Recursive Structures in Involutive Bases Theory[☆]

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Abstract

We study characterisations of involutive bases using a recursion over the variables in the underlying polynomial ring and corresponding completion algorithms. Three key ingredients are (i) an old result by Janet recursively characterising Janet bases for which we provide a new and simpler proof, (ii) the Berkesch–Schreyer variant of Buchberger's algorithm and (iii) a tree representation of sets of terms also known as Janet trees. We start by extending Janet's result to a recursive criterion for minimal Janet bases leading to an algorithm to minimise any given Janet basis. We then extend Janet's result also to Janet-like bases as introduced by Gerdt and Blinkov. Next, we design a novel recursive completion algorithm for Janet bases. We study then the extension of these results to Pommaret bases. It yields a novel recursive characterisation of quasi-stability which we use it for deterministically constructing "good" coordinates more efficiently than in previous works. A small modification leads to a novel deterministic algorithm for putting an ideal into Nœther position. Finally, we provide a general theory of involutive-like bases with special emphasis on Pommaret-like bases and study the syzygy theory of Janet-like and Pommaret-like bases.

Keywords: Polynomial ideals, Gröbner bases, involutive bases, Janet bases, Janet-like bases, Pommaret bases, Pommaret-like bases, completion algorithms, recursion, Schreyer's theorem, quasi-stable ideals, Janet trees, Nœther position
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1. Introduction

Gröbner bases are a fundamental concept in computational commutative algebra and algebraic geometry and their efficient determination has been an important topic for a long time. *Involutive bases* are a special kind of Gröbner bases with additional

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[☆] This article is dedicated to the memory of Vladimir P. Gerdt, one of the inventors of involutive bases.

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combinatorial properties. The basic ideas underlying them stem from Janet's works on general systems of partial differential equations (Janet, 1920, 1929). The first rigorous definition of what is nowadays called a Pommaret basis was given by Zharkov and Blinkov (1996); the general definition of an involutive basis is due to Gerdt and Blinkov (1998). This work also contains a general algorithm for their construction; a more efficient algorithm allowing for the construction of minimal involutive bases was presented in (Gerdt and Blinkov, 1998). For implementations of these algorithms and extensive benchmarks, we refer to the website `http://invo.jinr.ru`. For a comprehensive study and for applications of the theory of involutive bases to commutative algebra and to partial differential equations, we refer to (Seiler, 2010).

In this work, we are concerned with *recursive structures* in the theory of involutive bases where the recursion will mainly be over the number of variables in the underlying polynomial ring. The starting point is an old result by Janet providing a recursive criterion for a set of terms to form a Janet basis (Theorem 3.1). We will give a simpler proof for a slightly more general version of it (Theorem 3.4). As a first extension, we will prove a corresponding recursive criterion for *minimal* Janet bases (Theorem 3.10) and use it provide an algorithm to minimise an arbitrary Janet basis (Algorithm 3). Currently, the main algorithm for computing a minimal Janet basis is the TQ-algorithm of Gerdt and Blinkov (1998) which determines the basis from scratch. While it is in principle possible to give this algorithm a Janet basis as input, it will not benefit from this (in fact, this is even bad input). By contrast, our novel algorithm efficiently minimises any given Janet basis.

Combining our recursive criteria with a variant of the Buchberger algorithm presented by Berkesch and Schreyer (2015), we develop novel recursive algorithms for the construction of monomial and polynomial Janet and Janet-like bases (Algorithms 5 and 7). Then we proceed to the construction of Pommaret bases where a key issue is to find "good" coordinates, i. e. to obtain a *quasi-stable position* for the given ideal (see (Hashemi et al., 2018) for an extensive discussion of this topic). We provide first recursive criteria both for Pommaret bases (Theorem 5.1) and for quasi-stability (Corollary 5.5) and then a deterministic algorithm for the construction of "good" coordinates (Algorithm 9). Compared with the results by Hashemi et al. (2018), the novel approach is not only much more efficient, but also the termination proof becomes much simpler. Minor modifications of the underlying ideas lead to recursive criterion for Nœther position (Proposition 5.13) which also translates immediately into a corresponding deterministic algorithm.

We proceed then to *Janet-like bases* which were introduced by Gerdt and Blinkov (2005a,b) to obtain more compact bases, in particular in situations where the degrees of the leading terms in some variables differ greatly (as e. g. in toric ideals). Again we will give a recursive criterion for a set to be a (minimal) Janet-like basis (Theorems 3.14 and 3.17). While Gerdt and Blinkov extended solely the Janet division to the Janet-like division, we will introduce the general concept of an involutive-like division (Definition 6.1) and related notions like continuity or constructivity. Our main emphasis will be on Janet-like and Pommaret-like bases and how they are related to each other and to Janet and Pommaret bases, respectively (Propositions 6.8 and 6.16, Theorem 6.19). But we will also start developing a syzygy theory for these bases by providing a variant of Schreyer's theorem (Theorem 7.7).

The structure of the paper is as follows. In the next section, we give basic notations and definitions that are used throughout the paper. In Section 3, we present recursive criteria for a monomial set being either a Janet or a Janet-like basis. Similar tests for the minimality of these bases are discussed in this section as well. In Section 4, we describe a variant of the Berkesch–Schreyer algorithm to compute Janet(-like) bases. Section 5 is devoted to our new recursive test for Pommaret bases and its application to their construction. In the following two sections, we introduce involutive-like bases for arbitrary divisions and study their basic properties and their construction. Finally, some conclusions are given.

2. Preliminaries

In this section, we review some basic definitions and notations from the theory of Gröbner bases and involutive bases that will be used in the rest of the article. Throughout, we work in the polynomial ring $\mathcal{P} = \mathcal{K}[X] = \mathcal{K}[x_1, \ldots, x_n]$ over a field $\mathcal{K}$. We consider the polynomials $f_1, \ldots, f_k \in \mathcal{P}$ and the ideal $\mathcal{I} = \langle f_1, \ldots, f_k \rangle$ generated by them. We denote the total degree of and the degree with respect to a variable x_i of a polynomial $f \in \mathcal{P}$ by $\deg(f)$ and $\deg_i(f)$, respectively. We write $\mathcal{T} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0,\ 1 \leq i \leq n\}$ for the monoid of all terms in $\mathcal{P}$. A term ordering on $\mathcal{T}$ is denoted by $\prec$ and throughout we shall assume that $x_1 \prec \cdots \prec x_n$. The leading term of a given polynomial $f \in \mathcal{P}$ with respect to $\prec$ is denoted by $\operatorname{lt}(f)$. If $F \subset \mathcal{P}$ is a finite set of polynomials, we denote by $\operatorname{lt}(F)$ the set $\{\operatorname{lt}(f) \mid f \in F\}$. A finite set $G \subset \mathcal{P}$ is called a *Gröbner basis* for $\mathcal{I}$ with respect to $\prec$, if its leading ideal satisfies $\operatorname{lt}(\mathcal{I}) = \langle \operatorname{lt}(f) \mid f \in \mathcal{I} \rangle = \langle \operatorname{lt}(G) \rangle$. We refer e. g. to (Cox et al., 2015) for more details on Gröbner bases.

Instead of the standard Buchberger algorithm, we are more interested in a variant presented by Berkesch and Schreyer (2015). First, we shall need a particular form of the division algorithm based on the enumeration of the divisors. Let $f_1, \ldots, f_k \in \mathcal{P}$ be an ordered sequence of nonzero polynomials and $f \in \mathcal{P}$ a further polynomial. Then quotients $h_1, \ldots, h_k \in \mathcal{P}$ and a unique remainder $r \in \mathcal{P}$ exist such that:

- $f = h_1 f_1 + \cdots + h_k f_k + r$,
- No term in $h_i \operatorname{lt}(f_i)$ is divisible by any $\operatorname{lt}(f_j)$ with $j < i$,
- No term in r is a multiple of $\operatorname{lt}(f_i)$ for any i.

Buchberger's criterion is stated in this setting as follows: An (ordered) finite set $G = \{g_1, \ldots, g_m\} \subset \mathcal{P}$ is a Gröbner basis, if and only if for each index i and each term t in the minimal generating set of the monomial ideal $\langle \operatorname{lt}(g_1), \ldots, \operatorname{lt}(g_{i-1}) \rangle : \operatorname{lt}(g_i)$, the division of tg_i by G yields zero as remainder. Based on this result, we can now describe a variant of Buchberger's algorithm to compute Gröbner bases. In Algorithm 1, `Division`$(f, [f_1, \ldots, f_k])$ returns the remainder of the division of f by the list $[f_1, \ldots, f_k]$ by applying the above procedure. In addition, $\mathrm{G}(\mathcal{I})$ denotes the minimal generating set of the monomial ideal $\mathcal{I}$.

One of the advantages of the Berkesch–Schreyer approach, compared to the classical Buchberger's theory, is that one can give a simpler proof of the Schreyer theorem (Berkesch and Schreyer, 2015, Corollary 1.11): Keeping the above notations, there are $h_{ij} \in \mathcal{P}$ such that $tg_i = h_{i1} g_1 + \cdots + h_{im} g_m$. Then, the set of all syzygies

Algorithm 1: Berkesch–Schreyer Variant of Buchberger Algorithm

Data: A finite list of polynomials $F = [f_1, \dots, f_k]$ and a term ordering $\prec$
Result: A Gröbner basis G of $\langle F \rangle$
begin
    $G \longleftarrow F$
    $P \longleftarrow \{x^\mu f_i \mid x^\mu \in \mathrm{G}(\langle \mathrm{lt}(f_1), \dots, \mathrm{lt}(f_{i-1}) \rangle : \mathrm{lt}(f_i)), i = 2, \dots, k\}$
    while $P \neq \emptyset$ **do**
        Select and remove a product $x^\mu f_i$ from P
        $r \longleftarrow$ `Division`$(x^\mu f_i, G)$
        if $r \neq 0$ **then**
            $P \longleftarrow P \cup \{x^\mu r \mid x^\mu \in \mathrm{G}(\langle \mathrm{lt}(G) \rangle : \mathrm{lt}(r))\}$
            $G \longleftarrow$ `append`(G, r)
    return G

$t\mathbf{e}_i - h_{i1}\mathbf{e}_1 - \cdots - h_{im}\mathbf{e}_m$ for each i and for any choice of t forms a Gröbner basis for the syzygy module of $g_1, \dots, g_m$ with respect to the induced Schreyer ordering.

Next, we recall some for us relevant concepts for involutive divisions and bases, see (Gerdt, 2005; Seiler, 2010) for more details. We provide here a non-standard formulation of the basic definitions adapted to our later extension to involutive-like divisions and bases in Section 6.

Definition 2.1. An *involutive division* $\mathcal{L}$ on $\mathcal{T} \subset \mathcal{P}$ associates to any finite set $U \subset \mathcal{T}$ of terms and any term $u \in U$ a set of *$\mathcal{L}$-non-multipliers* $\bar{\mathcal{L}}(u, U)$ given by the terms contained in a prime monomial ideal. The variables generating this prime ideal are called the *non-multiplicative variables*

$\mathrm{NM}_{\mathcal{L}}(u, U) \subseteq X$ of $u \in U$. The set of *$\mathcal{L}$-multipliers* $\mathcal{L}(u, U)$ is given by the order ideal $\mathcal{T} \setminus \bar{\mathcal{L}}(u, U)$; it has as Dickson basis the set of *multiplicative variables* $\mathrm{M}_{\mathcal{L}}(u, U) = X \setminus \mathrm{NM}_{\mathcal{L}}(u, U)$. For any term $u \in U$, its *involutive cone* is defined as $\mathcal{C}_{\mathcal{L}}(u, U) = u \cdot \mathcal{L}(u, U)$. For an involutive division, the involutive cones must satisfy the following conditions:

(i) For two terms $v \neq u \in U$ with $\mathcal{C}_{\mathcal{L}}(u, U) \cap \mathcal{C}_{\mathcal{L}}(v, U) \neq \emptyset$, we have $u \in \mathcal{C}_{\mathcal{L}}(v, U)$ or $v \in \mathcal{C}_{\mathcal{L}}(u, U)$.
(ii) If a term $v \in U$ lies in an involutive cone $\mathcal{C}_{\mathcal{L}}(u, U)$, then $\mathcal{L}(v, U) \subset \mathcal{L}(u, U)$.
(iii) For any term u in a subset $V \subset U$, we have $\mathcal{L}(u, U) \subseteq \mathcal{L}(u, V)$.

We write $u \mid_{\mathcal{L}} w$ for a term $u \in U$ and an arbitrary term $w \in \mathcal{T}$, if $w \in \mathcal{C}_{\mathcal{L}}(u, U)$. In this case, u is called an *$\mathcal{L}$-involutive divisor* of w and w an *$\mathcal{L}$-involutive multiple* of u.

The first two conditions ensure that involutive cones can intersect only trivially. The third condition is often called the *filter axiom*. Obviously, it suffices for defining an involutive division to say what are the (non-)multiplicative variables for each term u in a finite set U. Note that involutive divisibility $u \mid_{\mathcal{L}} w$ implies ordinary divisibility, but not vice versa.

Definition 2.2. For a finite set of terms $U \subset \mathcal{T}$ and an involutive division $\mathcal{L}$ on $\mathcal{T}$, the *involutive span* of U is the union $\mathcal{C}_{\mathcal{L}}(U) = \bigcup_{u \in U} \mathcal{C}_{\mathcal{L}}(u, U)$. The set U is *involutively* *complete* or a *weak involutive basis*, if $C_{\mathcal{L}}(U) = U \cdot \mathcal{T}$. For a *(strong) involutive basis* the union is disjoint, i. e. every term in $C_{\mathcal{L}}(U)$ has a unique involutive divisor.

Example 2.3. One of the most important involutive divisions is the *Janet division* introduced by Janet (1929, pp. 16-17). Let $U \subset \mathcal{P}$ be a finite set of terms. For each sequence $d_1, \ldots, d_n$ of non-negative integers and for each index $1 \leq i \leq n$, we introduce the corresponding *Janet class* as the subset

$$U_{[d_i,\ldots,d_n]} = \{u \in U \mid \deg_j(u) = d_j,\ i \leq j \leq n\} \subseteq U\ . \qquad (2.1)$$

The variable x_n is *Janet multiplicative* (or shorter $\mathcal{J}$-multiplicative) for the term $u \in U$, if $\deg_n(u) = \max\{\deg_n(v) \mid v \in U\}$. For $i < n$ the variable x_i is Janet multiplicative for $u \in U_{[d_{i+1},\ldots,d_n]}$, if $\deg_i(u) = \max\{\deg_i(v) \mid v \in U_{[d_{i+1},\ldots,d_n]}\}$.

Definition 2.4. The set $F \subset \mathcal{P}$ of polynomials is *involutively head autoreduced* for the involutive division $\mathcal{L}$ and the term ordering $\prec$, if for no $f \in F$ there exists an $h \in F \setminus \{f\}$ with $\mathrm{lt}(h) \mid_{\mathcal{L}} \mathrm{lt}(f)$. Let $\mathcal{I} \subset \mathcal{P}$ be an ideal. An $\mathcal{L}$-involutively head autoreduced subset $G \subset \mathcal{I}$ is an *involutive basis* of $\mathcal{I}$ for $\mathcal{L}$ and $\prec$, if for any ideal element $f \in \mathcal{I}$ there exists a generator $g \in G$ such that $\mathrm{lt}(g) \mid_{\mathcal{L}} \mathrm{lt}(f)$.

If G is an involutive basis of the polynomial ideal $\mathcal{I}$, then $\mathrm{lt}(G)$ is an involutive basis of the monomial ideal $\mathrm{lt}(\mathcal{I})$. Thus any involutive basis is also a Gröbner basis.

Example 2.5. Consider the monomial ideal $\mathcal{I} = \langle x_1x_3^2,\ x_2x_3,\ x_1^2x_3\rangle \subset \mathcal{K}[x_1, x_2, x_3]$. The given minimal generating set is not a Janet basis of $\mathcal{I}$, but if we extend it to the set $\{x_1x_3^2,\ x_2x_3,\ x_1^2x_3,\ x_2x_3^2\}$, then we obtain one. One can show that any monomial ideal possesses a finite Janet basis, i. e. the Janet division is Nœtherian.

We will repeatedly use the idea of a *tree representation*[3] of a finite set U of terms. Assume that $U = \{x^{\mu_1}, \ldots, x^{\mu_k}\} \subset \mathcal{T}$ where $\mu_i = (\mu_{i1}, \ldots, \mu_{in})$ for each i and $x^{\mu_1} \prec_{lex} \cdots \prec_{lex} x^{\mu_k}$ with $x_1 \prec_{lex} \cdots \prec_{lex} x_n$. Then, to represent recursively this set as tree, we consider the root as level 0 and at the first level we write as the nodes the last entries of the μ_i's from the smallest to the largest one by removing the repeated elements. Now, assume that all the nodes at the level $n - i$ have been determined. To construct the nodes below a node μ_{ij} at the level $n - i + 1$, we represent the set $[\mu_{ij}, \ldots, \mu_{in}]$ by considering only the first $i - 1$ entries and the node corresponding to μ_{ij} as the root. As a simple example, in the polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$ let us consider the set $U = \{x_1^2x_3^3,\ x_2^4x_3^3,\ x_1^2x_3^5,\ x_2^2x_3^5\}$. Its tree representation is shown in Figure 1.

The level i in this representation corresponds to the variable x_i and one can read off the Janet multiplicative variables for any node. For example, for $x_1^2x_3^5 \in U$ the variables x_1 and x_3 are multiplicative, since the path from the root to the corresponding leave uses the respective last branch at the levels 1 and 3. However, this is not the case at level 2 and so x_2 is non-multiplicative.

The Janet trees introduced by Gerdt et al. (2001) correspond to a transformation of the above described tree into a binary tree. They are extensively used for the fast construction of Janet bases, as many necessary operations like searching for a Janet

[3] In some references, one speaks of a Janet tree, see e. g. (Seiler, 2010).

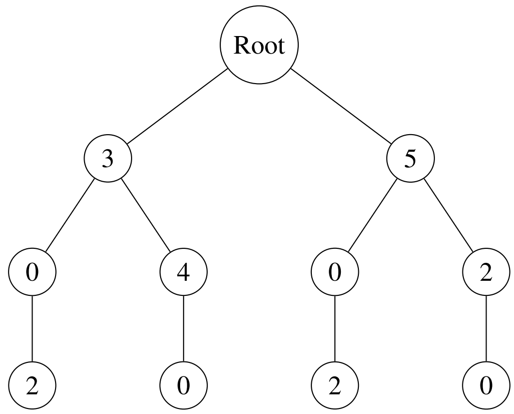

Figure 1: Tree representation of $U = \{x_1^2x_3^3,\ x_2^4x_3^3,\ x_1^2x_3^5,\ x_2^2x_3^5\}$.

divisor can be performed very efficiently with them. The bar codes introduced by Ceria (2019a) provide a similar representation of a set of terms using a two-dimensional diagram; the relation between the two representations is studied by Ceria (2019b). We refer to Lundqvist (2010) for the complexity of constructing a tree representation.

Gerdt (2005) proposed an efficient algorithm for the construction of involutive bases by a completion process where the products of elements of the current basis by non-multiplicative variables are reduced with respect to the basis. This process terminates for any Nœtherian division in finitely many steps. To further improve the computation of Gröbner bases for ideals where the Janet basis is much larger than the reduced Gröbner basis (toric ideals are a prototypical example), Gerdt and Blinkov (2005b) introduced a generalisation of Janet bases where not only non-multiplicative variables but also non-multiplicative powers are considered in the completion process.

Definition 2.6. Let $U \subset \mathcal{T}$ be a finite set of terms. For any term $u \in U$ and any index $1 \le i \le n$, we set

$$h_i(u, U) = \max\left\{\deg_i(v) \mid u, v \in U_{[d_{i+1},\dots,d_n]}\right\} - \deg_i(u)\ .$$

If $h_i(u, U) > 0$, the power $x_i^{k_i}$ with

$$k_i = \min\left\{\deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1},\dots,d_n]}, \deg_i(v) > \deg_i(u)\right\}$$

is called a *non-multiplicative* power of u for the *Janet-like division*. The set of all non-multiplicative powers of $u \in U$ is denoted by $\mathrm{NMP}(u, U)$. The elements of the set

$$\mathrm{NM}(u, U) = \{v \in \mathcal{T} \mid \exists w \in \mathrm{NMP}(u, U) : w \mid v\}$$

are called the *J-non-multipliers* for $u \in U$. The terms outside of it are the *J-multipliers* for u. An element $u \in U$ will be called a *Janet-like divisor* of $w \in \mathcal{T}$, if $w = u \cdot v$ with v a J-multiplier for u.

Although the Janet-like division is not an involutive division, it preserves all algorithmic properties of the Janet division and allows for the construction of Janet-like bases and in turn Gröbner bases. Indeed, the main algorithmic idea for the construction of Janet-like bases is similar to that of Janet bases, instead of multiplying with non-multiplicative variables one now multiplies with non-multiplicative powers.

We conclude the section by recalling the definition of the Pommaret division and the related notion of a quasi-stable monomial ideal which appears independent of involutive bases at many places in commutative algebra and algebraic geometry. Quasi-stable ideals are also known as ideals of nested type or of Borel type.

