations above, we find for each field $Y^{(q)}$ around its stationary point ρ_q qualitatively the same situation: a two-dimensional centre manifold and a one-dimensional (un)stable manifold (the stability depends on the sign of γ). As the eigenspace of the eigenvalue γ is spanned by the vector $(0,\dots,0,1)^T$, we can again identify the latter manifold with the fibre $\mathcal{F}_q$ which does not define a proper (for $q=1$ weak) generalised solution.

It follows from our above discussion of the generalised eigenspace for 0 that any centre manifold at ρ_q can be parametrised by the variables x and u, i. e. it can be expressed by equations $u^{(i)} = \eta_i(x,u)$ for $i = 1,\dots,q$. The reduced dynamics on any centre manifold is then governed by the two-dimensional system $\dot{x} = g(u)$ and

$\dot{u} = g(u)\eta_1(x,u)$. Its stationary points are the lines $u = \bar{u}$ with $g(\bar{u}) = 0$; in particular, the line $u = c_0$ consists of stationary points. Off the stationary points, the system is orbit equivalent to the system $\dot{x} = 1$ and $\dot{u} = \eta_1(x,u)$ which has no stationary points. Its trajectories are the graphs of the solutions of the ordinary differential equation $du/dx = \eta_1(x,u)$. By the theorem of Picard–Lindelöf, it possesses for every initial point (x_0,u_0) a unique solution.

We are interested in the initial point (y,c_0) which is a stationary point for the reduced dynamics. Let $u = \psi(x)$ be the corresponding solution of the equation $du/dx = \eta_1(x,u)$ defined for x in some interval $\mathcal{I} \subseteq \mathbb{R}$. After possibly shrinking $\mathcal{I}$, we can lift its graph to a curve in $\mathcal{J}_q$

$$\Psi_q(x) = \Bigl(x, \psi(x), \eta_1(x,\psi(x)), \dots, \eta_q(x,\psi(x))\Bigr). \tag{6.3}$$

By construction, $\operatorname{im}\Psi_q$ lies in the considered centre manifold and $\Psi_q(y) = \rho_q$. We must now distinguish two cases.

In the first case, $\operatorname{im}\Psi_q$ consists entirely of stationary points of $Y^{(q)}$ and thus is a subset of the curve of irregular singularities of $\mathcal{R}_q$. It defines a weak generalised solution, if and only if its tangent vectors are also contact vectors. If this is not the case, no proper weak generalised solution reaches ρ_q and thus the initial value problem (1.2) cannot possess any solution. As the irregular singularities lie in the hyperplane $u = c_0$, the tangent vectors to this curve have everywhere a vanishing u-component and we can obtain a proper weak generalised solution, if and only if $\operatorname{im}\Psi_q$ is the prolonged graph of the constant function $\varphi(x) \equiv c_0$ which is the case if and only if $\eta_1(x,u) \equiv 0$. As in the centre manifold no other solution of the reduced system reaches ρ_q, the vertical fibre is the only other weak generalised solution reaching ρ_q, but it is not proper. Hence in this case, the initial value problem (1.2) possesses a unique solution which is constant.

In the second case, $\operatorname{im}\Psi_q$ is an integral curve of $Y^{(q)}$ and thus trivially a weak generalised solution. Its tangent vector at ρ_q is of the form $\Psi_q'(y) = \bigl(1, \eta_1(y,c_0), \dots\bigr)^T$ with $\eta_1(y,c_0) = c_1 = 0$ by assumption. As its x-component does not vanish, we have even a proper weak generalised solution. It must furthermore lie in the generalised eigenspace for 0 and we can conclude from our above discussion that it must actually be a multiple of $\mathbf{v}^{(q)}$. It easily follows from Lemma 3.1, the form of $\hat{J}^{(q)}$ and the definition of the entries $a^{(q)}$ and $b_i^{(q)}$ that $\mathbf{v}^{(q)}$ is also tangent to the curve of irregular singularities of $\mathcal{R}_q$. Hence in this case $\operatorname{im}\Psi_q$ has a contact of at least order 1 with this curve. Within the centre manifold, $\operatorname{im}\Psi_q$ is the only weak generalised solution reaching ρ_q. As the third relevant eigenvalue of $J^{(q)}$ is γ and thus by assumption a non-zero real number, we find around ρ_q in $\mathcal{R}_q$ a saddle-node-like behaviour of the integral curves of $Y^{(q)}$.