Example 2.7. The *class* of a term $x^\mu \in \mathcal{T}$ with $\mu = (\mu_1, \dots, \mu_n)$ is defined as the index $\operatorname{cls}(x^\mu) = \min\{i \mid \mu_i \neq 0\}$. A variable x_i is *P*-multiplicative for x^μ, if $i \leq k$. Note that the thus defined *Pommaret division* is global, i. e. the assignment of multiplicative variables is independent of any finite set $U \subset \mathcal{T}$. In contrast to the Janet division, the Pommaret division is not Nœtherian, as e. g. the ideal $\mathcal{I} = \langle x_1 x_2 \rangle$ does not possess a finite Pommaret basis (it does not contain an element of class 2).

For sufficiently large fields $\mathcal{K}$, this non-Nœtherianity of the Pommaret division is only a problem of the used coordinates. After a generic linear change of variables any ideal $\mathcal{I} \subseteq \mathcal{P}$ admits a finite Pommaret basis (Seiler, 2010, Thm. 4.3.15). An in-depth study of this question can be found in (Hashemi et al., 2018) together with a deterministic algorithm for the explicit construction of "good" coordinates for any given ideal $\mathcal{I} \subset \mathcal{P}$. For Pommaret bases, we will always consider the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$, as it is the only class-respecting term ordering (Seiler, 2010, Lem. A.1.8).

Definition 2.8. A monomial ideal $\mathcal{I} \subset \mathcal{P}$ is called *quasi-stable*, if for any term $x^\mu \in \mathcal{I}$ and for any index $k = \operatorname{cls}(x^\mu) < i \leq n$ an exponent $s \geq 0$ exists such that $x_i^s x^\mu / x_k \in \mathcal{I}$. A polynomial ideal $\mathcal{I} \subset \mathcal{P}$ is in *quasi-stable position*, if $\operatorname{lt}(\mathcal{I})$ is quasi-stable.

One easily verifies that it suffices to consider in the definition of a quasi-stable ideal $\mathcal{I}$ only the terms x^μ in an arbitrary finite monomial generating set of $\mathcal{I}$. The notion of quasi-stability is closely related to the existence of finite Pommaret bases.

Proposition 2.9 ((Seiler, 2010, Prop. 5.3.4))**.** *A monomial ideal $\mathcal{I}$ possesses a finite Pommaret basis, if and only if it is quasi-stable.*

3. A recursive Janet basis test

Janet (1920, page 86) reported the following recursive criterion for a Janet basis as a consequence of a lengthy discussion of the properties of the Janet division (see also (Ceria, 2019, Cor. 4.11) from where we learned of this result). We will provide below a new proof for an improved variant.

Theorem 3.1. *Let $U = \{t_1, \dots, t_m\} \subset \mathcal{T}$ be a finite set of terms. We define $t_i' = t_i|_{x_n=1}$ for all i and $U' = \{t_1', \dots, t_m'\} \subset \mathcal{K}[x_1, \dots, x_{n-1}]$. If $\alpha = \max\{\deg_n(t_1), \dots, \deg_n(t_m)\}$, then we introduce for each degree $\lambda \leq \alpha$ the sets $I_\lambda = \{i \mid \deg_n(t_i) = \lambda\}$ and $U_\lambda' = \{t_i' \mid i \in I_\lambda\}$. Then, U is a Janet basis, if and only if the following two conditions are satisfied:*

(i) *For each $\lambda \le \alpha$ the set U'_λ is a Janet basis in $\mathcal{K}[x_1, \ldots, x_{n-1}]$.*
(ii) *Each term $t'_i \in U'_\lambda$ with $\lambda < \alpha$ lies in the Janet span of $U'_{\lambda+1}$.*

Example 3.2. In the polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$, we consider the following set of terms $U = \{x_2x_3^3,\ x_1^2x_3^3,\ x_2x_3^2,\ x_2^2x_3,\ x_1^3x_2x_3\}$. One observes that:
1. $U'_3 = \{x_2, x_1^2\}$, $M(x_2, U'_3) = \{x_1, x_2\}$ and $M(x_1^2, U'_3) = \{x_1\}$,
2. $U'_2 = \{x_2\}$ and $M(x_2, U'_2) = \{x_1, x_2\}$,
3. $U'_1 = \{x_2^2, x_1^3x_2\}$, $M(x_2^2, U'_1) = \{x_1, x_2\}$ and $M(x_1^3x_2, U'_1) = \{x_1\}$.

One readily checks that all sets U'_λ are Janet bases. In addition, we can see that $x_2^2, x_1^3x_2 \in U'_1$ belong to the Janet span of U'_2 and $x_2 \in U'_2$ lies in the Janet span of U'_3. Thus, the set U is a Janet basis by Janet's theorem.

We will improve Janet's Theorem 3.1 by a slight modification: instead of the Janet span, we use in (ii) the ordinary span which makes the condition easier to verify. For its proof, we shall need the following lemma which follows immediately from the definition of the Janet division.

Lemma 3.3. *In the situation of Theorem 3.1, for each term $t_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$ and for each variable x_j we have*
(i) *if $j = n$, then x_n is Janet non-multiplicative for t, if and only if $\alpha_n < \alpha$,*
(ii) *if $j < n$, then x_j is Janet non-multiplicative for $t_i \in U$, if and only if it is Janet non-multiplicative for $t'_i \in U'_\lambda$ with $\lambda = \alpha_n$.*

Theorem 3.4. *In the situation of Theorem 3.1, let $\beta = \min\{\deg_n(t_1), \ldots, \deg_n(t_m)\}$. Then, U is a Janet basis, if and only if the following conditions are satisfied:*
(i) *For each $\lambda \le \alpha$, U'_λ is a Janet basis in $\mathcal{K}[x_1, \ldots, x_{n-1}]$.*
(ii) *For each $\beta \le \lambda < \alpha$, we have $U'_\lambda \subset \langle U'_{\lambda+1} \rangle$.*

Proof. It is easy to see that any Janet basis U satisfies the given condition: the first one holds, as multiplying a term with a non-multiplicative variable x_j with $j < n$ does not lead outside the Janet span of U and the second one holds, as the same is true for $j = n$.

For the converse, consider a term $t_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$. Assume that x_j is Janet non-multiplicative for t_i. We distinguish two cases. If $j = n$, then $\lambda := \alpha_n < \alpha$. Since by (i) $U'_{\lambda+1}$ is a Janet basis of the ideal it generates and $t'_i \in U'_\lambda$ lies in this ideal by (ii), we may conclude that x_nt_i lies in the Janet span of the set $\{tx_n^{\lambda+1} \mid t \in U'_{\lambda+1}\} \subseteq U$. By Lemma 3.3, x_nt_i is thus in the Janet span of U as required for a Janet basis. If $j < n$, then, by Lemma 3.3, we know that x_j remains Janet non-multiplicative for $t'_i \in U'_\lambda$. Since U'_λ is a Janet basis, $x_jt'_i$ has an involutive divisor in U'_λ implying again by Lemma 3.3 that x_jt_i lies in the Janet span of $\{tx_n^\lambda \mid t \in U'_\lambda\}$ and thus of U. □

Example 3.5. We consider again the set U of Example 3.2. There we showed already that all sets U'_λ are Janet bases. One can see by direct inspection without determining any multiplicative variables that we have $U'_\lambda \subset \langle U'_{\lambda+1} \rangle$ for all $1 \le \lambda < 3$ and this shows that U is a Janet basis.

The criterion provided by Theorem 3.4 translate immediately into the simple recursive Algorithm 2 testing whether a monomial set is a Janet basis.

Theorem 3.6. *Algorithm 2 terminates in finitely many steps and is correct. Moreover, its arithmetic complexity is $O(dnm^2)$ where $d \ge 2$ denotes the average of the differences*

Algorithm 2: JanetTest

Data: A polynomial ring $\mathcal{P} = \mathcal{K}[x_1, \ldots, x_n]$ with n variables and a finite set $U = \{t_1, \ldots, t_m\} \subset \mathcal{P}$ of terms.

Result: True if U is a Janet basis for the ideal it generates in $\mathcal{P}$ and false otherwise.

begin
- $\alpha \longleftarrow \max\{\deg_n(t_1), \ldots, \deg_n(t_m)\}$
- **if** $n = 1$ **then**
  - $\beta \longleftarrow \min\{\deg(t_1), \ldots, \deg(t_m)\}$
  - **if** $\exists \beta < i < \alpha$ *with* $x_n^i \notin U$ **then**
    - **return** $(false)$
  - **else**
    - **return** $(true)$
- $\beta \longleftarrow \min\{\deg_n(t_1), \ldots, \deg_n(t_m)\}$
- **for** $i = \beta, \ldots, \alpha$ **do**
  - $U'_i \longleftarrow \{t \in \mathcal{K}[x_1, \ldots, x_{n-1}] \mid t \cdot x_n^i \in U\}$
- **if** `JanetTest`$(\mathcal{K}[x_1, \ldots, x_{n-1}], U'_\alpha) \neq true$ **then**
  - **return** $(false)$
- **for** $i = \alpha - 1, \ldots, \beta$ **do**
  - **if** `JanetTest`$(\mathcal{K}[x_1, \ldots, x_{n-1}], U'_i) \neq true$ **then**
    - **return** $(false)$
  - **else if** $\exists t \in U'_i \setminus \langle U'_{i+1} \rangle$ **then**
    - **return** $(false)$
- **return** $(true)$

between the maximal and minimal degrees of the elements of U with respect to each of the variables.

Proof. The correctness follows directly from Theorem 3.4 and the termination is trivial. To prove the complexity bound, we first note that using (Lundqvist, 2010, Thm. 4.2), one can construct the tree representation corresponding to the exponent vectors of the elements of U by using $O(m^2 + nm)$ comparisons. Suppose that $t_i = x^{\mu_i}$ with $\mu_i = (\mu_{i1}, \ldots, \mu_{in})$. Now, assume that we are given the tree representation of $\{\mu_1, \ldots, \mu_m\}$. Without loss of generality, we may assume that the cardinality of U'_i for each i is m/d. To check one inclusion $U'_\lambda \subset \langle U'_{\lambda+1} \rangle$, we need nm/d comparisons for the membership test of an element of U'_λ and thus in all we need nm^2/d^2 operations. It follows that to test the chain of inclusions $\langle U'_\beta \rangle \subseteq \cdots \subseteq \langle U'_\alpha \rangle$ we need $O(nm^2/d)$ operations. Therefore, by taking into account the fact that $d \geq 2$, the Janet test on U may be done within $O(nm^2/d + nm^2/d^2 + \cdots + nm^2/d^n) = O(dnm^2)$, which proves the claim. □

Remark 3.7. It is worth noting that the naive Janet test for the set $U = \{t_1, \ldots, t_m\}$ needs $O(n^2m^2)$ comparisons. Indeed, the tree representation corresponding to U is constructed within $O(m^2 + nm)$ comparisons. Using this representation, one is able to read off the non-multiplicative variables for each term $t_i \in U$. Having at most n non-multiplicative variables for each terms, one needs to perform nm operations to test whether a non-multiplicative product has a Janet divisor. Thus, all in all, we need $O(n^2m^2)$ comparisons for the Janet test of U. This shows that in the case that $n \gg d$ Algorithm 2 is more efficient than the classic approach. Note that the case $d \gg n$ is e. g. typical for toric ideals and it is well-known that in this case involutive bases are generally highly redundant, i. e. much larger than reduced Gröbner bases, and therefore should be avoided anyway.

Example 3.8. We illustrate the steps of Algorithm 3.4 for the set

$$U = \{x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3\} \subset \mathcal{K}[x_1, x_2, x_3]\ .$$

One obtains at the first recursion level the following sets in $\mathcal{K}[x_1, x_2]$ and Janet multiplicative variables of their elements:

1. $U'_3 = \{x_1^3, x_1^2\}$ with $M(x_1^3, U'_3) = \{x_1, x_2\}$ and $M(x_1^2, U'_3) = \{x_1\}$,
2. $U'_2 = \{x_1^2x_2^2\}$ with $M(x_1^2x_2^2, U'_2) = \{x_1, x_2\}$,
3. $U'_1 = \{x_1^2x_2^2, x_1x_2^2, x_2^2\}$ with $M(x_1^2x_2^2, U'_1) = \{x_1, x_2\}, M(x_1x_2^2, U'_1) = \{x_1\}$ and $M(x_2^2, U'_1) = \{x_1\}$.

These multiplicative variables show that U'_1, U'_2 and U'_3 are Janet bases. On the other hand, since we have $x_2^2 \in U'_1 \setminus \langle U'_2\rangle$, the algorithm returns correctly false, as indeed the non-multiplicative product $x_3 \cdot x_2^2x_3$ does not lie in the Janet span of U.

Since for the Janet division any monomial set is involutively autoreduced, the notion of a minimal Janet basis is crucial for efficiency reasons. We now adapt Theorem 3.4 to a test whether or not a given Janet basis is minimal.

Definition 3.9. An $\mathcal{L}$-involutive (or a Janet-like) basis $U \subset \mathcal{P}$ is called *minimal*, if no proper subset of U is an $\mathcal{L}$-involutive (or a Janet-like) basis of the ideal $\langle U\rangle$.

Theorem 3.10. *With the notations of Theorem 3.1, let U be a Janet basis for the ideal it generates. Then, U is minimal, if and only if the following conditions are satisfied:*

(i) For each $\lambda \leq \alpha$, U'_λ is a minimal Janet basis.
(ii) We have $\langle U'_{\alpha-1}\rangle \neq \langle U'_\alpha\rangle$.

Proof. Suppose that U is a minimal Janet basis. Then U'_λ is trivially a minimal Janet basis for each $\lambda \leq \alpha$, cf. Theorem 3.4. Now, assume that $\langle U'_{\alpha-1}\rangle = \langle U'_\alpha\rangle$. Since $U'_{\alpha-1}$ is a Janet basis by Theorem 3.4, $U \setminus \{t_i \mid t'_i \in U'_\alpha\}$ remains a Janet basis for $\langle U\rangle$, contradicting the minimality of U.

Conversely, assume that the properties (i) and (ii) hold for U, but that there exists a proper subset $V \subset U$ defining a minimal Janet basis for $\langle U\rangle$. Let x^μ be any element of $U \setminus V$. Then there exists a term $x^\nu \in V$ which involutively divides x^μ; we write $x^\mu = x^\eta x^\nu$. Assume that x_ℓ is the largest variable appearing in x^η. This implies that the two terms x^μ and x^ν lie in the same Janet class $U_{[\nu_{\ell+1},\dots,\nu_n]}$ with $\nu = (\nu_1, \dots, \nu_n)$. For each index i, write $u_i = t_i|_{x_{\ell+1}=\cdots=x_n=1}$ and define the set $W = \{u_i \mid t_i \in U_{[\nu_{\ell+1},\dots,\nu_n]}\} \subset \mathcal{K}[x_1, \dots, x_\ell]$. Applying property (i) recursively $n - \ell$ times to U, we see that W is a minimal Janet basis. Let γ be the largest x_ℓ-degree of a term $u_i \in W$. Then, similar to the notations above, we introduce the sets $W', W'_0, \dots, W'_\gamma$ and find $x_1^{\nu_1} \cdots x_{\ell-1}^{\nu_{\ell-1}} \in W'_{\nu_\ell}$ and $\nu_\ell < \mu_\ell \leq \gamma$. Furthermore, V cannot contain any element whose image under the map $\phi(u) = u|_{x_\ell=\cdots=x_n=1}$ lies in one of the sets $W'_{\nu_\ell+1}, \dots, W'_\gamma$, as otherwise x_ℓ could not be multiplicative for x^ν. This shows that $W'_{\nu_\ell+1}, \dots, W'_\gamma \subset \langle W'_{\nu_\ell} \rangle$. On the other hand, Theorem 3.4 entails that $\langle W'_{\nu_\ell} \rangle \subset \cdots \subset \langle W'_\gamma \rangle$ and in particular we get $\langle W'_{\gamma-1} \rangle = \langle W'_\gamma \rangle$, showing that property (ii) does not hold for the minimal Janet basis W, a contradiction. Hence, no proper subset of U can be a Janet basis of $\langle U \rangle$ and U is minimal. □

Theorem 3.10 leads immediately to the recursive Algorithm 3 for turning an arbitrary Janet basis into the minimal one. To the best of our knowledge, it represents the first such minimisation algorithm, as alternative approaches like the TQ-algorithm of Gerdt and Blinkov (1998) determine a minimal Janet basis directly from an arbitrary generating set and cannot exploit the knowledge of a non-minimal basis. It suffices that we describe the algorithm for monomial ideals, as also for a polynomial Janet basis the minimisation process depends only on the leading terms.

Algorithm 3: MinimalJanetBasis

Data: A polynomial ring $\mathcal{P} = \mathcal{K}[x_1, \dots, x_n]$ with n variables and a Janet basis $U = \{t_1, \dots, t_m\} \subset \mathcal{T}$.
Result: The minimal Janet basis of the ideal $\langle U \rangle$.

```
begin
    α ⟵ max {deg_n (t_1), ..., deg_n (t_m)}
    if n = 1 then
        β ⟵ min {deg (t_1), ..., deg (t_m)}
        return ({x_1^β})
    V ⟵ ∅
    β ⟵ min {deg_n (t_1), ..., deg_n (t_m)}
    for i = β, ..., α do
        U'_i ⟵ {t ∈ K[x_1, ..., x_{n-1}] | t · x_n^i ∈ U}
        U'_i ⟵ MinimalJanetBasis(K[x_1, ..., x_{n-1}], U'_i)
        V ⟵ V ∪ {t x_n^i | t ∈ U'_i}
    for i = α, ..., β + 1 do
        if ⟨U'_i⟩ = ⟨U'_{i-1}⟩ then
            V ⟵ V \ {t x_n^i | t ∈ U'_i}
        else
            return (V)
    return (V)
```

Theorem 3.11. *Algorithm 3 terminates in finitely many steps and is correct. Its arithmetic complexity is $O(dnm^2)$ with $d \geq 2$ the average difference between the maximal and minimal degrees of the elements of U with respect to each of the variables.*

Proof. The correctness follows by Theorem 3.10 and the termination is obvious. The complexity bound is obtained similarly to the proof of Theorem 3.6. □

Remark 3.12. The complexity bound presented in Remark 3.7 remains true for a naive algorithm to compute minimal Janet bases.

Example 3.13. We demonstrate the working of Algorithm 3 for the Janet basis

$$U = \{x_1x_2^2x_3^3,\ x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^2x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3\}\ . \tag{3.1}$$

Its tree representation can be seen in Figure 3.13.

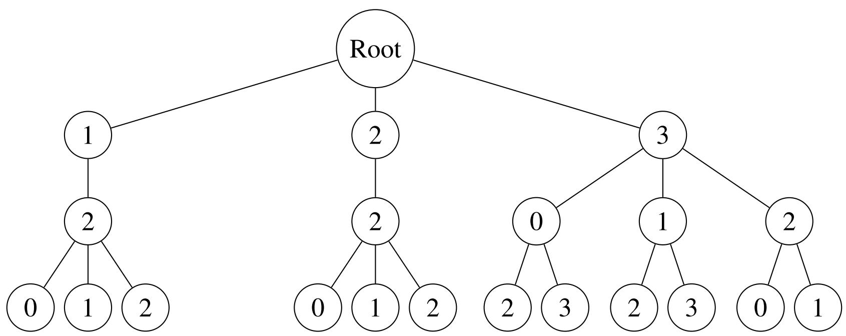

Figure 2: Tree representation of the Janet basis (3.1)

We consider the subset $V := U_3 = \{x_1x_2^2, x_2^2, x_1^3x_2, x_1^2x_2, x_1^3, x_1^2\}$. Then, we get $W := V_2 = \{1, x_1\}$. Finally, applying Theorem 3.10, we have $W_0 = W_1 = \{1\}$ which shows that W is not a minimal Janet basis. Following the structure of the algorithm, in order to minimise W, we must remove the branch W_1. This shows that we shall delete $x_1x_2^2$ from V and in turn $x_1x_2^2x_3^3$ from U. In the same way and by eliminating the extra terms from U, we see that $\{x_2^2x_3^3, x_1^2x_2x_3^3, x_1^2x_3^3, x_2^2x_3^2, x_2^2x_3\}$ is the minimal Janet basis of $\langle U\rangle$. Its tree representation shown in Figure 3 is obviously a subtree.

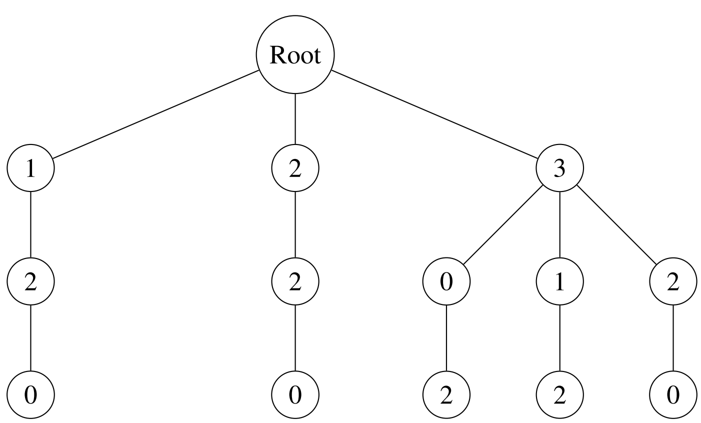

Figure 3: Tree representation of the minimal Janet basis

Janet's criterion can be generalised to Janet-like bases. First, we introduce some notations. If $U = \{t_1, \ldots, t_m\}$ is a set of terms, then there exist natural numbers $\beta \leq \alpha$ and a sequence of natural numbers $\lambda_0, \ldots, \lambda_\ell$ with ℓ depending on U such that each λ_i is the x_n-degree of some term $t_j \in U$ and such that conversely for each $t_j \in U$ there is a λ_i which is the x_n-degree of t_j.

Theorem 3.14. *Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{P}$ be a set of terms and let $\beta = \lambda_0 < \lambda_1 < \cdots < \lambda_\ell = \alpha$ be natural numbers encoding the x_n-degrees appearing in U. For each index $0 \leq i \leq \ell$, let $U_{\lambda_i} \subseteq U$ be the subset of terms of U having x_n-degree λ_i and set $U'_{\lambda_i} = \{t/x_n^{\lambda_i} \mid t \in U_{\lambda_i}\}$. Then U is a Janet-like basis of the ideal it generates, if and only if the following two conditions are satisfied:*

(i) Every set U'_{λ_i} is a Janet-like basis of the monomial ideal $\langle U'_{\lambda_i} \rangle \subseteq \mathcal{K}[x_1, \ldots, x_{n-1}]$.

(ii) For each $0 \leq i < \ell$, the inclusion $U'_{\lambda_i} \subset \langle U'_{\lambda_{i+1}} \rangle$ holds.

Proof. The necessity of the first condition follows from the observation that $x_k^{p_k}$ with $1 \leq k < n$ and $p_k \geq 0$ is a Janet-like non-multiplicative power of $t \in U$, if and only if it is a Janet-like non-multiplicative power of $t' \in U'_{\deg_n(t)} \subset \mathcal{K}[x_1, \ldots, x_{n-1}]$. The second condition is entailed by the fact that for each $0 \leq i < \ell$ the Janet-like non-multiplicative power of x_n of the term $t \in U_{\lambda_i}$ is exactly $x_n^{\lambda_{i+1}-\lambda_i}$ and that the product $x_n^{\lambda_{i+1}-\lambda_i} t$ can only be contained in the Janet-like span of U, if it lies in the Janet-like cone of a term $s \in U_{\lambda_{i+1}}$.

For the proof of the sufficiency of the two conditions, a main ingredient is again the observation that $x_k^{p_k}$ with $1 \leq k < n$ and $p_k \geq 0$ is a Janet-like non-multiplicative power of $t \in U$, if and only if it is a Janet-like non-multiplicative power of $t' \in U'_{\deg_n(t)} \subset \mathcal{K}[x_1, \ldots, x_{n-1}]$. We must check that all products by non-multiplicative powers are contained in the Janet-like span of U. We first consider Janet-like non-multiplicative powers of the form above: $x_k^{p_k}$ with $1 \leq k < n$. Let $x_k^{p_k} t$ be a product resulting from such a power. Then the x_n-degrees of t and its product are equal, say, to λ_i. Since we have in the polynomial subring with $n-1$ variables the relation $x_k^{p_k} t / x_n^{\lambda_i} \in \langle U'_{\lambda_i} \rangle$, we see by the first condition that $x_k^{p_k} t / x_n^{\lambda_i}$ is in the Janet-like span of U'_{λ_i}. But this implies easily that $x_k^{p_k} t$ is also in the Janet-like span of U in the polynomial ring with n variables.

We finally consider the Janet-like non-multiplicative powers of the form $x_n^{p_n}$. For them, there exists some index i with $0 \leq i < \ell$ such that $p_n = \lambda_{i+1} - \lambda_i$ and such that this non-multiplicative power belongs to a term $t \in U_{\lambda_i}$. By the second condition, we have in the polynomial subring with $n-1$ variables the relation $x_n^{p_n} t / x_n^{\lambda_{i+1}} \in \langle U'_{\lambda_{i+1}} \rangle$. By the first condition, $x_n^{p_n} t / x_n^{\lambda_{i+1}}$ is in the Janet-like span of $U'_{\lambda_{i+1}}$. It is easy to see that then $x_n^{p_n} t$ is in the Janet-like span of U. □

Example 3.15. We consider the set $U = \{x_1^2 x_3^3,\ x_1^2 x_2^2 x_3^3,\ x_2^4 x_3^3,\ x_1^2 x_3^5,\ x_2^2 x_3^5\}$ in the polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$. Evaluating the x_n-degrees appearing in U, we see that, in the terminology of Theorem 3.14, $\beta = 3 = \lambda_0 < \lambda_1 = 5 = \alpha$. We first check that the sets U'_{λ_i} are Janet-like complete:

1. $U'_5 = \{x_1^2, x_2^2\}$. Only one non-multiplicative power exists: $\mathrm{NMP}(x_1^2, U'_5) = \{x_2^2\}$. The product $x_1^2 x_2^2$ is in the Janet-like cone of x_2^2 so that U'_5 is Janet-like complete.
2. $U'_3 = \{x_1^2, x_1^2 x_2^2, x_2^4\}$. The term x_2^4 does not have non-multiplicative powers. Furthermore, $\mathrm{NMP}(x_1^2, U'_3) = \{x_2^2\}$. The corresponding product is already contained

in U'_3. Finally, $\mathrm{NMP}(x_1^2x_2^2, U'_3) = x_2^2$ and the corresponding product is in the Janet-like cone of x_2^4. Hence, U'_3 is a Janet-like basis.

In addition, we have $U'_3 = \{x_1^2, x_1^2x_2^2, x_2^4\} \subset \langle U'_5\rangle = \langle x_1^2, x_2^2\rangle$ and thus the given set U is a Janet-like basis by Theorem 3.14.

Example 3.16. Consider the set $V = \{x_2^2x_3^5,\ x_2^4x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^3\}$ obtained by removing the term $x_1^2x_3^5$ from the set U of Example 3.15. We still find $\beta = \lambda_0 = 3 < 5 = \lambda_1 = \alpha$. The singleton set $V'_5 = \{x_2^2\}$ is obviously Janet-like complete; the set V'_3 equals the set U'_3 of Example 3.15 and thus is complete. However, we have $V'_3 \nsubseteq \langle V'_5\rangle$, since $x_1^2 \notin \langle x_2^2\rangle$. Thus V is not Janet-like complete by Theorem 3.14. Moreover, observe that $\mathrm{NMP}(x_1^2x_3^3, V) = \{x_3^2\}$ and that the corresponding product is the eliminated term $x_1^2x_3^5$, which is not contained in the Janet-like span of V. Thus, one can compute the Janet-like completion U of V by adding this product to V.

Theorem 3.17. *Keeping the notations of Theorem 3.14, let U be a Janet-like basis for the ideal it generates. Then, U is minimal, if and only if the following conditions are satisfied:*

(i) For each $i \leq \ell$, U'_{λ_i} is a minimal Janet-like basis.

(ii) For each $i < \ell$, we have $\langle U'_{\lambda_i}\rangle \neq \langle U'_{\lambda_{i+1}}\rangle$.

Proof. The proof is similar to the one of Theorem 3.10. If U is a minimal Janet-like basis, then it is clear that for each $i \leq \ell$, U'_{λ_i} is a minimal Janet-like basis, see Theorem 3.14. To prove the second condition, assume that $\langle U'_{\lambda_i}\rangle = \langle U'_{\lambda_{i+1}}\rangle$ for some $i < \ell$. Then, $U \setminus \{t'_i x_n^{\lambda_{i+1}} \mid t'_i \in U'_{\lambda_{i+1}}\}$ is a Janet-like basis as well, contradicting the minimality of U.

Conversely, assume that the properties (i) and (ii) hold for U, but that there exists a proper subset $V \subset U$ forming a minimal Janet-like basis for $\langle U\rangle$. Let x^μ be any element of $U \setminus V$. There must exist $x^\nu \in V$ which divides x^μ for the Janet-like division; we write $x^\mu = x^\eta x^\nu$. Assume that x_ℓ is the largest variable appearing in x^η. This implies that x^μ and x^ν lie in the same Janet class $U_{[\nu_{\ell+1},\ldots,\nu_n]}$ with $\nu = (\nu_1, \ldots, \nu_n)$. For each index i let $u_i = t_i|_{x_{\ell+1}=\cdots=x_n=1}$ and set $W = \{u_i \mid t_i \in U_{[\nu_{\ell+1},\ldots,\nu_n]}\} \subset \mathcal{K}[x_1, \ldots, x_\ell]$. Applying recursively property (i) $n - \ell$ times to U, we see that W is a minimal Janet-like basis. Let δ be the largest x_ℓ-degree of a term $u_i \in W$. Then, similar to the notations above, we can introduce the sets $W', W'_{\gamma_0}, \ldots, W'_{\gamma_t}$ with $\gamma_t = \delta$. Thus $x_1^{\nu_1}\cdots x_{\ell-1}^{\nu_{\ell-1}} \in W'_{\nu_\ell}$ and $\nu_l < \delta$. We set $u = x_1^{\nu_1}\cdots x_{\ell-1}^{\nu_{\ell-1}}$ and $v = x_1^{\mu_1}\cdots x_{\ell-1}^{\mu_{\ell-1}}$ with $\mu = (\mu_1, \ldots, \mu_n)$ and assume that $\gamma_i = \nu_\ell$. Then, two cases may occur. If $v \in W'_{\gamma_{i+1}}$, then we can remove the terms in U whose images lie in $W'_{\gamma_{i+1}}$ and this shows that $W'_{\gamma_i} = W'_{\gamma_{i+1}}$ which contradicts property (ii). Otherwise, v belongs to W'_{γ_j} with $j > i + 1$. By Theorem 3.14, we have $u \in \langle W'_{\gamma_j}\rangle$ and thus W'_{γ_j} is not minimal contradicting property (i). Therefore, no proper subset of U can be a Janet-like basis of $\langle U\rangle$ and U is minimal. □

Example 3.18. We consider the Janet-like basis U given in Example 3.15 and verify if it is a minimal Janet-like basis. The tree representation of U is shown in Figure 4. We observe that $V := U'_3 = \{x_1^2, x_1^2x_2^2, x_2^4\}$ and check whether it is minimal or not. We know that $V'_0 = \{x_1^2\}$, $V'_2 = \{x_1^2\}$ and $V'_4 = \{1\}$. Since $\langle V'_0\rangle = \langle V'_2\rangle$, U is not a minimal Janet-like basis. It follows that we get the minimal Janet-like basis, if we remove the useless branch $x_1^2x_2^2x_3^3$.

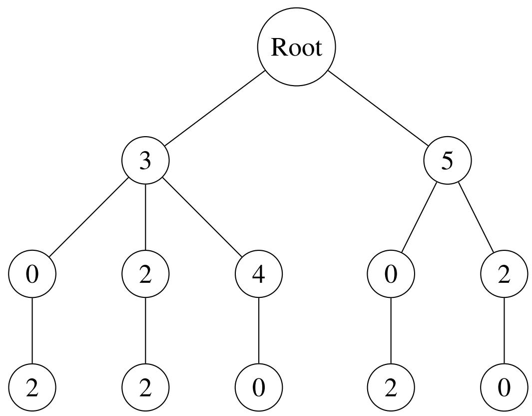

Figure 4: Tree representation of U

4. Janet Completion Procedure

We now show how our results from the previous section allow us to design a variant of the Berkesch–Schreyer algorithm for Gröbner bases (Berkesch and Schreyer, 2015) which can compute Janet(-like) bases. We take the lexicographic term ordering induced by $x_1 \prec_{lex} \cdots \prec_{lex} x_n$. Let $U := (t_1, \ldots, t_m) \in \mathcal{T}^m$ be a sequence of terms such that we have $t_m \prec_{lex} \cdots \prec_{lex} t_1$. We associate to U the following $(m-1)$-tuple $Q(U)$ of terms:

$$Q(U) := (t_{m-1} : t_m, \ldots, t_1 : t_m) .$$

Here $u : v = \mathrm{lcm}(u, v)/v$ for any two terms u and v. In addition, using the tree representation of U, one can compute $Q(U)$ efficiently. However, since such questions are not the main subject of this work, we do not give further details. The tuple $Q(U)$ is related to the Janet non-multiplicative variables of the term t_m. Assume that $Q(U) = (u_1, \ldots, u_{m-1})$. We know that there exist a positive integer r, indices $1 \le a_1 < a_2 < \cdots < a_r \le n$ and indices $1 = b_1 < b_2 < \cdots < b_r < b_{r+1} = m-1$ such that the highest variable dividing u_j is x_{a_ℓ} where $\ell \in \{1, \ldots, r\}$ and $b_\ell \le j < b_{\ell+1}$. Moreover, for indices $b_\ell \le j_1 < j_2 < b_{\ell+1}$, we have $\deg_{a_\ell}(u_{j_1}) \le \deg_{a_\ell}(u_{j_2})$. With these notations, we obtain the following assertion.

Lemma 4.1. $NM_{\mathcal{J}}(t_m, \{t_1, \ldots, t_m\}) = \{x_{a_1}, \ldots, x_{a_r}\}$.

Proof. We first show that each x_{a_i} is Janet non-multiplicative for t_m. Since x_{a_i} is the highest variable appearing in the quotients $u_{b_i}, \ldots, u_{b_{i+1}}$, we have $t_m \in [d_{a_i+1}, \ldots, d_n]$ where $d_j = \deg_j(t_m)$ and $\deg_{a_i}(t_m)$ is not maximal among the x_{a_i}-degree of the elements of this set. Thus, using the fact that the sequence of $t_1, \ldots, t_m$ is sorted in lexicographical order, x_{a_i} is Janet non-multiplicative for t_m. Conversely, assume that x_ℓ is Janet non-multiplicative for t_m. Then we have $t_m \in [d_{\ell+1}, \ldots, d_n]$. We know that this set is non-empty and we can choose an element $t \neq t_m$ from this set. Hence, x_ℓ is the highest variable appearing in the quotient $t : t_m$ and this end the proof. □

This result induces a partition of the remaining terms $\{t_1, \dots, t_{m-1}\}$ into subsets $Q_{x_{a_\ell}}$ consisting for each Janet non-multiplicative variable x_{a_ℓ} of t_m of exactly those quotient terms u_j with x_{a_ℓ} as highest dividing variable.

Example 4.2. In the polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$, consider the set of terms $U = (x_2x_3^3,\ x_1^2x_3^3,\ x_2x_3^2,\ x_2^2x_3,\ x_1^3x_2x_3)$ which form a Janet basis of the ideal generated by them. We obtain for the quotients $Q(u) = (x_2,\ x_3,\ x_3^2,\ x_3^2)$. Now one sees that $r = 2$ and $NM_{\mathcal{J}}(x_1^3x_2x_3,\ U) = \{x_2, x_3\}$. In addition, $Q_{x_2} = \{x_2\}$ and $Q_{x_3} = \{x_3, x_3^2,\ x_3^2\}$.

Lemma 4.3. *Keeping the above notations, U is a Janet basis, if and only if the set* $\{t_1, \dots, t_i\}$ *is a Janet basis for each index* $1 \le i \le m$.

Proof. If $\{t_1, \dots, t_i\}$ is a Janet basis for each index i, then this is in particular true for $U = \{t_1, \dots, t_m\}$. Conversely, if $U = \{t_1, \dots, t_m\}$ is a Janet basis, then consider for some index $1 < j < m$ the set $U_j := \{t_1, \dots, t_j\}$. Let k be any index with $1 \le k \le j$ and let x_i be any Janet non-multiplicative variable for $t_k \in U_j$ with respect to the Janet division. The definition of Janet multiplicative variables implies that x_i is also not Janet multiplicative for $t_k \in U$. Since U is a Janet basis, there is some index $1 \le \ell \le m$ such that x_it_k is in the Janet cone of t_ℓ with respect to the set U. The index ℓ cannot be greater than j. Indeed, arguing by *reductio ad absurdum*, assume $\ell > j$. Since by assumption $t_k \succ_{\text{lex}} t_\ell$, there is a variable x_a such that the x_a-power of t_ℓ is less than that of t_k and in turn x_a is Janet non-multiplicative for $t_\ell \in U$. This contradicts the fact that x_it_k lies in the Janet cone of t_ℓ. Thus we have shown that x_it_k lies in the Janet cone with respect to U of an element of U_j. By the filter axiom of involutive divisions, x_it_k must then also lie in the Janet cone of the same element with respect to U_j. This proves that U_j is a Janet basis and this finishes the proof. □

Theorem 4.4. *With the above notations, U is a Janet basis, if and only if for each* $i > 1$ *the following condition holds. If we write* $Q(t_1, \dots, t_i) = (u_1, \dots, u_{i-1})$ *and partition*

$$\{u_1,\ \dots,\ u_{m-1}\} = \bigsqcup_{x \in NM_{\mathcal{J}}(t_i, \{t_1, \dots, t_i\})} Q_x\ ,$$

then there exists for each non-multiplicative variable $x_\ell \in NM_{\mathcal{J}}(t_i,\ \{t_1, \dots, t_i\})$ *a term* $u \in Q_{x_\ell}$ *such that* $\deg_\ell(u) = 1$. *Moreover, if in this situation* $x_\ell \ne x_1$, *then in the ring* $\mathcal{K}[x_1, \dots, x_{n-1}]$ *we have the relation*

$$t_i|_{x_\ell = \cdots = x_n = 1} \in \langle t_j|_{x_\ell = \cdots = x_n = 1} \mid t_j : t_i \in Q_{x_\ell},\ j < i,\ \deg_\ell(t_j : t_i) = 1 \rangle\ .$$

Proof. Let $U = \{t_1, \dots, t_m\}$ be a Janet basis. By Lemma 4.3, $U_k = \{t_1, \dots, t_k\}$ is also a Janet basis for all $1 \le k \le m - 1$. Now, let $i \in \{2, \dots, m\}$ be an arbitrary index; we need to show that $Q(t_1, \dots, t_i)$ satisfies the conditions stated. Let x_ℓ be the highest variable which is Janet non-multiplicative for $t_i \in U_i$. Write $t_i = x^\nu$ with $\nu = (\nu_1, \dots, \nu_n)$. It is easy to see that the Janet class $C = [\nu_{\ell+1}, \dots, \nu_n]$ is itself a Janet basis of the ideal it generates. Applying Theorem 3.4 to C, we see that the subset $V = \{t'_j = t_j|_{x_{\ell+1} = \cdots = x_n = 1} \mid t_j \in C\}$ of the polynomial ring $\mathcal{K}[x_1, \dots, x_\ell]$ is also a Janet basis. In addition, we can partition the set V into *non-empty* subsets V_λ where $\beta \le \lambda \le \alpha$, $\beta = \deg_\ell(t'_i)$ and $\alpha = \deg_\ell(t'_1)$. By Theorem 3.4, we know that $\langle V_\beta \rangle \subset \langle V_{\beta+1} \rangle$. This implies, for the variable x_ℓ, simultaneously the degree condition on the elements of $Q(t_1,\dots,t_i)$ having highest variable x_ℓ and the containment of $t'_i|_{x_\ell=1}$ in the ideal $\langle t'_j|_{x_\ell=1} \mid t'_j \in V_{\beta+1}\rangle$.

We have thus verified the conditions for the highest Janet non-multiplicative variable. Now we apply again Theorem 3.4 to obtain the Janet basis $\overline{U} = \{t'_j|_{x_\ell=1} \mid t'_j \in V_\beta\}$. By construction, the highest Janet non-multiplicative variable of $t'_i \in \overline{U}$ is equal to the second highest Janet non-multiplicative variable of $t_i \in U_i$ and exactly those terms $t_j \in U_i$ which yield quotients u_j in $Q(t_1,\dots,t_i)$ with highest variable lower than x_ℓ contribute terms to $\overline{U}$ via the projection $t_j \mapsto t_j|_{x_\ell=\dots=x_n=1}$. Proceeding as in the case of x_ℓ and then iteratively going through all Janet non-multiplicative variables of $t_i \in U_i$, we arrive at our claim.

Let us now assume that for each $1 \le i \le m$ the conditions on the quotient list $Q(t_1,\dots,t_i) = (u_1,\dots,u_{i-1})$ are satisfied. We want to show that U is a Janet basis. By Lemma 4.3, it suffices to show that each set $U_i = \{t_1,\dots,t_i\}$ is a Janet basis. Since it is clear that $\{t_1\}$ is a Janet basis, we may proceed by induction on i and assume that U_{i-1} is a Janet basis. We then need to show that $U_i = U_{i-1} \cup \{t_i\}$ is a Janet basis. For this, we verify the two conditions of Theorem 3.4 for U_i. We partition U_i into the sets $U_\beta,\dots,U_\alpha$ according to the x_n-degrees of the elements of U_i with $\beta \le \alpha$ for $\beta = \deg_n(t_i)$, and $\alpha = \deg_n(t_1)$. Firstly, we verify the inclusions $U'_\lambda \subset \langle U'_{\lambda+1}\rangle$ for any $\beta \le \lambda < \alpha$. If $\beta = \alpha$, there is nothing to do. Otherwise, x_n is a Janet non-multiplicative variable for $t_i \in U_i$ and hence x_n appears as the highest variable of some quotient term u_k in the list $Q(t_1,\dots,t_i)$. Furthermore, in at least one u_k it must appear with degree one. Moreover, the containment of the projection $t_i|_{x_n=1}$ in the $\mathcal{K}[x_1,\dots,x_{n-1}]$-ideal defined by the projections of all t_j such that the x_n-degree of the quotient term $t_j : t_i$ is one must hold. All these terms t_j come from $U'_{\beta+1}$. This implies $t_i|_{x_n=1} \in \langle U'_{\beta+1}\rangle$. On the other hand, U_{i-1} is a Janet basis and by applying Theorem 3.4 on this set, it is clear that $U'_\beta \setminus \{t_i|_{x_n=1}\} \subset \langle U'_{\beta+1}\rangle \cdots \subset \langle U'_\alpha\rangle$. All these observations together imply that U'_λ is not empty for any λ, that $U'_\beta \subset \langle U'_{\beta+1}\rangle$ and that the inclusion conditions on the monomial ideals $\langle U'_\lambda\rangle$ are fulfilled by U_i.