These considerations prove indeed the existence of smooth solutions. Any weak generalised solution found in some order q induces via projections weak generalised solutions in all lower orders. This implies that if $q_1 < q_2$ are two different orders, then the components η_i for $1 \leq i \leq q_1$ of the parameterisations of the respective centre manifolds are the same (more precisely, we can choose at any order the centre manifold such that this is the case). Furthermore, it is a general fact that the irregular singularities of $\mathcal{R}_{q_2}$ lie over those of $\mathcal{R}_{q_1}$ (see e. g. [1, Prop. 8]). Hence, it is not possible that for q_1 we are in the first case and for q_2 in the second case or the other way round. We are in all orders $q \geq 1$ in the same case and the found weak generalised solutions are graphs of prolongations of smooth functions. □

Example 6.3. We consider the equation $uu'' = u^2 + (u+1)u'$. As it is autonomous, the choice of y is irrelevant; we have taken $y = 0$. For $c_0 = 0$, we must have $c_1 = 0$, too, for a vanishing right hand side. Since $\gamma = c_1 + 1 = 1$, we have then indeed an initial value problem with $c_1 = 0$ and $\gamma \neq 0$. It is trivial to see that $\varphi(x) \equiv 0$ is a solution of this problem and we are in the first case of the proof of Theorem 6.2. The left plot in Fig. 5 shows it in red and in magenta the vertical fibre which is simultaneously the stable manifold of the inital point $\rho_1 = (0, 0, 0)$. We computed a Taylor approximation of order 4 of the centre manifolds and some proper generalised solutions on it (shown in black). They form a family of curves which approach the red curve (which is simultaneously the curve of stationary points of $Y^{(1)}$) asymptotically either for $x \to \infty$ or $x \to -\infty$. The plot shows in addition in blue some proper generalised solutions outside of the centre manifold approximation; each of them approaches this approximation exponentially fast as predicted by centre manifold theory and thus they behave asymptotically like the black curves. None of these proper generalised solutions can reach the red curve at a finite value of x; this happens only for fibres parallel to the one in magenta.

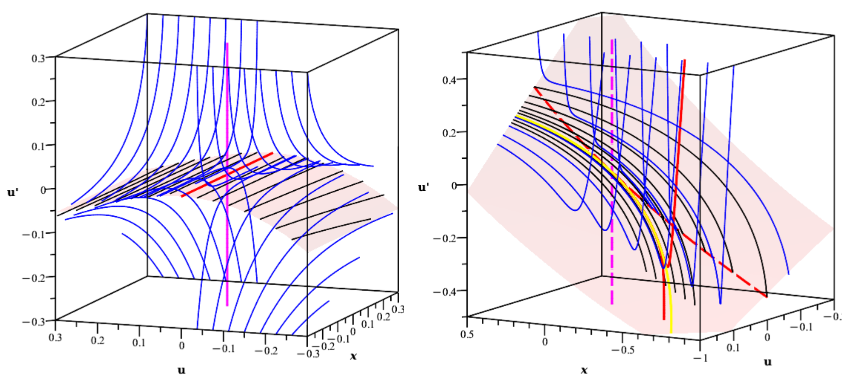


Figure 5: Approximate centre manifolds and generalised solutions on it for the initial value problems with $c_1 = 0$ in Ex. 6.3. Left: constant solution. Right: non-constant solution

Now we take the equation $uu'' = (u'-1)^2 - (u+1)^2 + x$. We choose $c_0 = 0$ and $c_1 = 1 - \sqrt{1-y}$ for values $y < 1$. This choice entails $\gamma = -2\sqrt{1-y}$. For $0 < y < 1$, we are then in the situation of Cor. 4.3 with a unique smooth solution. For $y < 0$, we obtain an initial value problem covered by Thm. 4.5 with infinitely many solutions which, however, except of a single smooth one are all of regularity C^1. We are here mainly interested in what happens for $y = 0$ where $c_1 = 0$ and $\gamma = -2 \neq 0$. It will turn out that it leads to the second case in the proof of Theorem 6.2.