We still have to show that each set U'_λ is a Janet basis. The sets U'_λ with $\lambda > \beta$ are Janet bases, as U_{i-1} is a Janet basis and thus fulfills the conditions of Theorem 3.4. If we have $U'_\beta = \{t_i|_{x_n=1}\}$, we are done. Otherwise, there exists some index $1 < a < i$ such that $U'_\beta = \{t_a|_{x_n=1},\dots,t_i|_{x_n=1}\}$. Removing the last element of this set, we obtain the set U'_β of the Janet basis U_{i-1}, which is again a Janet basis. And the quotient terms of the elements of this set by $t_i|_{x_n=1}$ inherit for the variables $x_1,\dots,x_{n-1}$ all properties which hold for the original quotient terms u_k with respect to these variables. By an induction on the number of variables in the ambient polynomial ring, we are done (the case of a polynomial ring with one variable being trivial). Thus we have shown that the individual sets U'_λ are Janet bases and verified the conditions of Theorem 3.4 for the set U_i. This finishes the proof. □

Example 4.5. Let $U = (x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3)$ be a sequence of terms in $\mathcal{P} := \mathcal{K}[x_1,x_2,x_3]$. In the following, we show how we can apply the above result to compute a Janet basis for U.

1. Since $Q(x_1^3x_3^3,\ x_1^2x_3^3) = (x_1)$, the two first elements form a Janet basis.
2. We have $Q(x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2) = (x_3,\ x_1x_3)$. The x_3-degrees of x_3, x_1x_3 are both one and so we check only whether $x_1^2x_2^2 \in \langle x_1^3, x_1^2\rangle$. Since this is the case,

the sequence of the first three elements forms a Janet basis.

3. Let us now consider $Q(x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1^2x_2^2x_3) = (x_3,\ x_3^2,\ x_1x_3^2)$. Since all elements in this quotient tuple contain x_3 as the highest variable, we shall consider only the first quotient which is linear. So, we check $x_1^2x_2^2 \in \langle x_1^2x_2^2\rangle$ which is true. Thus, the sequence the first four elements forms a Janet basis.
4. As next step we consider

$$Q(x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3) = (x_1,\ x_1x_3,\ x_1x_3^2,\ x_1^2x_3^2)\ .$$

We ignore the first quotient and check whether $x_1x_2^2 \in \langle x_1^2x_2^2\rangle$. As this does not hold, we add $x_1x_2^2x_3^2$ to U and obtain

$$U_1 = (x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3, x_2^2x_3)\ .$$

5. We now consider $Q(x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2) = (x_1,\ x_1x_3,\ x_1^2x_3)$. Since the x_1-degree of the first element is one, we ignore it. Hence, we must check whether $x_1x_2^2 \in \langle x_1^2\rangle$. As it does not hold, we add $x_1x_2^2x_3^3$ to U_1 and arrive at

$$U_2 = (x_1x_2^2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3)\ .$$

6. We next consider $Q(x_1x_2^2x_3^3,\ x_1^3x_3^3) = (x_2^2)$. Since the quotient is not linear, we must add $x_1^3x_2x_3^3$ to U_2 obtaining

$$U_3 = (x_1x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3)\ .$$

7. We now find $Q(x_1x_2^2x_3^3,\ x_1^3x_2x_3^3) = (x_2)$ and $Q(x_1x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^3x_3^3) = (x_2,\ x_2^2)$. Since $x_1^3 \in \langle x_1^3\rangle$, the first three terms of U_3 form a Janet basis.
8. Next, $Q(x_1x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3) = (x_1,\ x_1x_2,\ x_2^2)$. Since $x_1^2 \in \langle x_1^3x_2\rangle$, we add $x_1^2x_2x_3^3$ to U_3 obtaining

$$U_4 = (x_1x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^2x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3)\ .$$

9. We find that $Q(U_4) = (x_1,\ x_1^2,\ x_1x_3,\ x_1^2x_3,\ x_1^2x_3^2,\ x_1^3x_3^2,\ x_1^2x_3^2,\ x_1^3x_3^2,\ x_1x_3^2)$. We ignore the first two quotients and check whether $x_2^2 \in \langle x_1^2x_2^2,\ x_1x_2^2\rangle$. As this does not hold, we add $x_2^2x_3^2$ to U_4 obtaining

$$U_5 = (x_1x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^2x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3)\ .$$

10. We consider next

$$Q(x_1x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^2x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_2^2x_3^2) =\\ (x_1,\ x_1^2,\ x_1^2x_3,\ x_1^3x_3,\ x_1^2x_3,\ x_1^3x_3,\ x_1x_3)\ .$$

Since $x_2^2 \notin \langle x_1x_2^2,\ x_1^3x_2,\ x_2x_1^2,\ x_1^3, x_1^2\rangle$, we add $x_2^2x_3^3$ to U_5 finally reaching the set

$$U_6 = (x_1x_2^2x_3^3,\ x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^2x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3)$$

which satisfies all the condition of the above theorem and thus is a Janet basis of the ideal generated by U.

Theorem 4.4 translates straightforwardly into Algorithm 4 which checks whether a given monomial set is a Janet basis of the ideal generated by it. If the output is `false`, then the algorithm returns in addition an element which should be added. The correctness and the termination of the algorithm is obvious.

Algorithm 4: JanetTest

Data: A finite set $U \subset \mathcal{P}$ of terms.
Result: True if U is a Janet basis for the ideal it generates and false otherwise.
begin
  $flag \longleftarrow false$
  while $flag = false$ **do**
    $flag \longleftarrow true$; $(t_1, \dots, t_m) \longleftarrow \texttt{sort}(U, \prec)$
    for i **from** 2 **to** m **do**
      $(u_1, \dots, u_{i-1}) \longleftarrow Q(t_1, \dots, t_i)$
      if $\exists j$ *s.t. the highest variable* x_ℓ *in* u_j *is not linear* **then**
        return $(false, x_\ell t_i)$
      else if *the highest variable in the quotients* $t_1 : t_i, \dots, t_{is} : t_i$ *is* $x_\ell \neq x_1$ **then**
        if $t_i|_{x_\ell = \cdots = x_n = 1} \notin \langle t_{i_1}|_{x_\ell = \cdots = x_n = 1}, \dots, t_{i_s}|_{x_\ell = \cdots = x_n = 1} \rangle$ **then**
          return $(false, x_\ell t_i)$
  return $(true)$

The strategy applied in Example 4.5 for completing a monomial set to a Janet basis can then be easily extended to the general monomial completion Algorithm 5.

Algorithm 5: JanetMonomialCompletion

Data: A finite set $U \subset \mathcal{P}$ of terms.
Result: A Janet basis of $\langle U \rangle$.
begin
  $T \longleftarrow U$; $A \longleftarrow \texttt{JanetTest}(T)$
  while $A = (false, t)$ **do**
    $T \longleftarrow T \cup \{t\}$
    $A \longleftarrow \texttt{JanetTest}(T)$
  return (T)

Theorem 4.6. *Algorithm 5 algorithm terminates in finitely many steps and is correct.*

Proof. The termination of this algorithm is a consequence of the fact that Janet division is Nœtherian, see (Gerdt and Blinkov, 1998, Proposition 4.5). Its correctness is a corollary to Theorem 4.4 and the constructivity and continuity of the Janet division, see (Gerdt and Blinkov, 1998). □

We now proceed to the determination of a Janet basis for a given set of polynomials. Let $U = (t_1, \ldots, t_m)$ be a sequence of terms and $Q(U) = (u_1, \ldots, u_{m-1})$. By Lemma 4.1, we know that the highest variables in the u_i's are the Janet non-multiplicative variables for u_m as an element of U. Based on this observation and using the Janet polynomial completion algorithm (see e. g. (Gerdt, 2005, Section 4) or (Seiler, 2009, Alg. 3)), we can describe a variant of the Berkesch–Schreyer algorithm which computes a Janet basis for a polynomial ideal.

Again we begin with the auxiliary Algorithm 6 determining in the described manner the Janet non-multiplicative variables of the last polynomial in an ordered finite set. Its correctness is an immediate consequence of Lemma 4.1.

Algorithm 6: JanetNonMultVar

Data: An ordered finite set $F \subset \mathcal{P}$ of polynomials and a term ordering $\prec$.
Result: Set of Janet non-multiplicative variables of last polynomial in F.
begin
    $(t_1, \ldots, t_m) \longleftarrow \mathrm{lt}(F)$; $(u_1, \ldots, u_{m-1}) \longleftarrow Q(t_1, \ldots, t_m)$
    return *(set of highest variables for $\prec$ appearing in the u_i's)*

Based on this algorithm, we obtain the polynomial completion Algorithm 7. In it, we denote for any ordered set X by $X[i..j]$ the ordered subset containing all elements from the i-th one to the j-th one.

Algorithm 7: JanetPolynomialCompletion

Data: A finite set $F \subset \mathcal{P}$ of polynomials and a term ordering $\prec$.
Result: A Janet basis of $\langle F \rangle$.
begin
    $H \longleftarrow$ `sort`$(F, \prec)$ from the highest leading term to the lowest one
    while *true* **do**
        $flag \longleftarrow false$
        for i **from** 2 **to** $|H|$ **while** $flag = false$ **do**
            $A \longleftarrow$ `JanetNonMultVar`$(H[1..i], \prec)$
            foreach $a \in A$ **do**
                $g \longleftarrow$ an involutive normal form of $a \cdot H[i]$ with respect to H
                if $g \neq 0$ **then**
                    $H \longleftarrow$ `sort`$(H \cup \{g\}, \prec)$; $flag \longleftarrow true$
        if $flag = false$ **then**
            return (H)

Theorem 4.7. *Algorithm 7 terminates in finitely many steps and is correct.*

Proof. Since the structure of the algorithm is essentially that of (Seiler, 2009, Alg. 3), its termination and correctness follow by (Seiler, 2009, Thm 7.4). □

Remark 4.8. If H is already a sorted list of polynomials, then one can use an efficient insertion sort algorithm for sorting $H \cup \{g\}$. For the special case of lists of terms, we refer to Lundqvist (2008).

We conclude this section by providing a similar approach for the construction of Janet-like bases. Let again $U := (t_1, \ldots, t_m) \in \mathcal{T}^m$ be a sequence of terms such that $t_1 \succ_{\text{lex}} \cdots \succ_{\text{lex}} t_m$ and $Q(U) = (u_1, \ldots, u_{m-1})$. Then, there exist a positive integer r, indices $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and indices $1 = b_1 < b_2 < \cdots < b_r < b_{r+1} = m-1$ such that the highest variable dividing u_j is x_{a_ℓ} for all $\ell \in \{1, \ldots, r\}$ and for all indices j with $b_\ell \leq j < b_{\ell+1}$. Furthermore, we denote by d_{a_ℓ} the x_{a_ℓ}d-egree of u_{b_ℓ}. Keeping these notations, we obtain the next result analogous to Lemma 4.1.

Lemma 4.9. $\mathrm{NMP}(t_m, \{t_1, \ldots, t_m\}) = \left\{x_{a_1}^{d_{a_1}}, \ldots, x_{a_r}^{d_{a_r}}\right\}$.

Proof. From the proof of Lemma 4.1, we know that $\{x_{a_1}, \ldots, x_{a_r}\}$ is the set of all Janet non-multiplicative variables. On the other hand, from the underlying term ordering, we have that $\deg_{a_\ell}(u_{j_1}) \leq \deg_{a_\ell}(u_{j_2})$ for all indices $b_\ell \leq j_1 < j_2 < b_{\ell+1}$. It follows that $\deg_{a_\ell}(u_{b_\ell})$ has the minimal x_{a_ℓ}-degree among all elements $u_{b_\ell}, \ldots, u_{b_{\ell+1}-1}$. These observations imply the desired assertion. □

This lemma induces a partition of the set $\{u_1, \ldots, u_{m-1}\}$ into subsets $Q_{x_{a_\ell}}$ consisting for each Janet non-multiplicative variable x_{a_ℓ} of t_m exactly of those quotient terms u_j with x_{a_ℓ} as highest dividing variable.

Example 4.10. By Example 3.15, the sequence $U = (x_2^2x_3^5,\ x_1^2x_3^5,\ x_2^4x_3^3,\ x_1^2x_2^2x_3^3,\ x_1^2x_3^3)$ forms a Janet-like basis in the ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$. Here $Q(U) = (x_2^2,\ x_2^4,\ x_3^2,\ x_2^2x_3^2)$ and thus $\mathrm{NMP}(x_1^2x_3^3, U) = \{x_2^2, x_3^2\}$. Furthermore, we find the subsets $Q_{x_2} = \{x_2^2, x_2^4\}$ and $Q_{x_3} = \{x_3^2, x_2^2x_3^2\}$.

Lemma 4.11. *With the above notations, U is a Janet-like basis, if and only if the subsets* $\{t_1, \ldots, t_i\}$ *are Janet-like bases for each index* $1 \leq i \leq m$.

The proof of this lemma is analogous to the one of Lemma 4.3 and thus omitted. Finally, we adapt Theorem 4.4 to the Janet-like division. Taking Theorem 3.14 into account, its proof is similar to to the one of Theorem 4.4 and hence also not detailed.

Theorem 4.12. *With the above notations, U is a Janet-like basis, if and only if for each* $i > 1$ *the following condition holds. If we write* $Q(t_1, \ldots, t_i) = (u_1, \ldots, u_{i-1})$ *and partition*

$$\{u_1, \ldots, u_{m-1}\} = \bigsqcup_{x \in NM_{\mathcal{J}}(t_i, \{t_1, \ldots, t_i\})} Q_x\ ,$$

then there exists for each non-multiplicative power $x_\ell^{d_\ell} \in \mathrm{NMP}(t_i, \{t_1, \ldots, t_i\})$ *a term* $u \in Q_{x_\ell}$ *with minimal* x_ℓ*-degree* d_ℓ*. Moreover, if in this situation* $x_\ell \neq x_1$*, then in the ring* $\mathcal{K}[x_1, \ldots, x_{n-1}]$ *we have the relation*

$$t_i|_{x_\ell = \cdots = x_n = 1} \in \langle t_j|_{x_\ell = \cdots = x_n = 1} \mid t_j : t_i \in Q_{x_\ell},\ j < i,\ \deg_\ell(t_j : t_i) = d_\ell \rangle\ .$$

Remark 4.13. Based on these results, it is straightforward to provide also algorithms for computing Janet-like bases for both monomial and polynomial ideals by adapting Algorithms 5 and 7. We omit the obvious details.

5. A recursive Pommaret basis construction

So far, we have concentrated on Janet bases. We now provide a criterion similar to Theorem 3.4 for a finite set to be a Pommaret basis. As the existence of a finite Pommaret basis is equivalent to the ideal being quasi-stable, it is not surprising that the criterion can be extended to a recursive test of quasi-stability.

Theorem 5.1. *Let $U = \{t_1, \dots, t_m\}$ be a finite set of terms. We write $t'_i = t_i|_{x_n=1}$ for each index $1 \leq i \leq m$ and set $U' = \{t'_1, \dots, t'_m\}$ and $\alpha = \max\{\deg_n(t_1), \dots, \deg_n(t_m)\}$. For each degree $\lambda \leq \alpha$, we introduce the index set $I_\lambda = \{i \mid \deg_n(t_i) = \lambda\}$ and the set $U'_\lambda = \{t'_i \mid i \in I_\lambda\}$. Then U is a Pommaret basis, if and only if the following three conditions are satisfied:*

(i) For each degree $\lambda \leq \alpha$, the set U'_λ is a Pommaret basis.
(ii) For each degree $\lambda < \alpha$, we have the inclusion $U'_\lambda \subset \langle U'_{\lambda+1}\rangle$,
(iii) We have $U \cap \mathcal{K}[x_n] = x_n^\alpha$.

Proof. Assume first that U is a Pommaret basis. By Definition 2.4, U is autoreduced with respect to the Pommaret division. Let $\lambda \leq \alpha$ be a non-negative integer such that there exists a term $t \in U$ with $\deg_n(t) = \lambda$. We now show that U'_λ is a Pommaret basis of $\langle U'_\lambda\rangle \trianglelefteq \mathcal{K}[x_1, \dots, x_{n-1}]$. Note that U'_λ must be Pommaret autoreduced, too, as otherwise U could not be Pommaret autoreduced. Since the Pommaret division is continuous, we can check the involutivity of U'_λ by testing it for local involution. Choose a term $t'_i \in U'_\lambda$ and let x_k (with $k < n$) be a Pommaret non-multiplicative variable for it. Then, by definition of the Pommaret division, x_k is also not Pommaret multiplicative for $t_i \in U$. Since U is a Pommaret basis, there exists a Pommaret divisor $s \in U$ of $x_k \cdot t_i$. We claim that $s \in U_\lambda$. Indeed, $\deg_n(s) > \deg_n(t_i)$ is not possible because of $s \mid x_k \cdot t_i$. Also, $\deg_n(s) < \deg_n(t_i)$ is not possible because then the Pommaret divisibility of $x_k \cdot t_i$ by s implies that s is a pure power of x_n and hence a strict Pommaret divisor of t_i, in contradiction to the Pommaret autoreducedness of U. So, $s \in U_\lambda$ as claimed and s' is a Pommaret divisor of $x_k \cdot t'_i$ in the ring $\mathcal{K}[x_1, \dots, x_{n-1}]$.

That U satisfies Condition (ii) is easily seen: U is the unique Pommaret basis of the quasi-stable ideal $\langle U\rangle$ and hence also a Janet basis of $\langle U\rangle$. Condition (ii) now immediately follows by Theorem 3.4. Finally, Condition (iii) follows from the fact that the Pommaret autoreducedness of U implies that U contains exactly one pure x_n-power and this power must be x_n^α.

Now, we assume conversely that the set U satisfies Conditions (i) to (iii). We first show that U is Pommaret autoreduced. Arguing by *reductio ad absurdum*, suppose that there are terms $t_i, t_j \in U$ with $t_i \neq t_j$ and t_i is a Pommaret divisor of t_j. If $\deg_n(t_i) = \deg_n(t_j)$, then there is an integer λ such that $\{t'_i, t'_j\} \subseteq U'_\lambda$ and t'_i is a Pommaret divisor of t'_j in the ring $\mathcal{K}[x_1, \dots, x_{n-1}]$. This contradicts the Pommaret autoreducedness of U'_λ which is guaranteed by Condition (i). Otherwise, we have $\deg_n(t_i) < \deg_n(t_j)$ implying that t_i is a pure x_n-power. By Condition (iii), $t_i = x_n^\alpha$. But now necessarily $\deg_n(t_j) > \alpha$, in contradiction to the definition of α as the maximal x_n-degree appearing in U.

We still need to show the involutivity of U, which we do again via local involution. Consider a term $t \in U$ with $\deg_n(t) = \lambda$ and let x_k be a Pommaret non-multiplicative variable of t. Now, if $k < n$, then x_k is also a Pommaret non-multiplicative variable of t' in the ring $\mathcal{K}[x_1, \dots, x_{n-1}]$. Since U'_λ is a Pommaret basis by Condition (i), there is a term $s' \in U'_\lambda$ and a term $x^\mu \in \mathcal{K}[x_1, \ldots, x_{n-1}]$ Pommaret multiplicative for s' such that $t' = x^\mu \cdot s'$. This implies $t = x^\mu \cdot s$. It is easy to see that x^μ is also Pommaret multiplicative for $s \in \mathcal{K}[x_1, \ldots, x_n]$. Thus, in the case $k < n$ we are done. Now, assume $k = n$. Recall that $t \in U_\lambda$. Since, by Condition (ii), $t' \in \langle U'_{\lambda+1} \rangle$ and, by Condition (i), $U'_{\lambda+1}$ is a Pommaret basis, there are terms $t_\ell \in U_{\lambda+1}$ and x^ν Pommaret multiplicative for t'_ℓ in the ring $\mathcal{K}[x_1, \ldots, x_{n-1}]$ such that $t' = x^\nu \cdot t'_\ell$. This implies $x_n \cdot t = x^\nu \cdot t_\ell$. It is easy to see that x^ν is also Pommaret multiplicative for t_ℓ in the ring $\mathcal{K}[x_1, \ldots, x_n]$. This finishes the proof of local involutivity of U, and we are done. □

We provide two examples for the application of Theorem 5.1, a positive one and a negative one.

Example 5.2. In the trivariate polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$, we consider the set $U = \{x_3^3,\ x_2^2x_3^2,\ x_2^2x_3,\ x_1x_2x_3^2,\ x_1^2x_3^2,\ x_1^2x_2x_3,\ x_1^2x_3\}$. One observes that

1. $\beta = 1 \le 3 = \alpha$,
2. $U_3 = \{x_3^3\}$ and $U'_3 = \{1\}$, which is obviously a Pommaret basis,
3. $U'_2 = \{x_2^2,\ x_1x_2,\ x_1^2\}$, which is also a Pommaret basis,
4. $U'_1 = \{x_2^2,\ x_1^2x_2,\ x_1^2\}$, which is also a Pommaret basis,
5. $U'_1 \subset \langle U'_2 \rangle$, and finally
6. $U'_2 \subset \langle U'_3 \rangle$.

Hence, U is a Pommaret basis. Here, we have used that in two variables, one can identify Pommaret bases very easily. But in principle Theorem 5.1 requires to carry the recursion further, until it is only left to check subsets of $\mathcal{K}[x_1]$ for being a Pommaret basis, for which one applies Condition (iii), i. e., one must check whether one has a singleton set.

Example 5.3. In the same polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3]$, we consider now the set $U = \{x_3^3,\ x_2^2x_3^2,\ x_2^2x_3,\ x_1\}$. One observes that

1. $\beta = 0 \le 3 = \alpha$,
2. $U_3 = \{x_3^3\}$ and $U'_3 = \{1\}$, which is obviously a Pommaret basis,
3. $U'_2 = \{x_2^2\}$, which is also a Pommaret basis,
4. $U'_1 = \{x_2^2\}$, which is also a Pommaret basis,
5. $U'_0 = \{x_1\}$, which is not a Pommaret basis, as $U'_0 \cap \mathcal{K}[x_2] = \emptyset$.

Hence, U is not a Pommaret basis.

Remark 5.4. Theorem 3.10 holds for Pommaret bases, too, if one replaces everywhere in it "Janet basis" by "Pommaret basis". This follows immediately from the fact that any Pommaret basis is also a Janet basis for the ideal it generates.

Since quasi-stability is equivalent to the existence of a finite Pommaret basis by Proposition 2.9, we can use our results to derive also a recursive criterion for a monomial ideal to be quasi-stable using an arbitrary monomial generating set. This criterion, formulated in Corollary 5.5, translates directly into Algorithm 8 as an effective test for quasi-stability similar to Algorithm 4.

Corollary 5.5. *Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{P}$ be a set of terms with $\lambda_0 < \lambda_1 < \cdots < \lambda_\ell$ being the x_n-degrees of its elements. For each $0 \le i \le \ell$, we denote by $U_{\lambda_i} \subseteq U$ the subset containing those terms t with $\deg_n(t) = \lambda_i$ and we write $U'_{\lambda_i} = \{t|_{x_n=1} \mid t \in U_{\lambda_i}\}$. Then the monomial ideal $\langle U \rangle$ is quasi-stable, if and only if the following conditions hold:*

Algorithm 8: QuasiStableTest

Data: A finite set $U = \{t_1, \ldots, t_m\} \subset \mathcal{P}$ of terms.
Result: True if $\langle U \rangle$ is quasi-stable and false otherwise.
begin
  $(\lambda_0, \lambda_1, \ldots, \lambda_\ell) \longleftarrow$ the sequence of x_n-degrees of the terms t_i ordered such that $\lambda_0 < \lambda_1 < \cdots < \lambda_\ell$
  if $n = 1$ **then**
    return $(true)$
  if $U \cap \mathcal{K}[x_n] = \emptyset$ **then**
    return $(false)$
  for i **from** 0 **to** ℓ **do**
    $U'_{\lambda_i} \longleftarrow \{ t \in \mathcal{K}[x_1, \ldots, x_{n-1}] \mid t \cdot x_n^{\lambda_i} \in U \}$
    if QuasiStableTest$(\bigcup_{j=0}^{i} U'_{\lambda_j}) = false$ **then**
      return $(false)$
  return $(true)$

(i) For each $i \leq \ell$, the ideal $\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \rangle \subseteq \mathcal{K}[x_1, \ldots, x_{n-1}]$ is quasi-stable,
(ii) We have $U \cap \mathcal{K}[x_n] \neq \emptyset$.

Proof. Suppose first that $\langle U \rangle$ is quasi-stable. By Proposition 2.9, this ideal possesses thus a finite Pommaret basis H. By Theorem 5.1, H'_γ is a Pommaret basis for each $\gamma \leq \lambda_\ell$ and in addition $H'_\gamma \subset \langle H'_{\gamma+1} \rangle$ for each $\gamma < \lambda_\ell$. Since $\langle U \rangle = \langle H \rangle$, we have $\langle \bigcup_{j=0}^{i} U_{\lambda_i} \rangle = \langle \bigcup_{\gamma \leq \lambda_i} H_\gamma \rangle$ for each $i < \ell$. Projecting to $\mathcal{K}[x_1, \ldots, x_{n-1}]$ and using the inclusions $H'_\gamma \subset \langle H'_{\gamma+1} \rangle$, we get $\langle \bigcup_{j=0}^{i} U'_{\lambda_i} \rangle = \langle \bigcup_{\gamma \leq \lambda_i} H'_\gamma \rangle = \langle H'_{\lambda_i} \rangle$. Thus H'_{λ_i} is a Pommaret basis for $\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \rangle$ for each $i < \ell$. It follows from Proposition 2.9 that $\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \rangle$ is quasi-stable and this proves the first item. The second item follows directly from the definition of quasi-stability, as U must contain a pure power of x_n.

Conversely, assume that the two given conditions are satisfied and consider an arbitrary term $t = x_j^{\mu_j} \cdots x_n^{\mu_n} \in U$ with $\mu_j \neq 0$ for $j = \mathrm{cls}(t)$. A necessary condition for the quasi-stability of $\langle U \rangle$ is that there exists an exponent s such that $x_n^s t / x_j \in \langle U \rangle$. From the last condition, we know that some power x_n^a lies in $\langle U \rangle$ and hence we can simply chose any $s \geq a$. However, as a sufficient condition for the quasi-stability of $\langle U \rangle$, we must also check the membership $x_k^s t / x_j \in \langle U \rangle$ for any index $n > k > j$ and sufficiently high exponent s. For this, we must recursively descend via the first condition.

Similar to (2.1), we introduce $U_{(d_s, \ldots, d_n)} = \{ u|_{x_s = \cdots = x_n = 1} \mid u \in U, \deg_i(u) \leq d_i, i = s, \ldots, n \}$ and consider our term t as an element of the subset $V := U_{(\mu_{k+1}, \ldots, \mu_n)}$. Let $\gamma_0 < \gamma_1 < \cdots < \gamma_l$ be the x_k-degrees of the elements of V. By the first condition, $\cup_{j=0}^{\mu_k} V'_{\gamma_j}$ generates a quasi-stable ideal in $\mathcal{K}[x_1, \ldots, x_k]$ which by the second condition must contain a term x_k^b for some exponent b. Hence, the original set U must contain a term $x_k^b x_{k+1}^{\nu_{k+1}} \cdots x_n^{\nu_n}$ with $\nu_i \leq \mu_i$. Choosing $s \geq b$, this term is a divisor of $x_k^s t / x_j$ so that indeed $x_k^s t / x_j \in \langle U \rangle$ as required for the completion of the proof. □

Theorem 5.6. *Algorithm 8 terminates in finitely many steps and is correct. Moreover, its arithmetic complexity is $O(m^2 + nm)$.*

Proof. The termination of the algorithm is trivial due to the recursive structure of the algorithm and also the use of the **for**-loops. The correctness of the algorithm is a consequence of Corollary 5.5.

To determine the complexity of the algorithm, we construct first the tree representation corresponding to the exponents of the elements of U which needs $O(m^2+nm)$ comparisons by applying (Lundqvist, 2010, Thm 4.2). Then, we check for each $i = n, \dots, 2$ whether there is a branch with no child. If this is not the case for some index i, then we return false and true otherwise. Since for each i, the number of branches is at most m, these checks need $O(mn)$ comparisons and this completes the proof. □

Remark 5.7. For alternative approaches to testing quasi-stability, we refer to (Hashemi, 2010; Seiler, 2012). Hashemi (2010, Prop. 3.4) showed that the complexity of the there presented algorithm is $O(m^2n^2)$ and thus our new algorithm has a better performance.

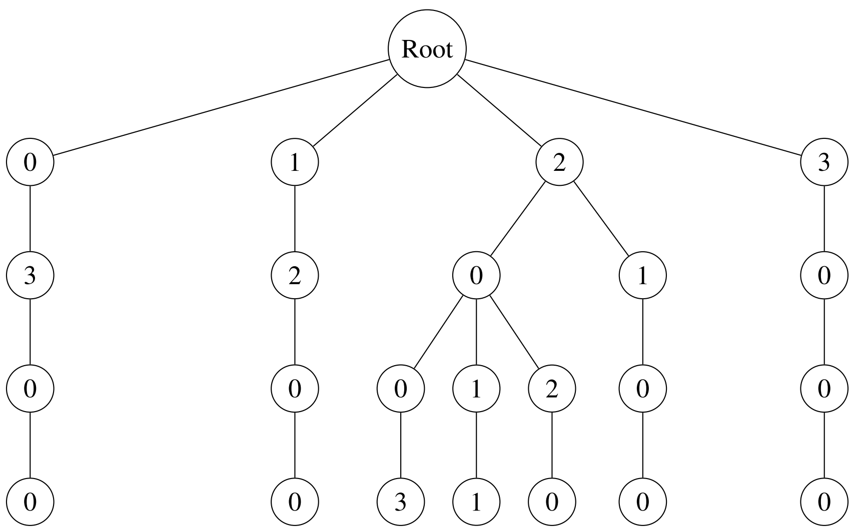

Figure 5: Tree representation of the set U in Example 5.8

Example 5.8. We consider $U = \{x_4^3,\ x_3x_4^2,\ x_2^2x_4^2,\ x_1x_2x_4^2,\ x_1^3x_4^2,\ x_3^2x_4,\ x_3^3\}$ in the polynomial ring $\mathcal{P} = \mathcal{K}[x_1, x_2, x_3, x_4]$. To illustrate the application of our test, we need the tree representation of U shown in Figure 5. One observes that

1. $\lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$,
2. $U'_3 = \{1\}$,
3. $U'_2 = \{x_3,\ x_2^2,\ x_1x_2,\ x_1^3\}$,
4. $U'_1 = \{x_3^2\}$,
5. $U'_0 = \{x_3^3\}$.

We have $x_3^3 \in U$ which satisfies the second condition in Corollary 5.5. In addition, each $\langle \cup_{j=0}^{i} U'_{\lambda_j} \rangle$ for each i is quasi stable. So, the ideal generated by U is quasi-stable.

Based on Corollary 5.5, we propose the simple Algorithm 9 to transform a given (non-necessarily homogeneous) ideal into quasi-stable position. Below, we consider the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$.

Algorithm 9: QuasiStableLinChange

Data: A finite set $F \subset \mathcal{P}$ of polynomials and a term ordering $\prec$.
Result: A linear change Φ so that $\langle \Phi(F) \rangle$ is in quasi-stable position.
begin
  $\Phi \longleftarrow \emptyset$; $G \longleftarrow F$; $U \longleftarrow \mathrm{lt}(G)$
  if $1 \in U$ **then**
    return (id)
  if *no pure power of x_n belongs to U* **then**
    choose a term $t = x_{i_1}^{\mu_1} \cdots x_{i_k}^{\mu_k}$ with minimal number of variables
    if *t does not contain x_n* **then**
      $\pi \longleftarrow$ permutation $x_{i_k} \longleftrightarrow x_n$
      $\phi_0 \longleftarrow$ map such that $x_{i_j} \mapsto x_{i_j} + x_n$ for $j < k$ and all other variables unchanged
    else
      $\pi \longleftarrow$ id
      $\phi_0 \longleftarrow$ map such that $x_{i_j} \mapsto x_{i_j} + x_n$ for $j < k$ and all other variables unchanged
    $\ell \longleftarrow 0$; $G \longleftarrow \pi(G)$
    repeat
      $H \longleftarrow \phi_\ell(G)$; $U \longleftarrow \mathrm{lt}(H)$; $\ell \longleftarrow \ell + 1$
      $\phi_\ell \longleftarrow$ map $x_{i_j} \mapsto x_{i_j} + (\ell^{2i_j} + 1)x_n$ for $j < k$
    until *U contains a pure power of x_n*
    $G \longleftarrow H$; $\Phi \longleftarrow \Phi \cup \{\phi_{\ell-1} \circ \pi\}$
  $G \longleftarrow$ The reduced Gröbner basis of $\langle G \rangle$; $U \longleftarrow \mathrm{lt}(G)$
  $(\lambda_0, \lambda_1, \ldots, \lambda_m) \longleftarrow$ the sequence of x_n-degrees of the elements of U with $\lambda_0 < \lambda_1 < \cdots < \lambda_m$
  for i **from** 0 **to** m **do**
    $G \longleftarrow \{f|_{x_n=1} \mid f \in G, \deg_n \mathrm{lt}(f) \leq \lambda_i\} \subset \mathcal{K}[x_1, \ldots, x_{n-1}]$
    $\Phi \longleftarrow \Phi \cup$ `QuasiStableLinChange`(G)
  return (Φ)

Theorem 5.9. *Algorithm 9 terminates in finitely many steps and is correct.*

Proof. The correctness of the algorithm is an obvious consequence of Corollary 5.5. Let us deal with its termination. For this purpose, consider a polynomial f of degree d whose leading monomial is $c_\mu x^\mu := c_\mu x_1^{\mu_1} \cdots x_n^{\mu_n}$. Assume that we perform a linear change ϕ of the form $x_i \mapsto x_i + (c^i + 1)x_n$ for each $i < n$ where c is a parameter. To prove the finite termination of the algorithm, we shall prove that there are finitely many integers c for which the term x^μ, after performing this linear change, vanishes. One observes that the coefficient of x_n^d in $\phi(x^\mu)$ is $c_\mu(c^2+1)^{\mu_1}\cdots(c^{2(n-1)}+1)^{\mu_{n-1}}$. Thus, the coefficient of x_n^d in $\phi(f)$ is a polynomial in $\mathcal{K}[c]$ of the form

$$p := c_\mu(c^2+1)^{\mu_1}\cdots(c^{2(n-1)}+1)^{\mu_{n-1}} + c_\nu(c^2+1)^{\nu_1}\cdots(c^{2(n-1)}+1)^{\nu_{n-1}} + \cdots$$

where $c_\nu x^\nu = c_\nu x_1^{\nu_1}\cdots x_n^{\nu_n}$ denotes another monomial of degree d in f. Assume that this polynomial is zero. Let us order $\mu, \nu, \ldots$ in the lexicographical ordering with $x_n \prec_{lex} \cdots \prec_{lex} x_1$. Without loss of generality, suppose that $\mu \succ_{lex} \nu \succ_{lex} \cdots$ and μ_i is the first non-zero component of μ. Let $\nu_i > 0$ be minimal among all the i-th non-zero components. Then, the sum of monomials containing $(c^{2i}+1)^{\nu_i}$ must be zero. The number of these monomials is at least two. So, we can consider only this sum and remove $(c^{2i}+1)^{\nu_i}$. By applying this argument repeatedly, we can find at the end two terms x^θ and x^η such that $\theta_j = \eta_j$ for each $j = 1, \ldots, n-1$. Since these two terms are of the same degree, we can conclude that $x^\theta = x^\eta$, leading to a contradiction.

Since p is a uni-variate polynomial, it has finitely many roots and it follows that after a finite number of iterations of the **repeat**-loop, one finds a value at which p is non-zero. This leads to the desired linear change, finishing the proof. □

Example 5.10. As a simple case, consider the set $U = \{x_3^3,\, x_2^2x_3^2,\, x_1\} \subset \mathcal{K}[x_1, x_2, x_3]$ introduced in Example 5.3. Since no pure power of x_2 lies in $U'_0 = \{x_1\} \subset \mathcal{K}[x_1, x_2]$, the ideal generated by this set is not quasi-stable and in turn the ideal generated by U is not quasi-stable. Following Algorithm 9, we apply the permutation $x_1 \longleftrightarrow x_2$ to $\mathcal{I}$ and obtain the ideal $\tilde{\mathcal{I}} = \langle x_3^3,\, x_1^2x_3^2,\, x_2\rangle$ which is quasi-stable.

Example 5.11. We consider the ideal treated by Eisenbud and Sturmfels (1994, Sect. 2) (see also (Seiler, 2012)) in their quest for the construction of optimal systems of parameters in the sense that they are as sparse as possible. Take

$$F = \{\, x_5x_6,\, x_4x_6,\, x_4x_5,\, x_3x_5,\, x_2x_5,\, x_3x_4,\, x_2x_4,\, x_2x_3,\, x_1x_3,\, x_1x_2 \,\} \subset \mathcal{K}[x_1, \ldots, x_6]\,.$$

Algorithm 9 performs first the linear change $x_5 \mapsto x_5 + x_6$ which yields a new leading ideal generated by

$$U := \{x_6^2,\, x_4x_6,\, x_3x_6,\, x_2x_6,\, x_4x_5,\, x_3x_4,\, x_2x_4,\, x_2x_3,\, x_1x_3,\, x_1x_2\}\,.$$

Then it considers the set $U'_0 = \{x_4x_5,\, x_3x_4,\, x_2x_4,\, x_2x_3,\, x_1x_3,\, x_1x_2\} \subset \mathcal{K}[x_1, \ldots, x_5]$ and performs the linear change $x_4 \mapsto x_4 + x_5$. This leads to a new leading ideal generated by

$$V := \{x_6^2,\, x_5x_6,\, x_3x_6,\, x_2x_6,\, x_5^2,\, x_3x_5,\, x_2x_5,\, x_2x_3,\, x_1x_3,\, x_1x_2\}\,.$$

We have $W := V'_0 = \{x_5^2,\, x_3x_5,\, x_2x_5,\, x_2x_3,\, x_1x_3,\, x_1x_2\}$. Since the ideal generated by $W'_0 = \{x_2x_3,\, x_1x_3,\, x_1x_2\} \subset \mathcal{K}[x_1, \ldots, x_4]$ is not quasi-stable and since W'_0 does not contain x_4, Algorithm 9 proceeds with the linear change $x_3 \longleftrightarrow x_4$ and $x_2 \mapsto x_2 + x_4$. The new leading ideal is generated by

$$Z := \{x_6^2,\, x_5x_6,\, x_4x_6,\, x_2x_6,\, x_5^2,\, x_4x_5,\, x_2x_5,\, x_4^2,\, x_1x_4,\, x_1x_2\}\,.$$

Set $T := Z'_0 = \{x_5^2,\, x_4x_5,\, x_2x_5,\, x_4^2,\, x_1x_4,\, x_1x_2\}$ and $R := T'_0 = \{x_4^2,\, x_1x_4,\, x_1x_2\}$. Algorithm 9 considers now the set $R'_0 = \{x_1x_2\} \subset \mathcal{K}[x_1, x_2, x_3]$. Since no term in it contains

x_3, it performs the linear change $x_2 \longleftrightarrow x_3$ and $x_1 \mapsto x_1 + x_3$ and obtains as new leading ideal

$$\langle x_6^2,\ x_5x_6,\ x_4x_6,\ x_3x_6,\ x_5^2,\ x_4x_5,\ x_3x_5,\ x_4^2,\ x_3x_4,\ x_3^2\rangle$$

which is quasi-stable. One sees that the number of elementary linear changes applied is 4, which is the same as for the transformation proposed in (Seiler, 2012).

Remark 5.12. Consider the ideal generated by $U = \{x_3^3,\ x_1^2x_3,\ x_2\} \subset \mathcal{K}[x_1, x_2, x_3]$. We have $U'_3 = \{1\}$, $U'_1 = \{x_1^2\}$ and $U'_0 = \{x_2\}$. One can see that $U'_0 \not\subset \langle U'_1\rangle$ and therefore the second condition of Theorem 5.1 does not hold. Indeed, although the ideal is quasi-stable, U is not its Pommaret basis.

We conclude this section by discussing a recursive test for being in Nœther position.[4] An ideal $\mathcal{I} \subset \mathcal{P}$ with the Krull dimension D is in *Nœther position*, if the ring extension $\mathcal{K}[x_1, \ldots, x_D] \hookrightarrow \mathcal{P}/\mathcal{I}$ is integral, i. e. the image in $\mathcal{P}/\mathcal{I}$ of x_i for any $i = D + 1, \ldots, n$ is a root of a polynomial of the form $X^s + g_1X^{s-1} + \cdots + g_s = 0$ where s is an integer and $g_1, \ldots, g_s \in \mathcal{K}[x_1, \ldots, x_D]$ (see e. g. (Eisenbud, 1995)). Bermejo and Gimenez (2001) proved that $\mathcal{I}$ is in Nœther position, if and only if for each $i = D + 1, \ldots, n$ there exists r_i such that $x_i^{r_i}$ belongs to the leading ideal of $\mathcal{I}$ with respect to $\prec$. Furthermore, they showed that this is equivalent to the fact that $\mathcal{I} + \langle x_1, \ldots, x_D\rangle$ is zero-dimensional. These observation show that $\mathcal{I}$ is in Nœther position, if and only if $\mathrm{lt}(\mathcal{I})$ is as well. In the next proposition, we give a recursive test for being in Nœther position using the minimal generating set of a monomial ideal.

Proposition 5.13. *Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{P}$ be a set of terms with $\lambda_0 < \lambda_1 < \cdots < \lambda_\ell$ the x_n-degrees of its elements. For each $0 \le i \le \ell$, we denote by $U_{\lambda_i} \subseteq U$ the subset of U containing the terms t with $\deg_n(t) = \lambda_i$ and set $U'_{\lambda_i} = \{\, t|_{x_n=1} \mid t \in U_{\lambda_i}\}$. Then the monomial ideal $\langle U\rangle$ is in Nœther position, if and only if the following conditions hold:*
(i) The ideal $\langle U'_{\lambda_0}\rangle \subseteq \mathcal{K}[x_1, \ldots, x_{n-1}]$ is in Nœther position,
(ii) $U \cap \mathcal{K}[x_n] \neq \emptyset$.