Our initial points define one branch of the parabola of stationary points of $Y^{(1)}$ given by $(t, 0, 1 - \sqrt{1-t})$ for $t \leq 1$; a curve which obviously cannot be interpreted as the prolonged graph of a function. The right plot in Fig. 5 shows it as a dashed red line and the vertical fibre as a dashed magenta line. A quadratic approximation of the centre manifolds in $\mathcal{J}_1$ is given by

$$u' = \eta_1(x, u) = \frac{1}{2}x - \frac{5}{4}u + \frac{1}{8}x^2 - \frac{7}{16}xu - \frac{9}{32}u^2 + \cdots \tag{6.4}$$

and the reduced dynamics on it is described by $du/dx = \eta_1(x, u)$. Our plot shows a quartic Taylor approximation and as black curves some weak generalised solutions on it. They form a family of curves (some crossing the dashed red curve) and exactly one member of the family – shown in yellow – goes through $\rho_1 = (0, 0, 0)$, i. e. solves the initial value problem we are interested in. As one can expect from the considerations in the proof of Theorem 6.2, it does not cross the dashed red curve, but has a contact of order 1 with it. A Taylor approximation of this solution can be determined in two different ways: either one lifts the corresponding series solution of the reduced dynamics or one computes via prolongations the irregular singularities $\rho_2 \in \mathcal{R}_2, \rho_3 \in \mathcal{R}_3, \ldots$ above ρ_1. Both approaches yield the same result, namely

$$\varphi(x) = \frac{1}{4}x^2 - \frac{1}{16}x^3 + \frac{1}{128}x^4 - \frac{1}{160}x^5 + O(x^6)\,. \tag{6.5}$$

One can now easily verify that a tangent vector of the curve at ρ_1 is $(1, 0, 1/2)^T$ which spans the nullspace of $J^{(1)}$ and is also the tangent vector of the dashed red curve.

The plot shows in addition in blue some further weak generalised solutions starting outside of the centre manifold approximation. Again they approach the approximated centre manifold exponentially fast. In particular, there exist infinitely many weak generalised solutions going into ρ_1 for $x < 0$ (the plot shows in red two of them). By contrast, only one unique weak generalised solution goes into ρ_1 for $x > 0$. Thus we have a situation as in a saddle-node: from one side we have infinitely many solutions, from the other side only one and together they define infinitely many two-sided solutions.

If we choose $y = 1$ and $c_0 = 0$ for this equation, then we obtain $c_1 = 1$ and $\gamma = 0$ and are thus in the situation of Thm. 6.1 guaranteeing the existence of a unique smooth solution. Fig. 6 shows this solution and its first derivative.

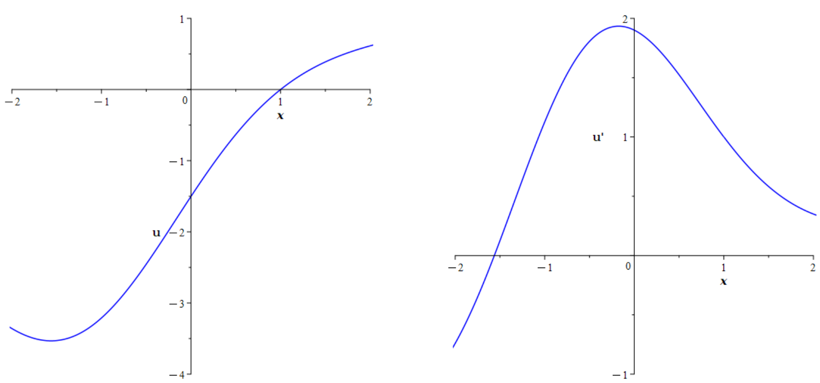


Figure 6: Initial value problem with $\gamma = 0$ from Ex. 6.3. Left: solution; right: first derivative

7. Conclusions

We applied the geometric approach from [1] where problems of the form $g(x)u'' = f(x, u, u')$ were treated to differential equations of the form $g(u)u'' = f(x, u, u')$; thus we proceeded from semilinear to truely quasilinear equations. We studied singular initial value problems $u(y) = c_0$ and $u'(y) = c_1$ with $f(y, c_0, c_1) = 0$ and where – as in the semilinear case – we assume that y is a simple zero of g and in the truely quasilinear case that c_0 is a simple zero of g. Somewhat surprisingly, it turns out that the results are very similar.

The properties of the initial value problem are determined by two parameters: $\delta = g'(y)$ or $\delta = g'(c_0)$, respectively, and $\gamma = f_{u'}(y, c_0, c_1)$. The main difference is the fact that in the truely quasilinear case, one must always consider the product δc_1 instead of δ alone. This leads to additional special cases when $c_1 = 0$ which are not present in the semilinear case.

The simplest case is when δc_1 and γ are both non-zero and of opposite sign. In this case one obtains as for regular initial value problems the existence of a unique smooth solution. If both values have the same sign, then in addition infinitely many solutions of finite regularity appear. Further complications arise, if in addition γ is an integer multiple of δc_1, i. e. when a resonance occurs. In the case of a smooth resonance, the additional infinitely many solutions are smooth, too. These results are essentially identical to the semilinear case.

Special cases arise, if either c_1 or γ vanishes. It turns out that the case $\gamma = 0$ and $c_1 \neq 0$ is even simpler as for semilinear equations: one always obtains a unique smooth solution. In the case $c_1 = 0$ and $\gamma \neq 0$, different possibilities exist: generically, one obtains a picture similar to a saddle-node with infinitely many smooth solutions on one side of the initial point; but it is also possible that only one or no solution exists.

We did not consider the case $c_1 = \gamma = 0$, as it leads to a Jacobian with a triple eigenvalue 0. One has to blow-up the vector field $Y^{(q)}$ at its stationary point ρ_q for determining the local phase portrait. This will probably lead again to many different subcases. For a concrete equation within the considered class, the procedure is quite clear. But we doubt that a meaningful finite classification exists for the whole class.

Appendix A. Numerical Computations

A nice feature of our geometric approach is that it leads very naturally to numerical computations (see [13] for a general discussion of mainly the first-order case and [14] for quasi-linear equations with a principal part of the form $g(x)u''$). We construct generalised solutions as trajectories of a vector field and thus most part of them can be easily obtained by a numerical integration. In the quasi-linear case, most singularities correspond to a stationary point of this vector field and generalised solutions going through it are by definition one-dimensional invariant manifolds. In particular, (one-dimensional) unstable manifolds are numerically very easy to construct. All plots in this article have been produced in this way with Maple using the commands `dsolve` with the `numeric` option and `odeplot`. In this appendix, we briefly indicate some aspects of the various examples.

The case of a unique smooth solution covered in Ex. 4.4 is rather straightforward. One uses the vector field $Y^{(1)}$ defined on the whole jet bundle $\mathcal{J}_1$ and computes an eigenvector $\mathbf{v}$ to the eigenvalue $c_1 = 1/2$. The prolonged graph of the solution is the unstable manifold of the stationary point $\rho_1 = (3/2, 0, 1/2)$. Starting a numerical integration at the points $\rho_1 \pm \epsilon\mathbf{v}$ for a sufficiently small value ϵ yields then immediately the desired graph (plus the graph of the first derivative, since we integrate on $\mathcal{J}_1$).

In Ex. 4.6, we have $c_1 = -2/3$ and the prolonged solutions lie on the *stable* manifold of $\rho_1 = (1/3, 0, -2/3)$. Hence, we use for the numerical integrations the vector field $-Y^{(1)}$. Its unstable manifold is two-dimensional. The smooth solution is again obtained by using an eigenvector to the eigenvalue $2/3$. For the solutions of finite regularity, we take linear combinations of this vector with an eigenvector of the other positive eigenvalue 1. In Fig. 2, we plotted the second derivative of some solutions. We obtained them directly by integrating instead of $-Y^{(1)}$ the prolonged vector field $-Y^{(2)}$ on $\mathcal{R}_2 \subset \mathcal{J}_2$. In our experience, it is not necessary to enforce that the trajectories lie on $\mathcal{R}_2$; our initial points are sufficiently close to the unstable manifold and the dynamics of $-Y^{(2)}$ on $\mathcal{J}_2$ automatically dampens any errors.