Proof. Suppose that the ideal $\langle U\rangle$ is in Nœther position and has dimension D. Then, by (Bermejo and Gimenez, 2001, Lem 4.1), we know that $\langle U\rangle + \langle x_1, \ldots, x_D\rangle$ is zero-dimensional and a pure power of x_n appears in U. Thus, $\langle U'_{\lambda_0}\rangle \subseteq \mathcal{K}[x_1, \ldots, x_{n-1}]$ is an ideal of dimension D and $\langle U'_{\lambda_0}\rangle + \langle x_1, \ldots, x_D\rangle$ is zero-dimensional, proving the first item. On the other hand, for any $i = D + 1, \ldots, n$ there exists r_i such that $x_i^{r_i} \in U$ and this proves the second item.

Conversely, to prove that $\langle U\rangle$ is in Nœther position, we note that a pure power of x_n belongs to U. It follows that $\langle U\rangle$ and $\langle U'_{\lambda_0}\rangle \subseteq \mathcal{K}[x_1, \ldots, x_{n-1}]$ share the same dimension D. From the fact that $\langle U'_{\lambda_0}\rangle$ is in Nœther position, we conclude that $\langle U'_{\lambda_0}\rangle + \langle x_1, \ldots, x_D\rangle$ is zero-dimensional and hence that $\langle U\rangle + \langle x_1, \ldots, x_D\rangle$ is zero-dimensional too, proving the claim. □

Example 5.14. Consider the ideal $\mathcal{I} = \langle x_1^3,\ x_2x_3,\ x_3^2\rangle \subset \mathcal{K}[x_1, x_2, x_3]$. With $U := \{x_1^2,\ x_2x_3,\ x_3^2\}$, one sees that the ideal $\langle U'_0\rangle = \langle x_1^2\rangle \subset \mathcal{K}[x_1, x_2]$ is not in Nœther position and hence, by Proposition 5.13, $\mathcal{I}$ is also not in Nœther position.

[4] While Nœther position is implied by quasi-stable position, the converse is not true.

Remark 5.15. We can adapt Algorithm 9 to transform a given ideal into Nœther position by simply performing the last **for**-loop only for $m = 0$. If we consider the ideal presented in Example 5.11, then one finds the same linear change to transform the ideal into Nœther position. As it has been mentioned, this approach allows us to perform permutations of the variables to get a sparser linear change.

6. Pommaret-like division

Gerdt and Blinkov (2005a,b) introduced only the concept of a Janet-like basis. It is natural to expect that the underlying idea can be applied to other divisions, too, but to the best of our knowledge this has never been done so far. We will give here the general definition of an involutive-like division based on non-multiplicative powers and then extend some related notions. Our main emphasis will, however, lie on the special case of the Pommaret-like division.

Definition 6.1. An *involutive-like division* L on $\mathcal{T} \subset \mathcal{P}$ associates to any finite set $U \subset \mathcal{T}$ of terms and any term $u \in U$ a set of *L-non-multipliers* $\bar{L}(u, U)$ given by the terms contained in an irreducible monomial ideal. The powers generating this irreducible ideal are called the *non-multiplicative powers* $\mathrm{NMP}_L(u, U)$ of $u \in U$. The set of *L-multipliers* $L(u, U)$ is given by the order ideal $\mathcal{T} \setminus \bar{L}(u, U)$. For any term $u \in U$, its *involutive cone* is defined as $C_L(u, U) = u \cdot L(u, U)$. For an involutive division, the involutive cones must satisfy the following conditions:

(i) For two terms $v \neq u \in U$ with $C_L(u, U) \cap C_L(v, U) \neq \emptyset$, we have $u \in C_L(v, U)$ or $v \in C_L(u, U)$.

(ii) If a term $v \in U$ lies in an involutive cone $C_L(u, U)$, then $L(v, U) \subset L(u, U)$.

There are two differences between this definition and Definition 2.1 of an involutive division. Firstly, the non-multipliers are now only required to generate an irreducible ideal instead of a prime one. Therefore we must speak of non-multiplicative powers instead of non-multiplicative variables. Secondly, we have dropped the filter axiom, as we were not able to come up with a Pommaret-like division respecting it in its classical form. The filter axiom is relevant for completion algorithms for the Janet and closely related divisions and for the existence of a strong basis within each weak basis. As we will show below, all these applications are still possible within our framework.

Definition 6.2. For a finite set of terms $U \subset \mathcal{T}$ and an involutive-like division L on $\mathcal{T}$, the *involutive span* of U is the union $C_L(U) = \bigcup_{u \in U} C_L(u, U)$. The set U is *involutively complete* or a *weak involutive basis*, if $C_L(U) = U \cdot \mathcal{T}$. For a *(strong) involutive basis* the union is disjoint, i. e. every term in $C_L(U)$ has a unique involutive divisor.

Definition 6.3. Let L be an involutive-like division on $\mathcal{T}$ and let $U \subset \mathcal{T}$ be a finite set of terms. The terms $t \cdot \mathrm{NMP}_L(t, U)$ with $t \in U$ are minimal among those terms of the monomial ideal $\langle U \rangle$ which are possibly not contained in the involutive span of U. Those terms which are indeed not contained in $C_L(U)$ are called *L-obstructions* of U and we write

$$\mathrm{Obstr}_L(U) = \Big(\bigcup_{t \in U} t \cdot \mathrm{NMP}(t, U)\Big) \setminus C_L(U).$$

The set of minimal elements of $\mathrm{Obstr}_L(U)$ with respect to divisibility is denoted by $\mathrm{MinObstr}_L(U)$.

Example 6.4. The *Janet-like division* assigns non-multiplicative powers to a term x^μ contained in a finite set $U \subset \mathcal{T}$ as follows:

$$\mathrm{NMP}_J(x^\mu, U) = \left\{ x_a^{p(J,x^\mu,U,a)} \mid x_a \in \mathrm{NM}_{\mathcal{J}}(x^\mu, U) \right\},$$

where the exponents are given by

$$p(J, x^\mu, U, a) = \min \{\nu_a - \mu_a \mid x^\nu \in U_{[\mu_{a+1},\dots,\mu_n]} \wedge \nu_a > \mu_a\}.$$

Here, the letter J stands for the Janet-like division, while the classical involutive Janet division from which it is derived is denoted by the calligraphic letter $\mathcal{J}$. We will always use calligraphic letters to denote involutive divisions and roman letters to denote the involutive-like divisions derived from them.

We extend now important notions for involutive divisions like Nœtherianity, continuity and constructivity to involutive-like divisions.

Definition 6.5. The involutive-like division L is called

(i) *Nœtherian*, if for every finite set of terms $U \subset \mathcal{T}$ there exists a finite set $\overline{U} \subset \mathcal{T}$ with $U \subseteq \overline{U}$ such that $\overline{U}$ is an L-basis of the monomial ideal $\langle U \rangle$; such a set $\overline{U}$ is called an *L-completion* of U;

(ii) *continuous*, if for every finite set $U \subset \mathcal{T}$ every sequence $(t_1, t_2, \dots, t_k) \in U^k$ such that $t_i \cdot \mathrm{NMP}_L(t_i, U) \cap C_L(t_{i+1}, U) \neq \emptyset$ for each index $i \in \{1, \dots, k-1\}$ consists of k distinct terms, i. e. $t_i \neq t_j$ for all $1 \leq i < j \leq k$;

(iii) *constructive*, if it is continuous and if additionally for every finite set of terms $U \subset \mathcal{T}$ and for each term $s \in \mathrm{MinObstr}_L(U)$ no term $s' \in C_L(U)$ exists such that $s \in C_L(U \cup \{s'\})$.

Remark 6.6. The above definitions of Nœtherianity and continuity are straightforward generalisations of their classical counterparts. However, the definition of constructivity uses a more restrictive condition than in the classical theory. Because of the filter axiom, one only has to control there the involutive cone of the newly added term s'. In the involutive-like case without such an axiom, we must at the same time also control the involutive cones of all the other terms $t \in U$, as they might get larger when adding s' to U.

The following property of an involutive-like division will serve us as a substitute for the missing filter axiom in some situations.

Definition 6.7. Let L be an involutive-like division on the set of terms $\mathcal{T} \subset \mathcal{P}$. We say that L satisfies the *strong basis property* if for every weak L-basis $U \subset \mathcal{T}$ of the monomial ideal $\langle U \rangle$, there is a subset $\tilde{U} \subseteq U$ such that $\tilde{U}$ is a strong L-basis of the same monomial ideal.

Proposition 6.8. *The Janet-like division is a Nœtherian, continuous and constructive involutive-like division. Moreover, it satisfies the strong basis property.*

Proof. The first statement is due to Gerdt and Blinkov (2005b, Prop. 2, Thms. 1–3). For the strong basis property, we simply remark that every finite set of terms $H \subset \mathcal{T}$ is autoreduced with respect to the Janet-like division; this follows also from (Gerdt and Blinkov, 2005b, Prop. 2). □

Theorem 6.9. *For a continuous involutive-like division L, the finite set of terms $U \subset \mathcal{T}$ is a weak L-basis of the monomial ideal $\langle U \rangle$, if and only if* $\mathrm{MinObstr}_L(U) = \emptyset$.

Proof. The proof is a straightforward generalisation of the proof of the analogous result for involutive divisions. □

Proposition 6.10. *The Janet-like division is related to the Janet division as follows:*

(i) For each term t contained in a finite set $U \subset \mathcal{T}$, we have $C_{\mathcal{J}}(t, U) \subseteq C_J(t, U)$.

(ii) Every Janet basis of a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ is also a Janet-like basis.

(iii) From a Janet-like basis U of the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, one can obtain a Janet basis U' of the same ideal as follows:

$$U' = \left\{ t \cdot x^\mu \mid t \in U \wedge x^\mu \mid \Pi_{x_a^{p_a} \in \mathrm{NMP}_J(t,U)} x_a^{p_a - 1} \right\} .$$

Proof. The first item follows directly from the definitions. The second item is a direct consequence of the first one.

The third item holds, if we can prove that $C_J(U) \subseteq C_{\mathcal{J}}(U')$. Let $v \in C_J(U)$ be an arbitrary term in the Janet-like span of U. Then there exists a term $t \in U$, a term x^μ dividing $\Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{J}}(t,U)} x_a^{p_a - 1}$ and a term $x^\rho \in \mathcal{K}[\mathrm{M}_{\mathcal{J}}(t, U)]$ such that $v = t \cdot x^\mu \cdot x^\rho$. By definition of U', we see that $t \cdot x^\mu \in U'$. It remains to show that $x^\rho \in \mathcal{K}[\mathrm{M}_{\mathcal{J}}(t \cdot x^\mu, U')]$. For this, it suffices to show that $\mathrm{M}_{\mathcal{J}}(t, U) \subseteq \mathrm{M}_{\mathcal{J}}(t \cdot x^\mu, U')$. We do this iteratively by ordering the set of variables $\mathrm{M}_{\mathcal{J}}(t, U)$ descendingly according to their indices and showing the containments $x_j \in \mathrm{M}_{\mathcal{J}}(t \cdot x^\mu, U')$ one after the other.

So let x_j be the variable with the highest index in $\mathrm{M}_{\mathcal{J}}(t, U)$. By definition of the Janet division, we have that $\deg_j(t)$ is maximal among the x_j-degrees of the Janet class $U_{[\deg_{j+1}(t),\ldots,\deg_n(t)]}$. We know that $\deg_j(t \cdot x^\mu) = \deg_j(t)$ and we have to show that it is maximal among the x_j-degrees of the Janet class $U'_{[\deg_{j+1}(t \cdot x^\mu),\ldots,\deg_n(t \cdot x^\mu)]}$. To see this, we now analyse which elements $s \in U$ induce elements $s \cdot x^\theta$ in this Janet class of U'. We consider first those terms $s \in U$ which are *not* in the same Janet class of U as t. If s is lexicographically smaller than t, then by analysing the highest variable index ℓ where s and t differ, we see, by definition of Janet-like non-multiplicative powers, that all terms $u = s \cdot x^\theta \in U'$ induced by s have $\deg_\ell(u) < \deg_\ell(t)$. However, $\deg_\ell(t \cdot x^\mu) \geq \deg_\ell(t)$. Hence, $s \cdot x^\theta$ and $t \cdot x^\mu$ are *not* in the same Janet class of U'. If s is lexicographically larger than t, then again by analysing the highest variable index ℓ where s and t differ, we see that $p(J, t, U, \ell) \leq \deg_\ell(s) - \deg_\ell(t)$ and hence, $\deg_\ell(t \cdot x^\mu) < \deg_\ell(s)$, whereas, obviously, $\deg_\ell(s \cdot x^\theta) \geq \deg_\ell(s)$ for all terms $s \cdot x^\theta$ induced by s in U'. Hence, $t \cdot x^\mu$ and $s \cdot x^\theta$ are *not* in the same Janet class of U'.

It remains to analyse the case of a term $s \in U$ which is in the same Janet class $U_{[\deg_{j+1}(t),\ldots,\deg_n(t)]}$ as t. If $\deg_j(s) < \deg_j(t)$, then it is easy to see that also for the induced term $s \cdot x^\theta$, $\deg_j(s \cdot x^\theta) < \deg_j(t)$. If, on the other hand, $\deg_j(s) \geq \deg_j(t)$, then by the Janet multiplicativity of x_j for t, we have in fact $\deg_j(s) = \deg_j(t)$ and x_j is also Janet multiplicative for s. Moreover, the Janet-like powers of variables x_a with

$a > j$ are the same for s and t. So s can induce terms $s \cdot x^{\theta}$ which are in the same Janet class of U' as $t \cdot x^{\mu}$, namely exactly for those x^{θ} which have the same projection on the subring $\mathcal{K}[x_{j+1}, x_{j+2}, \dots, x_n]$ as x^{μ}. But since x_j is Janet multiplicative for s, we must have $\deg_j(x^{\theta}) = 0$ by definition of U'. This proves that $\deg_j(t)$ is still maximal among all x_j-degrees of elements of the Janet class $U'_{[\deg_{j+1}(t \cdot x^{\mu}), \dots, \deg_n(t \cdot x^{\mu})]}$.

Thus, we have shown that $x_j \in \mathrm{M}_{\mathcal{J}}(t \cdot x^{\mu}, U')$. The iteration over the variables of $\mathrm{M}_{\mathcal{J}}(t \cdot x^{\mu}, U')$ which have lower indices than j can now be performed using similar arguments, making use of the equality $\deg_j(t) = \deg_j(t \cdot x^{\mu})$. □

We introduce now an involutive-like division based on the Pommaret division. Note that it is no longer a global division. This is not very surprising, as the very idea of involutive-like divisions consists of comparing different terms in a given set.

Definition 6.11. The *Pommaret-like division P* assigns to each term $t \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T}$ non-multiplicative powers as follows: For each x_a with $a > \operatorname{cls}(t)$, set

$$p(P, t, U, a) = \begin{cases} 1, & \text{if } \; x_a \in \mathrm{M}_{\mathcal{J}}(t, U), \\ p(J, t, U, a), & \text{if } \; x_a \in \mathrm{NM}_{\mathcal{J}}(t, U). \end{cases}$$

Note that no non-multiplicative power is assigned to any variable x_b with $b \leq \operatorname{cls}(t)$.

Proposition 6.12. *The Pommaret-like division is an involutive-like division.*

Proof. Let $U \subset \mathcal{T}$ be a finite set of terms. Let s and t be two terms in U whose Pommaret-like cones have a non-empty intersection: $\mathcal{C}_P(s, U) \cap \mathcal{C}_P(t, U) \neq \emptyset$. Without loss of generality, $\operatorname{cls}(s) \leq \operatorname{cls}(t)$. Consider an arbitrary variable $x_j \in \mathrm{M}_{\mathcal{J}}(t, U)$ with $j > \operatorname{cls}(t)$. By definition of the Pommaret-like division, we have $\deg_j(u) = \deg_j(t)$ for all terms $u \in \mathcal{C}_P(t, U)$. Thus, if we pick a term $v \in \mathcal{C}_P(s, U) \cap \mathcal{C}_P(t, U)$, we also have $\deg_j(v) = \deg_j(t)$. By definition, s divides v and hence $\deg_j(s) \leq \deg_j(t)$. If the strict inequality $\deg_j(s) < \deg_j(t)$ were true, then this would imply $x_j \in \mathrm{NM}_{\mathcal{J}}(s, U)$ and $\mathrm{NMP}_P(s, U) \leq \deg_j(t) - \deg_j(s)$, in contradiction to $v \in \mathcal{C}_P(s, U)$. Hence, we can conclude that $\deg_j(s) = \deg_j(t)$.

Now let x_{ℓ} be a variable such that $\ell > \operatorname{cls}(t)$ and $\ell \in \mathrm{NM}_{\mathcal{J}}(t, U)$. A power of it is a Pommaret-like non-multiplicative power of t and we have that $\deg_{\ell}(t) \leq \deg_{\ell}(u) < \deg_{\ell}(t) + p(P, t, U, \ell)$ for all terms $u \in \mathcal{C}_P(t, U)$. In particular, these inequalities hold for any term $v \in \mathcal{C}_P(s, U) \cap \mathcal{C}_P(t, U)$. Now let $\hat{\ell}$ be the greatest index of such a variable. Then, since by the first paragraph of this proof $s \in U_{[\deg_{\hat{\ell}+1}(t), \dots, \deg_n(t)]}$ and since $\deg_{\hat{\ell}}(s) \leq \deg_{\hat{\ell}}(v) < \deg_{\ell}(t) + p(P, t, U, \ell)$ by the definition of Janet-like non-multiplicative powers, $\deg_{\hat{\ell}}(s) \leq \deg_{\hat{\ell}}(t)$. But a strict inequality is not possible here (apply again the definition of Janet-like non-multiplicative powers). Hence, $\deg_{\hat{\ell}}(s) = \deg_{\hat{\ell}}(t)$. It is now possible to apply the same arguments to the next highest index ℓ and so on obtaining after finitely many steps that $\deg_j(s) = \deg_j(t)$ for all $j > \operatorname{cls}(t)$.

It now only remains to analyse the degrees at the variable $x_{\operatorname{cls}(t)}$. First, let us assume additionally that $\operatorname{cls}(s) < \operatorname{cls}(t)$. For any term $v \in \mathcal{C}_P(s, U) \cap \mathcal{C}_P(t, U)$, we have that $\deg_{\operatorname{cls}(t)}(v) \geq \deg_{\operatorname{cls}(t)}(t)$. If the strict inequality $\deg_{\operatorname{cls}(t)}(s) < \deg_{\operatorname{cls}(t)}(t)$ held, then, using the fact that $s \in U_{[\deg_{\mathrm{cls}(t)+1}(t),\dots,\deg_n(t)]}$ and the definition of Janet-like non-multiplicative powers, we would obtain the inequality $p(P, s, U, \mathrm{cls}(t)) \leq \deg_{\mathrm{cls}(t)}(t) - \deg_{\mathrm{cls}(t)}(s)$, in contradiction to the constraints on $\deg_{\mathrm{cls}(t)}(v)$ found above.