Resonances were treated in Ex. 5.2. As the resonances appeared at order 2, we use again the vector field $Y^{(2)}$ on $\mathcal{R}_2$ (this time, the relevant eigenvalue $c_1 = 1/2$ is positive, so that the solutions lie on the unstable manifold). In the smooth case, we choose two linearly independent eigenvectors and construct initial points out of linear combinations. In the critical case, the eigenspace is one-dimensional and leads to the unique smooth solution. For the solutions of finite regularity, one takes linear combinations of an eigenvector and a linearly independent generalised eigenvector for the construction of initial points. Because of the integration on $\mathcal{R}_2$, we obtain again simultaneously the graphs of the solutions and their first two derivatives.

For the initial value problem with $\gamma = 0$ and $c_1 \neq 0$ treated at the end of Ex. 6.3, we performed the numerical integration on $\mathcal{R}_2 \subset \mathcal{J}_2$, although we only produced plots of the solution and its first derivative. The reason lies in the arguments in the proof of Thm. 6.1: only for $q \geq 2$, the prolonged graph can be realised as an unstable manifold. If we integrated on the first-order jet bundle $\mathcal{J}_1$, the prolonged graph would correspond to some trajectory on a centre manifold and the integration would be less robust.

Acknowledgments

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References

[1] W. Seiler, M. Seiß, Singular initial value problems for scalar quasi-linear ordinary differential equations, J. Diff. Eqs. 281 (2021) 258–288.

[2] J. Liang, A singular initial value problem and self-similar solutions of a nonlinear dissipative wave equation, J. Diff. Eqs. 246 (2009) 819–844.

[3] V. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations, 2^{nd} Edition, Grundlehren der mathematischen Wissenschaften 250, Springer-Verlag, New York, 1988.

[4] A. Remizov, Multidimensional Poincaré construction and singularities of lifted fields for implicit differential equations, J. Math. Sci. 151 (2008) 3561–3602.

[5] D. Saunders, The Geometry of Jet Bundles, London Mathematical Society Lecture Notes Series 142, Cambridge University Press, Cambridge, 1989.

[6] W. Seiler, Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra, Algorithms and Computation in Mathematics 24, Springer-Verlag, Berlin, 2010.

[7] U. Kant, W. Seiler, Singularities in the geometric theory of differential equations, in: W. Feng, Z. Feng, M. Grasselli, X. Lu, S. Siegmund, J. Voigt (Eds.), Dynamical Systems, Differential Equations and Applications (Proc. 8th AIMS Conference, Dresden 2010), Vol. 2, AIMS, Springfield, 2012, pp. 784–793.

[8] W. Seiler, Singularities of implicit differential equations and static bifurcations, in: V. Gerdt, W. Koepf, E. Mayr, E. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing — CASC 2013, Lecture Notes in Computer Science 8136, Springer-Verlag, Cham, 2013, pp. 355–368.

[9] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Mathematical Sciences 42, Springer-Verlag, New York, 1990.

[10] J. Sijbrand, Properties of center manifolds, Trans. AMS 289 (1985) 431–469.

[11] L. Perko, Differential Equations and Dynamical Systems, 3rd Edition, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2001.

[12] M. Guysinsky, B. Hasselblatt, V. Rayskin, Differentiability of the Hartman-Grobman linearization, Discr. Cont. Dyn. Sys. 9 (2003) 979–984.

[13] E. Braun, W. Seiler, M. Seiß, On the numerical analysis and visualisation of implicit ordinary differential equations, Math. Comput. Sci. 14 (2020) 281–293.

[14] W. Seiler, M. Seiß, On the numerical integration of singular initial and boundary value problems for generalised Lane–Emden and Thomas–Fermi equations, Appl. Math. Comp. 466 (2024) 128446.