Finally, consider the case $\mathrm{cls}(s) = \mathrm{cls}(t)$. Then, by definition of the Pommaret-like division, there exists neither for s nor for t a non-multiplicative power of the variable $x_{\mathrm{cls}(t)}$ and, keeping in mind the conclusion of the second paragraph of this proof, we get that $C_P(s, U) \subset C_P(t, U)$ in the case that $\deg_{\mathrm{cls}(t)}(t) < \deg_{\mathrm{cls}(t)}(s)$ and $C_P(t, U) \subset C_P(s, U)$ in the case that $\deg_{\mathrm{cls}(t)}(s) < \deg_{\mathrm{cls}(t)}(t)$. This finishes the proof. □

Proposition 6.13. *The Pommaret-like division is not Nœtherian.*

Proof. The monomial ideal $\mathcal{I} = \langle x_1 \rangle \trianglelefteq \mathcal{K}[x_1, x_2]$ does not possess a finite Pommaret-like basis. To see this, observe that for any finite set of terms $U \subset \mathcal{I}$ and for all terms $t \in U$ with $\deg_2(t) =: D$ maximal, $x_2 \in \mathrm{M}_{\mathcal{J}}(t, U)$, and hence $x_2 \in \mathrm{NMP}_P(t, U)$. For all terms $s \in U$ with the degree $\deg_2(s)$ non-maximal in U, we have $\deg_2(v) \leq D$ for all $v \in C_P(s, U)$. So, for all terms $u \in \mathcal{I}$ with $\deg_2(u) > D$, we have $u \notin C_P(U)$. □

Proposition 6.14. *The Pommaret-like division is continuous.*

Proof. The proof is a generalisation of the proof of the analogous result for the classical Pommaret division. Let $U \subset \mathcal{T}$ be a finite set of terms and $(t_1, \dots, t_k) \in U^k$ a sequence of terms as in the definition of involutive-like continuity. Let $i \in \{1, \dots, k-1\}$ be an arbitrary index and let $v_i \in t_i \cdot \mathrm{NMP}_P(t_i, U)$ be a prolongation with $v_i \in C_P(t_{i+1}, U)$. We know that $v_i = t_i \cdot x_j^p$ for some $j > \mathrm{cls}(t_i)$. The divisibility of v_i by t_{i+1} implies that $\mathrm{cls}(t_{i+1}) \geq \mathrm{cls}(t_i)$ and if $\mathrm{cls}(t_{i+1}) = \mathrm{cls}(t_i)$, then $\deg_{\mathrm{cls}(t_i)}(t_{i+1}) \leq \deg_{\mathrm{cls}(t_i)}(t_i)$. Finally, let us assume that $\deg_{\mathrm{cls}(t_i)}(t_{i+1}) \leq \deg_{\mathrm{cls}(t_i)}(t_i)$. Then t_{i+1} is in the Janet class $U_{[\deg_{j+1}(t_i),\dots,\deg_n(t_i)]}$. Indeed, for all indices $b > j$ the divisibility of v_i by t_{i+1} gives $\deg_b(t_{i+1}) \leq \deg_b(t_i)$ and if any of these inequalities were strict, then we would get a Pommaret-like non-multiplicative power for t_{i+1} at that index, in contradiction to $v_i \in C_P(t_{i+1}, U)$. Moreover, a similar argument now gives that $\deg_j(t_{i+1}) = \deg_j(v_i)$. Thus, $t_{i+1} \succ_{\mathrm{lex}} t_i$. In conclusion, the sequence $(t_1, \dots, t_k)$ must consist of pairwise distinct terms, finishing the proof. □

Proposition 6.15. *The Pommaret-like division is constructive.*

Proof. Let $U \subset \mathcal{T}$ be a finite set of terms, $t \in U$ a term in it and $s = t \cdot x_j^p$ a product of t with $x_j^p \in \mathrm{NMP}_P(t, U)$ such that $s \in \mathrm{MinObstr}_P(t, U)$. We must show that for no term $s' \in C_P(U) \setminus U$ the relation $s \in C_P(U \cup \{s'\})$ holds. Before coming to the main part of the proof, let us show that whenever $u \in \mathrm{Obstr}_P(U)$, we must have $u \in C_P(v, U \cup \{v\})$ for any term $v \in C_P(U)$ with $u \in C_P(U \cup \{v\})$. To see this, first note that the only way how there can be a term $h \in U$ for which a term r exists with $r \in C_P(h, U \cup \{v\}) \setminus C_P(h, U)$ is if there exists an index $\ell > \mathrm{cls}(h)$ with $x_\ell \in \mathrm{M}_{\mathcal{J}}(h, U)$ such that $x_\ell \in \mathrm{NM}_{\mathcal{J}}(h, U \cup \{v\})$. This means that $\deg_\ell(h)$ is maximal among the x_j-degrees of the Janet class $U_{[\deg_{\ell+1}(h),\dots,\deg_n(h)]}$, that there is the additional term v in the Janet class $(U \cup \{v\})_{[\deg_{\ell+1}(h),\dots,\deg_n(h)]}$ and that $\deg_\ell(v) > \deg_\ell(h)$. There is a term $w \in U$ with $v \in C_P(w, U)$. We now distinguish two cases. If $\mathrm{cls}(w) > \ell$, then, by definition of Pommaret-like non-multiplicative powers and by the fact that w

divides the term $v \in (U \cup \{v\})_{[\deg_{\ell+1}(h),\dots,\deg_n(h)]}$, we have that w is in the Janet class $U_{[\deg_{\mathrm{cls}(w)+1}(h),\dots,\deg_n(h)]}$ and that $\deg_{\mathrm{cls}(w)}(w) \le \deg_{\mathrm{cls}(w)}(h)$. But from this it follows in particular that $\mathrm{NMP}_P(w,U) = \mathrm{NMP}_P(h,U) \cap \mathcal{K}[x_{\mathrm{cls}(w)+1},\dots,x_n]$. But, by construction, also

$$\mathrm{NMP}_P(h,U) \cap \mathcal{K}[x_{\mathrm{cls}(w)+1},\dots,x_n] = \mathrm{NMP}_P(h,U\cup\{v\}) \cap \mathcal{K}[x_{\mathrm{cls}(w)+1},\dots,x_n]\ .$$

We can conclude that $C_P(h,U\cup\{v\}) \subseteq C_P(w,U)$ and now the assumption $u \in C_P(h,U\cup\{v\})$ would lead to the contradiction $u \in C_P(w,U)$. Thus, it is not possible that $\mathrm{cls}(w) > \ell$. If, on the other hand, $\mathrm{cls}(w) \le \ell$, then, arguing similarly as above, we get that w is in the Janet class $U_{[\deg_{\ell+1}(h),\dots,\deg_n(h)]}$. Recall that v is in the Pommaret-like cone $C_P(w,U)$ and that $\deg_\ell(v) > \deg_\ell(h)$. However, we also know that $\deg_\ell(h)$ is maximal among the x_j-degrees of the Janet class $U_{[\deg_{\ell+1}(h),\dots,\deg_n(h)]}$. Thus $\deg_\ell(w) \le \deg_\ell(h)$. But this implies that the exponent $p(P,w,U,\ell)$ of the Pommaret-like non-multiplicative power for w at x_ℓ is less or equal to $\deg_\ell(h) - \deg_\ell(w)$. Hence, no term in the Pommaret-like cone $C_P(w,U)$ can have an x_ℓ-degree greater than $\deg_\ell(h)$. This contradicts $\deg_\ell(v) > \deg_\ell(h)$. Thus, we have shown that should it at all be possible to lift the minimal obstruction $s \in \mathrm{MinObstr}_P(U)$ by adding an element $s' \in C_P(U)$ to U, we must have $s \in C_P(s', U\cup\{s'\})$.

Let us return to the terms $s = t \cdot x_j^p$ and s'. We distinguish two cases, in accordance with the case distinction of the assignment of Pommaret non-multiplicative variables. First, assume that $x_j^p \in \mathrm{NMP}_J(t,U)$. By the definition of the Pommaret-like division, we know additionally that $j > \mathrm{cls}(t)$ and that $\mathrm{cls}(t) = \mathrm{cls}(s)$. Arguing by *reductio ad absurdum*, assume that there does exist a term $s' \in C_P(U)$ with $s \in C_P(U\cup\{s'\})$. Here again, we can distinguish two cases. First, let us assume that $\mathrm{cls}(s') \le \mathrm{cls}(s)$. Then, by taking the projections of all the terms in U, of s and of s' to the subring $\mathcal{K}[x_{\mathrm{cls}(s)+1},\dots,x_n]$, we obtain a configuration which is a counterexample to the constructivity of the Janet-like division. Indeed, denoting all projections by adding a bar on top of the symbols, we have then $\overline{s} \in \mathrm{MinObstr}_J(\overline{U})$, $\overline{s'} \in C_J(\overline{U})$ and $\overline{s} \in C_J(\overline{U}\cup\{\overline{s'}\})$. So we are left with the other case, i.e. with the case $\mathrm{cls}(s') > \mathrm{cls}(s)$. Note that by construction, s' is a proper divisor of s, as it must obviously be a divisor and the two terms cannot be equal since $s \in \mathrm{Obstr}_P(U)$. In particular, this implies $s' \prec_{\mathrm{lex}} s$. So there is a maximal index ℓ where the x_ℓ-degrees of s and s' differ and we have $\deg_\ell(s') < \deg_\ell(s)$. Again we must distinguish two cases. Firstly, let us assume that $\ell \le \mathrm{cls}(s')$. An immediate consequence is $\ell \le \mathrm{cls}(v)$ for any term v with $s' \in C_P(v,U)$. But then also $s \in C_P(v,U)$, which is not possible as $s \in \mathrm{Obstr}_P(U)$. Secondly, consider the case $\ell > \mathrm{cls}(s')$. Then $x_\ell \in \mathrm{NM}_{\mathcal{J}}(s', U\cup\{s'\})$, because either $\deg_\ell(s) = \deg_\ell(t)$ and then $t \in U$ causes x_ℓ to be Janet non-multiplicative for s' or $j = \ell$, $\deg_\ell(s) = \deg_\ell(t) + p(J,t,U,\ell)$ and the same term, say, $r \in U$, which causes the Janet-like non-multiplicative power for t at x_ℓ in U causes x_ℓ to be Janet non-multiplicative also for s' in $U\cup\{s'\}$. The exponent of the corresponding Janet-like non-multiplicative power then satisfies the inequality $p(J,s',U\cup\{s'\},\ell) \le \deg_\ell(s) - \deg_\ell(s')$. This of course then also holds for the induced Pommaret-like non-multiplicative power. This contradicts $s \in C_P(s', U\cup\{s'\})$. The analysis of the case $x_j^p \in \mathrm{NMP}_J(t,U)$ is now finished.

Let us turn to the analysis of the case $x_j \in \mathrm{M}_{\mathcal{J}}(t,U)$. Here $p = 1$ and $x_j^p = x_j$. So, $s = t \cdot x_j$. By the definition of the Pommaret-like division, we know additionally that

$j > \mathrm{cls}\,(t)$ and that $\mathrm{cls}\,(t) = \mathrm{cls}\,(s)$. Arguing again by *reductio ad absurdum*, assume that there does exist a term $s' \in C_P(U)$ with $s \in C_P(U \cup \{s'\})$. Similarly to the situation in the last paragraph, we must have $s' \prec_{\mathrm{lex}} s$ and there is a maximal index ℓ where the x_ℓ-degrees of s' and s differ. We know then that $\deg_\ell(s') < \deg_\ell(s)$. We now distinguish several cases which reflect the relation of the indices j and ℓ.

The first main case is $\ell > j$. Then it follows that s' is in the Janet class $(U \cup \{s'\})_{[\deg_{\ell+1}(t),\dots,\deg_n(t)]}$ and that $\deg_\ell(t) = \deg_\ell(s) > \deg_\ell(s')$. Hence $x_\ell \in \mathrm{NM}_{\mathcal{J}}(s', U \cup \{s'\})$ and $p(J, s', U \cup \{s'\}, \ell) \leq \deg_\ell(s) - \deg_\ell(s')$ leading to a contradiction if this Janet-like non-multiplicative power is also a Pommaret-like non-multiplicative power for s'. Otherwise, we would have $\mathrm{cls}\,(s') \geq \ell$ and from this it is not hard to show that for the term $w \in U$ with $s' \in C_P(w, U)$ we would also have $s \in C_P(w, U)$, in contradiction to $s \in \mathrm{Obstr}_P(U)$.

The second main case is $\ell = j$. It follows that s' is in the Janet class $(U \cup \{s'\})_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$. Again, if $\mathrm{cls}\,(s') \geq j$, then it is not hard to obtain a contradiction to $s \in \mathrm{Obstr}_P(U)$. So we may assume that $\mathrm{cls}\,(s') < j$. Since $\deg_j(s) = \deg_j(t) + 1$ and $s \in C_P(s', U \cup \{s'\})$, we must have $\deg_j(s') = \deg_j(t)$. But this implies that $x_j \in \mathrm{M}_{\mathcal{J}}(s', U \cup \{s'\})$ and we get the Pommaret-like non-multiplicative power $x_j \in \mathrm{NMP}_P(s', U \cup \{s'\})$, in contradiction to $s \in C_P(s', U \cup \{s'\})$.

The third main case is $\ell < j$. We then get that s' is in the Janet class $(U \cup \{s'\})_{[\deg_j(t)+1,\deg_{j+1}(t),\dots,\deg_n(t)]}$. Again, if $\mathrm{cls}\,(s') \geq j$, then it is not hard to obtain a contradiction to $s \in \mathrm{Obstr}_P(U)$. So we may assume that $\mathrm{cls}\,(s') < j$. But then again, we know that there is a term $w \in U$ with $s' \in C_P(w, U)$ and since this term divides s', one can see quite easily that it must belong to the Janet class $U_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$. If it has $\mathrm{cls}\,(w) \geq j$, then again it is not hard to obtain a contradiction to $s \in \mathrm{Obstr}_P(U)$, and if $\mathrm{cls}\,(w) < j$, then, via the fact that $x_j \in \mathrm{M}_{\mathcal{J}}(t, U)$, we get that $\deg_j(w) \leq \deg_j(t)$ and thus a contradiction to $s' \in C_P(w, U)$. This finishes the proof. □

Proposition 6.16. *The Pommaret-like division is related to the Pommaret division as follows:*

(i) For each term $t \in \mathcal{T}$ in a finite set $U \subset \mathcal{T}$, we have $C_{\mathcal{P}}(t, U) \subseteq C_P(t, U)$.
(ii) Every Pommaret basis of a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ is also a Pommaret-like basis.
(iii) From a Pommaret-like basis U of the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, one can obtain a Pommaret basis U' of the same ideal as follows:

$$U' = \left\{ t \cdot x^\mu \mid t \in U \wedge x^\mu \mid \Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{P}}(t,U)} x_a^{p_a - 1} \right\} .$$

(iv) A monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ is quasi-stable, if and only if it possesses a finite Pommaret-like basis.

Proof. The first item follows directly from the definitions; the second one is a immediate consequence of it. The third item follows, if we can show that $C_P(U) \subseteq C_{\mathcal{P}}(U')$. Let $u \in C_P(U)$ be an arbitrary term in the Pommaret-like span. Then there exists a term $t \in U$, a divisor x^μ of $\Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{P}}(t,U)} x_a^{p_a - 1}$ and a term $x^\rho \in \mathcal{K}[x_1, \dots, x_{\mathrm{cls}\,(t)}]$ such that $u = t \cdot x^\mu \cdot x^\rho$. We have to show that there is a Pommaret divisor of u in the set U'. We know that $t \cdot x^\mu \in U'$. It is clear that $\mathrm{cls}\,(t \cdot x^\mu) = \mathrm{cls}\,(t)$. Hence, $u \in C_{\mathcal{P}}(t \cdot x^\mu)$ and we have proved the third item.

The fourth item is a direct consequence of the second and third items, as a monomial ideal is quasi-stable, if and only if it possesses a finite Pommaret basis. □

Lemma 6.17. *A finite set of terms $U \subset \mathcal{T}$ is Pommaret-like autoreduced, if and only if it is Pommaret autoreduced.*

Proof. The only if direction is obvious. So let $U \subset \mathcal{T}$ be a finite set of terms which is Pommaret autoreduced. We want to show that it is also Pommaret-like autoreduced. We argue by *reductio ad absurdum*. Assume that U is Pommaret, but not Pommaret-like autoreduced. Then there exist two terms $s \neq t \in U$ such that $s \in C_P(t, U)$. Let $k = \operatorname{cls}(t)$. We know that $\deg_\ell(s) \geq \deg_\ell(t)$ for each index $k < \ell \leq n$. There must exist an index $k < j \leq n$ with $\deg_j(s) > \deg_j(t)$, since otherwise t would be a Pommaret divisor of s, contradicting the assume Pommaret autoreducedness. We pick the maximal such index j. Then there exists a Janet-like non-multiplicative power $x_j^{p(\mathcal{J},t,U,j)} \in \mathrm{NMP}_{\mathcal{J}}(t, U)$ with $1 \leq p(\mathcal{J}, t, U, j) \leq \deg_j(s) - \deg_j(t)$. This gives also a Pommaret-like non-multiplicative power for t at x_j with the same exponent. Hence, $s \notin C_P(t, U)$ contradicting our assumptions. □

Corollary 6.18. *The Pommaret-like division satisfies the strong basis property.*

Proof. Let the finite set of terms $U \subset \mathcal{T}$ be a weak Pommaret-like basis of the monomial ideal $\langle U \rangle$. If it is a strong basis, then we are done. Otherwise, it is not Pommaret-like autoreduced, and hence it is also not Pommaret autoreduced by Lemma 6.17. We claim that the Pommaret autoreduction $\tilde{U} \subset U$ is a strong Pommaret-like basis of $\langle U \rangle$. More precisely, we will show that $C_P(u, \tilde{U}) = C_P(u, U)$ for each term $u \in \tilde{U}$ which is equivalent to $\mathrm{NMP}_P(u, \tilde{U}) = \mathrm{NMP}_P(u, U)$. The latter statement can be reduced to an analysis of Janet-like non-multiplicative powers: We have to show that

$$\mathrm{NMP}_J(u, \tilde{U}) \cap \mathcal{K}[x_{\operatorname{cls}(u)+1}, \ldots, x_n] = \mathrm{NMP}_J(u, U) \cap \mathcal{K}[x_{\operatorname{cls}(u)+1}, \ldots, x_n] .$$

The set $\tilde{U}$ arises from U by removing elements which possess strict Pommaret divisors in U. It is clear that the removal of strict Pommaret multiples of u does not change the Janet-like non-multiplicative powers of u lying in $\mathcal{K}[x_{\operatorname{cls}(u)+1}, \ldots, x_n]$. Let $v \in \tilde{U} \setminus \{u\}$ be any term for which a strict Pommaret multiple $t \in U \setminus \tilde{U}$ has been removed. If this removal would change a Janet-like non-multiplicative power of u lying in $\mathcal{K}[x_{\operatorname{cls}(u)+1}, \ldots, x_n]$, then there would be some index $\ell > \operatorname{cls}(u)$ such that t lies in the Janet class $U_{[\deg_{\ell+1}(u),\ldots,\deg_n(u)]}$. Since the removal of t changes the non-multiplicative power of u at x_ℓ and we know that $\deg_\ell(v) \leq \deg_\ell(t)$, we must have $\deg_\ell(v) \leq \deg_\ell(u) < \deg_\ell(t)$. Since v is a Pommaret divisor of t, it follows that v is in the same Janet-class of U as u and t and additionally, $\operatorname{cls}(v) \geq \ell$. This in turn shows that v is a strict Pommaret divisor of u, which is impossible, since u and v both survived the Pommaret autoreduction of U. □

Theorem 6.19. *The Pommaret-like and the Janet-like divisions are related as follows:*

(i) Let $U \subset \mathcal{T}$ be a finite set of terms which is autoreduced with respect to the Pommaret-like division. Then $C_P(t, U) \subseteq C_J(t, U)$ for each $t \in U$.
(ii) Let $U \subset \mathcal{T}$ be a Pommaret-like basis of the monomial ideal $\langle U \rangle$. Then U is also a Janet-like basis of the same ideal.
(iii) Any minimal Janet-like basis is Pommaret-like autoreduced.
(iv) The unique minimal Janet-like basis of a quasi-stable monomial ideal is also a Pommaret-like basis of the same ideal.

(v) In the situation of (ii), the set U is the unique minimal Pommaret-like basis of $\langle U\rangle$, if and only if it is the unique minimal Janet-like basis of this ideal.

Proof. For the first item, let U be a finite Pommaret-like autoreduced set of terms, $t \in U$ a term and $j \le \operatorname{cls}(t)$ an index. Then x_j is Pommaret-like multiplicative for t and we must show that it is also Janet-like multiplicative. If not, then there exists a term $s \in U$ in the Janet class $U_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$ with $\deg_j(s) > \deg_j(t)$. By the definition of Pommaret-like multiplicative powers, we see that $s \in C_P(t,U) \setminus \{t\}$, in contradiction to the Pommaret-like autoreducedness of U. It only remains to observe that, by the definition of Pommaret-like multiplicative powers, it is clear that for indices $j > \operatorname{cls}(t)$ the Janet-like non-multiplicative powers of t are either identical to the Pommaret non-multiplicative powers or that the Pommaret non-multiplicative powers are linear while the Janet-like division does not pose any restriction on the given variable. This concludes the proof of the first item.

The second item is a direct consequence of the first item, because a Pommaret-like basis is by definition autoreduced with respect to the Pommaret-like division.

For the third item, we only need to show that any minimal Janet-like basis is Pommaret autoreduced in view of Lemma 6.17. So for a given minimal Janet-like basis $U \subset \mathcal{T}$ of the monomial ideal $\mathcal{I} = \langle U\rangle$, we must show that the Pommaret autoreduction of U is still a Janet-like basis of $\mathcal{I}$. If U is already Pommaret autoreduced, there is nothing to prove. If not, then there exists a disjoint partition $U = U_1 \sqcup U_2 \sqcup \ldots \sqcup U_r$ such that, $C_{\mathcal{P}}(s) \cap C_{\mathcal{P}}(t) = \emptyset$ for any two indices $i \neq j \in \{1,\dots,r\}$ and any two terms $s \in U_i$ and $t \in U_j$ and for each $i \in \{1,\dots,r\}$ there exists a unique term $t_i \in U_i$ such that $U_i \subset C_{\mathcal{P}}(t_i)$, i.e. t_i is a strict Pommaret divisor of every term $s \in U_i \setminus \{t_i\}$. We must show that $\{t_1,\dots,t_r\}$ is still a Janet-like basis of $\langle U\rangle$. It suffices to show that we have $C_J(t_i,\{t_1,\dots,t_r\}) \supseteq \bigcup_{s\in U_i} C_J(s,U)$ for each i. To this end, fix an index i and look at the Janet-like non-multiplicative powers of t_i for a variable x_ℓ with $\ell > \operatorname{cls}(t_i)$. We have $p(J,s,U,\ell) = p(J,t_i,U,\ell)$ for each $s \in U_i$, since $\deg_\ell(s) = \deg_\ell(t_i)$ by Pommaret divisibility. Hence, using (Gerdt and Blinkov, 2005b, Prop. 3), $p(J,s,U,\ell) \le p(J,t_i,\{t_1,\dots,t_r\},\ell)$ for all $s \in U_i$. Since $\{t_1,\dots,t_r\}$ is Pommaret autoreduced and hence also Pommaret-like autoreduced, using (i), we see that there are no Janet-like non-multiplicative powers for t_i in $\{t_1,\dots,t_r\}$ at any variable x_ℓ with $\ell \le \operatorname{cls}(t_i)$. Putting everything together, we get $C_J(t_i,\{t_1,\dots,t_r\}) \supseteq C_J(s,U)$ for all $s \in U_i$, which suffices to prove our claim.

For the fourth item, let U be the minimal Janet-like basis of the quasi-stable monomial ideal $\langle U\rangle$. The U is Pommaret autoreduced by (iii). By Proposition 6.10 (iii), we can construct a Janet basis $\overline{U} \supseteq U$ of $\langle U\rangle$. We claim that $\overline{U}$ is also Pommaret autoreduced. Indeed, let $\bar{s},\bar{t} \in \overline{U}$ be two distinct terms which arise as multiples of the terms $s,t \in U$. If $s = t$, then it is not hard to show that $C_{\mathcal{P}}(\bar{s}) \cap C_{\mathcal{P}}(\bar{t}) = \emptyset$. So assume that $s \neq t$. Without loss of generality, let $\operatorname{cls}(s) \le \operatorname{cls}(t) = k$. Then s cannot be in the Janet class $U_{[\deg_k(t),\dots,\deg_n(t)]}$, because otherwise t would be a strict Pommaret divisor of s, contradicting the Pommaret autoreducedness of U. Hence, there is a maximal index ℓ with $k \le \ell \le n$ where $\deg_\ell(s) \neq \deg_\ell(t)$.

If now $\ell = k = \operatorname{cls}(t)$, then we must have $\deg_\ell(s) < \deg_\ell(t)$ and $\operatorname{cls}(s) < k$, again since U is Pommaret autoreduced. Hence, s has a Janet-like non-multiplicative power with respect to the set U at x_ℓ with exponent $p(J,s,U,\ell) \le \deg_\ell(t) - \deg_\ell(s)$. By construction of the Janet basis $\overline{U}$, this implies $\deg_\ell(\overline{s}) < \deg_\ell(t) \le \deg_\ell(\bar{t})$ and, since $\mathrm{cls}(\overline{s}) = \mathrm{cls}(s)$, we get $\mathcal{C}_{\mathcal{P}}(\overline{s}) \cap \mathcal{C}_{\mathcal{P}}(\bar{t}) = \emptyset$. If $\ell > k = \mathrm{cls}(t)$, then $\ell > \max(\mathrm{cls}(s), \mathrm{cls}(t))$ and, similarly as in the last paragraph, we find that $\mathcal{C}_{\mathcal{P}}(\overline{s}) \cap \mathcal{C}_{\mathcal{P}}(\bar{t}) = \emptyset$. Hence, we have proved that the Janet basis $\overline{U}$ is Pommaret autoreduced.

Since the ideal $\langle U \rangle$ is quasi-stable, $\overline{U}$ must be the Pommaret basis of $\langle U \rangle$. This implies that for each term $\bar{t} \in \overline{U}$, its Janet multiplicative variables with respect to $\overline{U}$ agree with its Pommaret multiplicative variables (Gerdt and Blinkov, 2005b). On the other hand, it is not hard to show that the Janet non-multiplicative variables of $\bar{t} \in \overline{U}$ also agree with the Janet non-multiplicative variables of the term $t \in U$ used for the construction of $\bar{t}$. This finally shows that for all $u \in U$, $C_P(u, U) = C_J(u, U)$, which means that U is a Pommaret-like basis of $\langle U \rangle$.

The fifth item follows from (ii) and (iv). □

7. Syzygy Theory of Janet-like and Pommaret-like bases

In the theory of involutive bases, it is well-known that from a given Pommaret or Janet basis, respectively, of a polynomial ideal, one can obtain a Pommaret or Janet basis, respectively, of the syzygy module of this basis with respect to a suitable module term ordering (Seiler, 2010). The goal of this section is to generalise these results also to Pommaret-like and Janet-like involutive bases. We start with an analysis of the set of non-multiplicative powers associated to some term t contained in a finite set of terms U which is not assumed to be an involutive-like basis.

Lemma 7.1. *Let a term $t \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T}$ be given. Then the set $\mathrm{NMP}_P(t, U)$ is a Pommaret-like basis of the monomial ideal $\langle \mathrm{NMP}_P(t, U) \rangle$ generated by it and the set $\mathrm{NMP}_J(t, U)$ is a Janet-like basis of the monomial ideal $\langle \mathrm{NMP}_J(t, U) \rangle$.*

Proof. Let us first consider $\mathrm{NMP}_P(t, U)$, which is of the form $\{x_a^{p(a)}, x_{a+1}^{p(a+1)}, \ldots, x_n^{p(n)}\}$ where $a = \mathrm{cls}(t) + 1$ and $p(b) \in \mathbb{Z}_{>0}$ for all $a \le b \le n$. Let $x_j^p \in \mathrm{NMP}_P(t, U)$. Then one can easily see that $\mathrm{cls}(x_j^p) = j$, $\mathrm{NM}_{\mathcal{J}}(x_j^p, \mathrm{NMP}_P(t, U)) = \{x_{j+1}, x_{j+2}, \ldots, x_n\}$, and

$$\mathrm{NMP}_P(x_j^p, \mathrm{NMP}_P(t, U)) = \mathrm{NMP}_J(x_j^p, \mathrm{NMP}_P(t, U)) = \{v \in \mathrm{NMP}_P(t, U) \mid \mathrm{cls}(v) > j\} .$$

Trivially, x_j^p is Pommaret-like multiplicative for all terms in $v \in \mathrm{NMP}_P(t, U)$ with $\mathrm{cls}(v) > j$ and hence $x_j^p \cdot \mathrm{NMP}(x_j^p, \mathrm{NMP}_P(t, U)) \subset C_P(\mathrm{NMP}_P(t, U))$, proving the statement for the Pommaret-like division.

We now consider $\mathrm{NMP}_J(t, U)$, which is of the form $\{x_a^{p(a)} \mid x_a \in \mathrm{NM}_{\mathcal{J}}(t, U)\}$. Let $x_j^p \in \mathrm{NMP}_J(t, U)$ be a non-multiplicative power. One can see easily that

$$\mathrm{NM}_{\mathcal{J}}(x_j^p, \mathrm{NMP}_J(t, U)) = \{x_b \in \mathrm{NM}_{\mathcal{J}}(t, U) \mid b > j\} ,$$
$$\mathrm{NMP}_J(x_j^p, \mathrm{NMP}_J(t, U)) = \{x_b^{p(b)} \in \mathrm{NMP}_J(t, U) \mid b > j\} .$$

Furthermore, x_j^p is Janet-like multiplicative for all terms $x_b^{p(b)} \in \mathrm{NMP}_J(t, U)$ with $b > j$ and hence $x_j^p \cdot \mathrm{NMP}_J(x_j^p, \mathrm{NMP}_J(t, U)) \subset C_J(\mathrm{NMP}_J(t, U))$, proving the statement for the Janet-like division. □

Up to now, we have only considered involutive-like bases of monomial ideals. For general polynomial ideals in $\mathcal{P}$, we use the following definition.

Definition 7.2. Let L be an involutive-like division, $\prec$ a term ordering on $\mathcal{T}$ and $\mathcal{I} \trianglelefteq \mathcal{P}$ a polynomial ideal. Then a finite set $H \subset \mathcal{I} \setminus \{0\}$ is called an *L-involutive-like basis* of $\mathcal{I}$ with respect to the term ordering $\prec$, if the set of leading terms $\operatorname{lt}(H)$ is a strong L-involutive-like basis of the leading ideal $\operatorname{lt}(\mathcal{I})$ and $|H| = |\operatorname{lt}(H)|$ (i. e. the leading terms of the elements of H are distinct).

Remark 7.3. Let $H \subset \mathcal{T}$ be an L-involutive like basis for the ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ with respect to some involutive-like division L and some term ordering $\prec$. Then H is also a Gröbner basis of $\mathcal{I}$ for $\prec$, since $\operatorname{lt}(H)$ is a generating system of $\operatorname{lt}(\mathcal{I})$. It is also straightforward to introduce the notion of an involutive-like standard representation for every polynomial in an ideal generated by an involutive-like basis and to show that it is unique.

We now recall a construction due to Schreyer, which, given a Gröbner basis H of a polynomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, yields a module term ordering $\prec_H$ on the free module $\mathcal{P}^{|H|}$ and a Gröbner basis of the syzygy module $\operatorname{Syz}(H) \subset \mathcal{P}^{|H|}$ for this ordering $\prec_H$.

Construction 7.4. *Let $H = \{H_1, \dots, H_r\}$ be a Gröbner basis of the ideal $\mathcal{I} \trianglelefteq \mathcal{P}$. On the module $\mathcal{P}^r$ define the* Schreyer module term ordering $\prec_H$ *by*

$$x^\mu \mathbf{e}_i \prec_H x^\nu \mathbf{e}_j \Longleftrightarrow x^\mu \operatorname{lm}(g_i) \prec x^\nu \operatorname{lm}(g_j) \vee (x^\mu \operatorname{lm}(g_i) = x^\nu \operatorname{lm}(g_j) \wedge j < i)\,. \tag{7.1}$$

By Buchberger's criterion, the S-polynomial $\operatorname{S}(h_i, h_j)$ of two generators $h_i, h_j \in H$ has a standard representation $\sum_{\ell=1}^r q_\ell h_\ell$ where $\operatorname{lt}(q_\ell)\operatorname{lm}(h_\ell) \preceq \operatorname{lt}(\operatorname{S}(h_i, h_j))$ for all $\ell \in \{1, \dots, r\}$ with $q_\ell \neq 0$. The leading module term of the corresponding syzygy

$$\mathbf{S}_{ij} := \frac{\operatorname{lcm}(\operatorname{lt}(h_i), \operatorname{lt}(h_j))}{\operatorname{lc}(h_i)\operatorname{lt}(h_i)}\mathbf{e}_i - \frac{\operatorname{lcm}(\operatorname{lm}(h_i), \operatorname{lm}(h_j))}{\operatorname{lc}(h_j)\operatorname{lt}(h_j)}\mathbf{e}_j - \sum_{\ell=1}^r q_\ell \mathbf{e}_\ell \tag{7.2}$$

is $\frac{\operatorname{lcm}(\operatorname{lt}(g_i), \operatorname{lt}(g_j))}{\operatorname{lt}(g_i)}\mathbf{e}_i$ and the set $\{\mathbf{S}_{ij} \mid 1 \leq i < j \leq s\}$ is a Gröbner basis of $\operatorname{Syz}(H)$ *for the module term ordering $\prec_H$.*

Let us now consider the special case when H is an L-involutive-like basis of the polynomial ideal $\langle H \rangle$ for a continuous involutive-like division L. Analogously to the case of involutive bases, we can construct a directed graph with one node for each leading term $\operatorname{lt}(h_j) \in \operatorname{lt}(H)$ and a directed edge from $\operatorname{lt}(h_j)$ to $\operatorname{lt}(h_i)$ exactly when there is an L-non-multiplicative power $x_k^p \in \operatorname{NMP}_L(\operatorname{lt}(h_j), \operatorname{lt}(H))$ such that $x_k^p \cdot \operatorname{lt}(h_j) \in C_L(\operatorname{lt}(h_i), \operatorname{lt}(H))$. We call it the *$L$-graph of* $\operatorname{lt}(H)$. Note that it is acyclic because of the continuity of L. This leads to the concept of L-orderings.

Definition 7.5. Let $U \subset \mathcal{T}$ be a strong L-involutive-like basis for the monomial ideal $\langle U \rangle$ for a continuous involutive-like division L. Then an *L-ordering* of U is an enumeration $U = \{u_1, \dots, u_r\}$ for which $i < j$ whenever there exists a non-multiplicative power $x_k^p \in \operatorname{NMP}_L(u_i, U)$ such that $x_k^p \cdot u_i \in C_L(u_j, U)$.

The following proposition is immediate from the above discussion.

Proposition 7.6. *Let the involutive-like division L be continuous. Then for each strong L-involutive-like basis U there exists an L-ordering.*

We continue the analysis of the syzygies of an L-involutive-like basis H of the polynomial ideal $\langle H \rangle$ with respect to a continuous involutive-like division L. Assume that $H = \{h_1, \ldots, h_r\}$ is enumerated according to an L-ordering. Let $x_k^p \in \mathrm{NMP}_L(\mathrm{lt}(h_i), \mathrm{lt}(H))$ be a non-multiplicative power of a leading term $\mathrm{lt}(h_i) \in \mathrm{lt}(H)$. Then there exists a unique generator $h_j \in H \setminus \{h_i\}$ such that $x_k^p \in C_L(\mathrm{lt}(h_j), \mathrm{lt}(H))$. The polynomial $h_{ij} := x_k^p h_i - c(x_k^p \,\mathrm{lt}(h_i)/\,\mathrm{lt}(h_j))h_j \in \mathcal{I}$ has, for a suitably chosen scalar $c \in \mathcal{K}$, the leading term $\mathrm{lt}(h_{ij}) \prec x_k^p \,\mathrm{lt}(h_i)$. Then, the standard representation obtained by involutive-like reduction $h_{ij} = \sum_{h_\alpha \in H} q_\alpha \cdot h_\alpha$ yields the syzygy $\mathbf{S}_{i;k} = x_k^p \mathbf{e}_i - c(x_k^p \,\mathrm{lt}(h_i)/\,\mathrm{lt}(h_j))\mathbf{e}_j - \sum_{h_\alpha \in H} q_\alpha \mathbf{e}_\alpha$. We have $\mathrm{lt}(\mathbf{S}_{i;k}) = x_k^p \mathbf{e}_i$ with respect to the Schreyer ordering $\prec_H$. We now show that in the case that L is either the Pommaret-like or the Janet-like division the collection of the thus obtained syzygies is an involutive-like basis of the syzygy module $\mathrm{Syz}(H)$.

Theorem 7.7. *Let $H = \{h_1, \ldots, h_r\} \subset \mathcal{P}$ be a strong Janet-like or Pommaret-like, respectively, basis of the polynomial ideal $\langle H \rangle$ enumerated according to a J- or P-ordering, respectively. Then the set*

$$H_{\mathrm{Syz}} = \left\{ \mathbf{S}_{i;k} \mid 1 \le i \le r \,\wedge\, \exists x_k^p \in \mathrm{NMP}_L(\mathrm{lt}(h_i), \mathrm{lt}(H)) \right\}$$

of syzygies induced by non-multiplicative powers from a Janet-like or Pommaret-like, respectively, basis of the syzygy module $\mathrm{Syz}(H)$ *with respect to the Schreyer module term ordering* $\prec_H$.

Proof. By construction, $H_{\mathrm{Syz}} \subseteq \mathrm{Syz}(H)$. Let $\mathbf{0} \neq \mathbf{S} = \sum_{i=1}^{|H|} s_i \mathbf{e}_i \in \mathrm{Syz}(H)$ be any non-zero syzygy. Then there exists a module term $x^\mu \mathbf{e}_\ell \in \mathrm{supp}(\mathbf{S})$ such that $x^\mu \,\mathrm{lt}(h_\ell) \notin C_L(\mathrm{lt}(h_\ell), \mathrm{lt}(H))$, as otherwise the leading terms of the summands $s_i h_i$ are distinct and the highest appearing term cannot cancel out. Thus, a non-multiplicative power $x_k^p \in \mathrm{NMP}_L(\mathrm{lt}(h_\ell), \mathrm{lt}(H))$ exists such that x_k^p divides x^μ and thus $\mathrm{lt}(\mathbf{S}_{\ell;k})$ divides $x^\mu \mathbf{e}_\ell$. This means that any non-zero syzygy is reducible with respect to H_{Syz} which implies the existence of a standard representation of $\mathbf{S}$ with respect to H_{Syz}. Hence, H_{Syz} is a Gröbner basis of $\mathrm{Syz}(H)$ with respect to $\prec_H$. It is in fact an involutive-like basis because of Lemma 7.1. □

8. Conclusions

We studied a not much known recursive criterion for Janet bases already proven by Janet himself. We provided a slightly modified form of it with a novel proof and exploited this for the design of a novel algorithm for the construction of Janet bases. Right now, we cannot make any statements about the efficiency of this algorithm compared to the classical one. From a theoretical point of view it is interesting to note that the novel approach also leads to an algorithm for turning a given Janet basis into a minimal one. To the best of our knowledge, this is the first such algorithm; previous algorithms only permit the direct construction of minimal Janet bases and cannot really exploit the previous knowledge of a (non-minimal) Janet basis.

We extended the recursive approach also to Pommaret bases. In their construction, a crucial part is always to determine "good" coordinates. We showed that the novel recursive criterion also permits the effective construction of such coordinates. Compared with the results by Hashemi et al. (2018), the proof of the termination of this process becomes much simpler, which is a great theoretical advantage. It still remains to be checked whether the novel approach is also more efficient in practical computations. There are some indications that this might be the case, as Algorithm 9 e. g. naturally incorporates permutations (which helps to preserve sparsity) and groups several elementary moves into one transformations. However, the decisive factor is how sparse the finally obtained linear change really is and it is difficult to predict how different approaches fare in this respect.

We then studied Janet-like bases and provided also for them a recursive criterion leading to a corresponding completion algorithm. Janet-like bases are of interest, as they are typically smaller than Janet bases, but still permit most of the typical applications of the latter ones. We then extended the idea behind Janet-like bases to arbitrary involutive divisions and studied in detail the case of the Pommaret-like division. We had to drop the filter axiom in this process, but replaced it with the strong basis property which is e. g. satisfied by the Pommaret-like division. We could show that the Pommaret-like division possesses the crucial properties for algorithms: continuity and constructivity.

Involutive-like bases are of interest, as they are generally smaller than the corresponding involutive bases, but still can be used for most applications. We showed in particular a Schreyer theorem for Janet-like and Pommaret-like bases. Such a result represents a first step towards the construction of free resolutions. It is well-known that Pommaret bases induce nice, though generally non-minimal resolutions (Seiler, 2010; Albert et al., 2015). It can be expected that Pommaret bases induce smaller resolutions and thus allow for the more efficient determination of Betti numbers.

References

M. Janet, Sur les systèmes d'équations aux dérivées partielles., C. R. Acad. Sci., Paris 170 (1920) 1101–1103.

M. Janet, Leçons sur les Systèmes d'Équations aux Dérivées Partielles, Cahiers Scientifiques, Fascicule IV, Gauthier-Villars, Paris, 1929.

A. Zharkov, Y. Blinkov, Involution approach to investigating polynomial systems., Math. Comput. Simul. 42 (1996) 323–332.

V. P. Gerdt, Y. A. Blinkov, Involutive bases of polynomial ideals., Math. Comput. Simul. 45 (1998) 519–541.

V. P. Gerdt, Y. A. Blinkov, Minimal involutive bases, Math. Comput. Simul. 45 (1998) 543–560.

W. M. Seiler, Involution — The formal theory of differential equations and its applications in computer algebra, Berlin: Springer, 2010.

C. Berkesch, F.-O. Schreyer, Syzygies, finite length modules, and random curves., in: Commutative algebra and noncommutative algebraic geometry. Volume I: Expository articles, Cambridge: Cambridge University Press, 2015, pp. 25–52.

A. Hashemi, M. Schweinfurter, W. M. Seiler, Deterministic genericity for polynomial ideals., J. Symb. Comput. 86 (2018) 20–50.

V. P. Gerdt, Y. A. Blinkov, Janet-like Gröbner bases, in: Computer algebra in scientific computing. 8th international workshop, CASC 2005, Kalamata, Greece, September 12–16, 2005. Proceedings, Berlin: Springer, 2005a, pp. 184–195.

V. P. Gerdt, Y. A. Blinkov, Janet-like monomial division., in: Computer algebra in scientific computing. 8th international workshop, CASC 2005, Kalamata, Greece, September 12–16, 2005. Proceedings, Berlin: Springer, 2005b, pp. 174–183.

D. A. Cox, J. Little, D. O'Shea, Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. 4th revised ed., Cham: Springer, 2015.

V. P. Gerdt, Involutive algorithms for computing Gröbner bases., in: Computational commutative and non-commutative algebraic geometry. Proceedings of the NATO Advanced Research Workshop, 2004, Amsterdam: IOS Press, 2005, pp. 199–225.

V. P. Gerdt, Y. A. Blinkov, D. A. Yanovich, Construction of Janet bases. I: Monomial bases, in: Computer algebra in scientific computing, CASC 2001. Proceedings of the 4th international workshop, Konstanz, Germany, September 22–26, 2001, Berlin: Springer, 2001, pp. 233–247.

M. Ceria, Bar code for monomial ideals, J. Symb. Comput. 91 (2019a) 30–56.

M. Ceria, Bar code vs Janet tree, Atti della Accademia Peloritana dei Pericolanti 97 (2019b) 1–12.

S. Lundqvist, Vector space bases associated to vanishing ideals of points., J. Pure Appl. Algebra 214 (2010) 309–321.

M. Ceria, Applications of Bar Code to involutive divisions and a greedy algorithm for complete sets., arXiv:1910.02802 (2019) 1–18.

W. M. Seiler, A combinatorial approach to involution and δ-regularity. I: Involutive bases in polynomial algebras of solvable type, Appl. Algebra Eng. Commun. Comput. 20 (2009) 207–259.

S. Lundqvist, Complexity of comparing monomials and two improvements of the Buchberger-Möller algorithm, in: Mathematical methods in computer science. Essays in memory of Thomas Beth, Berlin: Springer, 2008, pp. 105–125.

A. Hashemi, Strong Nœther position and stabilized regularities, Commun. Algebra 38 (2010) 515–533.

W. M. Seiler, Effective genericity, δ-regularity and strong Noether position., Commun. Algebra 40 (2012) 3933–3949.

D. Eisenbud, B. Sturmfels, Finding sparse systems of parameters, J. Pure Appl. Algebra 94 (1994) 143–157.

D. Eisenbud, Commutative algebra. With a view toward algebraic geometry, volume 150, Berlin: Springer-Verlag, 1995.

I. Bermejo, P. Gimenez, Computing the Castelnuovo-Mumford regularity of some subschemes of $\mathbb{P}^n_K$ using quotients of monomial ideals, J. Pure Appl. Algebra 164 (2001) 23–33.

M. Albert, M. Fetzer, E. Sáenz-de Cabezón, W. Seiler, On the free resolution induced by a Pommaret basis, J. Symb. Comp. 68/2 (2015) 4–26.